Non-homogeneous linear ODEs with constant coefficients

\[ a_\ell y^{(\ell)} + a_{\ell-1} y^{(\ell-1)} + \cdots + a_1 y' + a_0 y = G(x), \] where \( a_j \) are real numbers, \( a_\ell \neq 0 \)

**Note:** We can solve this only when \( G(x) \) looks like \( P(x)e^{kx} \cos(mx) \) or \( P(x)e^{kx} \sin(mx) \), where \( P \) is a polynomial in \( x \) (could be 0 or 1 or any other constant), \( k \) is any real number (including 0) and \( m \) is any integer (this could be 0 as well). In case \( m = 0 \), \( G(x) = P(x)e^{kx} \).

1. Write down the associated homogeneous equation

\[ a_\ell y^{(\ell)} + a_{\ell-1} y^{(\ell-1)} + \cdots + a_1 y' + a_0 y = 0 \]

and find its general solution \( y_h \), as follows.

(a) Write down the characteristic polynomial \( p(s) = a_\ell s^\ell + \cdots + a_0 \).
(b) Find its roots \( r_1, \ldots, r_\ell \).
(c) Write down the general solution to the homogeneous equation

\[ y_h = C_1 y_1 + \cdots + C_\ell y_\ell. \]

Take into account multiplicities (double/triple/etc... roots) and pairs of complex roots.

2. Find a particular solution \( y_p \) of the non-homogeneous equation using the method of undetermined coefficients.

If \( G(x) = P(x)e^{kx} \cos(mx) \) or \( G(x) = P(x)e^{kx} \sin(mx) \), try

\[ y_p = x^s \left[R_1(x)e^{kx} \cos(mx) + R_2e^{kx} \sin(mx) \right], \]

where \( \deg(R_1) = \deg(R_2) = \deg(P) \) and \( s \) is the smallest nonnegative integer such that no term in \( y_p \) duplicates a term in \( y_h \). For instance,

- if \( G(x) = de^{kx} \), then try \( y_p = Ax^s e^{kx} \);
- if \( G(x) = P(x)e^{kx} \), then try \( y_p = x^s R(x)e^{kx} \) with \( R \) a polynomial in \( x \) of the same degree as \( P \);
- if \( G(x) = P(x) \), then try \( y_p = x^s R(x) \) with \( R \) a polynomial in \( x \) of the same degree as \( P \);
- if \( G(x) = \cos(mx) \) or \( G(x) = \sin(mx) \) or \( G(x) = \cos(mx) + \sin(mx) \), then try \( y_p = x^s (A \cos(mx) + B \sin(mx)) \);
- if \( G(x) = e^{kx} \cos(mx) + \sin(mx) \), then try \( y_p = x^s A e^{kx} (A \cos(mx) + B \sin(mx)) \);
- if \( G(x) = P(x) \cos(mx) \) or \( G(x) = P(x) \sin(mx) \) or \( G(x) = P(x) (\cos(mx) + \sin(mx)) \), then try \( y_p = x^s (R_1(x)e^{kx} \cos(mx) + R_2e^{kx} \sin(mx)) \), with \( R_1 \) and \( R_2 \) polynomials in \( x \) of the same degree as \( P \).

If \( G(x) = G_1(x) + G_2(x) \), then find \( \tilde{y}_j \), \( j = 1, 2 \) particular solution for the equation

\[ a_\ell \tilde{y}_j^{(\ell)} + a_{\ell-1} \tilde{y}_j^{(\ell-1)} + \cdots + a_1 \tilde{y}_j' + a_0 \tilde{y}_j = G_j(x) \]

and then take \( y_p = \tilde{y}_1 + \tilde{y}_2 \).

3. Write down the general solution \( y = y_h + y_p \).

4. If it is an initial value problem or a boundary problem, plug in the given values and solve for \( C_1, \ldots, C_\ell \). Don’t forget to take the derivative of \( y \) in the case of an initial value problem (chain rule!).