This problem set concerns the Γ-function and its properties. It is inspired by a homework given by Noah Snyder at Harvard in 2002 to a bunch of high-school students.

Define \( \Gamma(x) = \int_0^\infty e^{-t} t^x \frac{dt}{t} \).

1. Show that this definition makes sense for all real \( x > 0 \). Furthermore, show by integration by parts that \( \Gamma(x+1) = x\Gamma(x) \). Compute \( \Gamma(n) \) for a positive integer \( n \).

2. Show that
\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^y}{(1+t)^{x+y}} \frac{dt}{t}.
\]

Conclude that \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and prove the “duplication formula”
\[
\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right).
\]

3. Show that, for \( 0 < a < 1 \),
\[
\int_0^\infty \frac{t^a}{1+t} \frac{dt}{t} = \frac{\pi}{\sin \pi a}.
\]

**Hint:** Consider \( \int \frac{z^{a-1}}{1-z} dz \), taken over a well-chosen path. For instance, it could consist of two circles of radii \( R \) and \( \rho \) respectively, joined along the negative real axis from \(-R\) to \(-\rho\).

4. Combine your earlier results to show that
\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{for} \ 0 < x < 1.
\]

5. Show that the definition of \( \Gamma \) makes sense for all complex numbers with positive real part. Show that \( \Gamma \) has meromorphic continuation to the whole complex plane. Show that all the formulas you have proved for \( \Gamma \) for real values must be true for all complex values. Find the poles of \( \Gamma(z) \), their orders, and compute the residues at those points. Show that \( \frac{1}{\Gamma(z)} \) has no poles. Conclude that \( \Gamma \) has no zeros.
6. Prove the product expansions

\[
\sin \pi z = \pi z \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{z/n} = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right).
\]

and

\[
\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-z/n},
\]

where \( \gamma \) is Euler’s constant given by \( \gamma = \lim_n \frac{1}{n} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n. \)

7. Stirling’s Asymptotic Formula

Show that for \( |z| \to \infty \) and \(-\pi + \delta < \arg z < \pi - \delta\), where \( \delta \) is any fixed positive number, we have

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log 2\pi + O(|z|^{-1})
\]

and

\[
\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O(|z|^{-1}).
\]

Here \( \log z \) denotes the principal branch of the logarithm.