SECOND ORDER LINEAR HOMOGENEOUS ODEs

\[ mx'' + bx' + kx = 0, \quad m, b, k \text{ constants, } m \neq 0 \]

1. If the equation describes a physical spring–dashpot system, the coefficients \( m, b, k \) are non-negative.

2. An initial value problem consists of a differential equation and an initial condition \( x(t_0) = A, \; x'(t_0) = B \). How many solutions does it have?

3. A boundary problem consists of a differential equation and a boundary condition \( x(t_0) = A, \; x(t_1) = B \). How many solutions does it have?

4. The general solution is of the form \( x(t) = c_1 x_1(t) + c_2 x_2(t) \), where \( x_1 \) and \( x_2 \) are two linearly independent solutions (none of them can be written as a constant multiple of the other).

5. The characteristic polynomial of this equations is \( p(s) = ms^2 + bs + k \).

6. The exponential solutions of this equation are \( c_1 e^{r_1 t} \) and \( c_2 e^{r_2 t} \), where \( r_1, r_2 \) are the roots (real or complex) of the characteristic polynomial and \( c_1, c_2 \) are arbitrary constants. If \( r_1 = r_2 = r \), there is only one family of exponential solutions, namely \( ce^{rt} \).

How to solve:

1. Write down the characteristic equation \( ms^2 + bs + k = 0 \).

2. Compute its discriminant \( \Delta = b^2 - 4mk \).

3. There are three possible situations:
   - overdamped: If \( \Delta > 0 \), the quadratic equation has two distinct real solutions, \( r_1 \) and \( r_2 \). Find them. (You might need to use the quadratic formula.) The general solution of the differential equation is
     \[ x = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \]
   - critically damped: If \( \Delta = 0 \), the quadratic equation has only one real root \( r \). Find it. The general solution of the differential equation is
     \[ x = C_1 e^{rt} + C_2 te^{rt}. \]
   - underdamped: If \( \Delta < 0 \), the quadratic equation does not have any real roots, it has two complex conjugate roots \( r_{1,2} = \alpha \pm i\beta \), where \( \alpha = -\frac{b}{2m} \) and \( \beta = \frac{\sqrt{|\Delta|}}{2m} \). The general solution of the differential equation is
     \[ x = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t) \quad \text{or} \quad x = Ae^{\alpha t} \cos(\beta t - \phi). \]

   In the second expression, \( A \) is any real number and \( \phi \) any number in \([0, 2\pi)\).

4. If it is an initial value problem or a boundary problem, plug in the given values and solve for \( C_1 \) and \( C_2 \) (or for \( A \) and \( \phi \)). Don’t forget to take the derivative of \( x \) in the case of an initial value problem (chain rule!).