Midterm Solutions

Average 30.8
STD 9.
Median 33

(1) (a) Let $g \in L^2([0,2\pi])$ be a 2π-periodic function. Then

$$\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2\,dx$$

(b) $f \in C^2 \implies$ Fourier series of f converges absolutely + uniformly to f

Hence we can write $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$

$$= \lim_{N\to\infty} S_N f(x)$$

$$\therefore \quad |S_N f(x) - f(x)| = \Big|\sum_{|n|>N} \hat{f}(n) e^{inx}\Big|$$

$$\le \sum_{|n|>N} |\hat{f}(n)|$$

Recall that $\widehat{f'}(n) = in\hat{f}(n)$ so by Parseval

$$1 = \frac{1}{2\pi}\int_0^{2\pi} |f'(x)|^2\,dx = \sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2$$

$$\therefore \quad |S_N f(x) - f(x)| \le \sum_{|n|>N} |\hat{f}(n)| \underset{\substack{\uparrow \\ \text{Cauchy} \\ \text{-Schwarz}}}{\le} \Big(\sum_{|n|>N} n^2 |\hat{f}(n)|^2\Big)^{1/2} \times \Big(\sum_{|n|>N} n^{-2}\Big)^{1/2}$$

$$\cancel{E} \leq \sqrt{2\pi} \left(\sum_{|n| \geq N} n^{-2} \right)^{1/2}$$

$$\lesssim C N^{-1/2} \quad \text{as desired.}$$

(2) Let $E_\lambda := \{ x : |f(x)| > \lambda \}$ $\lambda > 0$

Then $0 \leq \lambda \chi_{E_\lambda}(x) \leq |f(x)| \chi_{E_\lambda}(x)$

Hence $\lambda \int \chi_{E_\lambda}(x)\,dx \leq \int |f| \chi_{E_\lambda}$

or in other words

$$\lambda \mu(E_\lambda) \leq \int |f|$$

Fix $\varepsilon > 0$

(3) Let $x, y \in \mathbb{R}$ and say $x < y$.

Then

$$|f(y) - f(x)| = \left| \int_{[x,y]} f(z)\,dz \right| \leq \int_{[x,y]} |f(z)|\,dz$$

Let $N \in \mathbb{N}$ and set $\cancel{f}$

$$E_N = \{ x : |f(x)| \leq N \}$$

set $f_N(x) = |f(x)| \chi_{E_N}(x)$

Then $f_N \leq f_{N+1}$ and by Monotone convergence

$$\int f_N \longrightarrow \int |f|$$

Fix $N > 0$ large enough s.t.

$$\int \big| |f| - f_N \big| < \varepsilon/2$$

Now

$$\int_{[x,y]} |f| = \int_{[x,y]} |f| - f_N + \int_{[x,y]} f_N$$

$$\leq \int \big||f| - f_N\big| + N|x-y|$$

$$\leq \varepsilon/2 + N|x-y|$$

Choose δ small enough st $N|x-y| < \varepsilon/2$
when $|x-y| < \delta$

$\Rightarrow \int_{[x,y]} |f| \leq \varepsilon$ whenever $|x-y| < \delta$.

(4) The Fourier series of f is absolutely convergent.

i.e $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$

Hence $S_N f(x) \longrightarrow f(x)$ uniformly in x.
and we can justify writing

$$|f(x) - f(y)| \leq \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} |\hat{f}(n)|\,|e^{inx} - e^{iny}|$$

$$\leq \sum_{|n| > \frac{1}{|x-y|}} |\hat{f}(n)|\,|e^{inx} - e^{iny}| + \sum_{1\leq |n| \leq \frac{1}{|x-y|}} |\hat{f}(n)|\,|e^{inx} - e^{iny}|$$

Note that $|e^{inx} - e^{iny}| \leq \min\left(|n|\,|x-y|, 2\right)$

so above is

$$\leq \sum_{|n| > \frac{1}{|x-y|}} 2\,|n|^{-3/2} + \sum_{1\leq|n|\leq\frac{1}{|x-y|}} |x-y|\,|n|^{-1/2}$$

$$\leq C\,|x-y|^{-1/2}$$

$\leq \quad C \; |x-y|^{1/2}$