This Pset works through the proof that the weight 2 Eisenstein series is a quasi-modular form.

Define \( G_2 : \mathcal{H} \rightarrow \mathbb{C} \) by

\[
G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n)q^n,
\]

where \( q = e^{2\pi i \tau} \). Note that the series is absolutely convergent for \( |q| < 1 \), i.e. for \( \Im(\tau) > 0 \). Hence \( G_2 \) is analytic on \( \mathcal{H} \).

1. Show that \( G_2(\tau + 1) = G_2(\tau) \) (this is the transformation under \( T \)).

2. To see how \( G_2 \) transforms under the action of the other generator, \( S \), start by showing that

\[
G_2(\tau) = 2\zeta(2) + \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau)^2}.
\]

3. Now prove that

\[
\frac{1}{\tau^2} G_2 \left( -\frac{1}{\tau} \right) = 2\zeta(2) + \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m + n\tau)^2}.
\]

*Be careful with the order of summation!*

4. Show that

\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m + n\tau)^2} = -8\pi^2 \int_{0}^{\infty} \cos(2\pi mt)g_r(t)dt,
\]

where \( g_r(t) = t \sum_{n=1}^{\infty} e^{2\pi int\tau} \) for \( t > 0 \)

and \( g_r(0) = \lim_{t \to 0^+} g_r(t) = -\frac{1}{2\pi i \tau} \).

5. Now use the previous point to prove that

\[
\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(m + n\tau)^2} = \sum_{n \in \mathbb{Z} \setminus \{0\}} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau)^2} - \frac{2\pi i}{\tau}.
\]

*Note: In particular this shows that the double sum is not absolutely convergent.*
6. Prove that $G_2$ transforms under the action of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as follows:

$$G_2 \left( -\frac{1}{\tau} \right) = \tau^2 G_2(\tau) - 2\pi i \tau.$$ 

7. Quasi-modularity: Prove that for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and any $\tau \in \mathcal{H}$ we have

$$G_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 G_2(\tau) - 2\pi ic(c\tau + d).$$

*Note:* You have to show that $c(c\tau + d)$ and $(c\tau + d)^2$ behave well w.r.t. matrix multiplication.

8. Set $\tilde{G}_2(\tau) = -\frac{1}{8\pi^2} G_2(\tau)$. Show that

(a) $\tilde{G}_2(\tau) = \frac{1}{2} \zeta(-1) + \sum_{n=1}^{\infty} \sigma_1(n) q^n$

(b) $\tilde{G}_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \tilde{G}_2(\tau) - \frac{1}{4\pi i} c(c\tau + d)$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ and $\tau \in \mathcal{H}$. 