FIRST ORDER DIFFERENTIAL EQUATIONS

1. A first order differential equation is an equation of the form
   \[ F(x, y, y') = 0. \]
   A solution of the differential equation is a function \( y = y(x) \) that satisfies the equation. A differential equation has \textbf{infinitely many} solutions.

2. An equilibrium solution is a constant solution, i.e. a constant function \( y(x) = K \) that satisfies the equation.

3. An initial value problem consists of a differential equation and an initial condition \( y(x_0) = y_0 \). It has a \textbf{unique} solution, which has to be a function. For instance \( y = \pm \sqrt{x} \) cannot be the solution of an i.v.p.

A. Separable Equations

   \[ \frac{dy}{dx} = f(x)g(y) \]

   1. Separate the variables: \( \frac{dy}{g(y)} = f(x)dx. \)

   2. Integrate both sides: \( \int \frac{dy}{g(y)} = \int f(x)dx \) and get \( G(y) = C + F(x) \).

   3. Solve for \( y \) (if possible).

   4. If it is an initial value problem, plug in the given values and solve for \( C \).

B. Linear equations

   \[ \frac{dy}{dx} + P(x)y = Q(x) \]

   1. Write the equation in the above form.

   2. Compute the integrating factor \( I(x) = e^{\int P(x)dx} \).

   3. Multiply both sides of the equation by this integrating factor: \( I(x)\frac{dy}{dx} + I(x)\cdot P(x)y = I(x)\cdot Q(x) \).

   4. Integrate both sides and get \( I(x)y = C + \int I(x)Q(x)dx \).

   5. Solve for \( y \).

   6. If it is an initial value problem, plug in the given values and solve for \( C \).

C. Homogeneous Equations

   \[ \frac{dy}{dx} = F\left(\frac{y}{x}\right) \]

   1. Make the substitution \( v = \frac{y}{x} \). \textbf{Attention:} \( \frac{dy}{dx} = x\cdot \frac{dv}{dx} + v. \)
2. Solve the new equation, not forgetting the constant involved. (It will probably be a separable equation.)
3. Find \(y(x) = xv(x)\).
4. If it is an initial value problem, make sure you find the constant.

**D. Bernoulli Equations**

\[
\frac{dy}{dx} + P(x)y = Q(x)y^n
\]

1. Make the substitution \(v = y^{1-n}\).
2. Solve the new linear equation: \(\frac{dv}{dx} + (1-n)P(x)y = (1-n)Q(x)\).
3. Find \(y(x)\).
4. If it is an initial value problem, make sure you find the constant.

**E. Other Substitutions**

Some equations need some other substitution to transform them in a known type. These substitutions vary greatly and there are no general formulas that can help. However, the more you practice, the better your intuition becomes in these matters.

**F. Linear first order equations with sinusoidal input**

\[
y' + ky = B\cos(\omega t) \text{ or } B\sin(\omega t)
\]

Such an equation can be seen as the real or imaginary part of the complex differential equation

\[
z' + kz = Be^{i\omega t}
\]

The general idea is that the solutions to (*) are of the form

\[y = y_p + y_h,\]

where \(y_p\) is a particular solution of the original equation (*), and \(y_h\) is the general solution to the associated homogeneous equation (see below).

1. Solve the corresponding homogeneous equation \(y' + ky = 0\). The solution is \(y_h = Ce^{-kt}\). This is where the arbitrary constant appears!
2. Find a particular solution \(z_p\) of (**). It will usually be of the same form as the input, namely \(Ae^{i\omega t}\). So try that and determine \(A\).
3. Write \(y_p\) by taking either the real or imaginary part of \(z_p\). If the input in (*) is cos, you should take the real part, if it is sin, the imaginary part.
4. Write down the general solution to (*), \(y = y_p + y_h\)
5. If it is an initial value problem, find the constant.