18.103 MIDTERM REVIEW

MIDTERM WILL TAKE PLACE FRIDAY NOV 4TH IN CLASS

1. Reading

The midterm will cover there material from Chapters 2, 3 of Fourier Analysis by Stein and Shakarchi, the material from Chapters 1 of Real Analysis by Stein and Shakarchi, and Sections 1, 2 from Chapter 2 of Real Analysis.

The test will consist of 3 or 4 problems.

2. Homework Exercises to Review

Some of the questions on the midterm may be drawn from your HW problems including the recommended problems.

3. Practice problems

(1) Let $f \in C^2(\mathbb{R})$ be a 2π periodic function. Suppose that f' satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(x)|^2 \, dx = 1 \tag{3.1}$$

Let $S_N f$ denote the Nth partial Fourier sum of f. Prove that there exists a constant $C > 0$ so that

$$|S_N f(x) - f(x)| \le C N^{-\frac{1}{2}}$$

Hint: You may want to use Parseval's theorem here along with the Cauchy Schwarz inequality to make use of (3.1) – recall that the space $\ell^2(\mathbb{Z})$ from example 1 on page 73 of Stein and Shakarchi is a Hilbert Space, and that Cauchy Schwarz inequality here is given by

$$\left|\sum a_n \bar{b}_n\right| \le \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |b_n|^2\right)^{\frac{1}{2}}$$

(2) Consider the sequence $\{a_k\} \in \ell^2(\mathbb{Z})$ given by

$$a_k = \begin{cases} \frac{1}{k} & \text{if } \ k \ge 1 \\ 0 & \text{if } \ k \le 0 \end{cases}$$

Show that although $\{a_k\} \in \ell^2(\mathbb{Z})$ there is no Riemann integrable function whose kth Fourier coefficient is given by a_k. Hint: think about Abel means and the Poisson kernel.

(3) Suppose $f \in C^0(\mathbb{R})$ and is 2π periodic. Is it true that

$$\int_0^{2\pi} |S_N f(x) - f(x)| \, dx \to 0 \ \text{ as } \ N \to \infty$$

Give a justification of your answer.

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a measurable function that is integrable, in the sense that $\int_{\mathbb{R}} |f| < \infty$. For all $x \geq 0$ define

$$F(x) = \int_{[0,x]} f$$

Show that F is a continuous function. Hint: you might want to argue as in the proof of Proposition 1.12 in Chapter 2 of the Real Analysis book.

(5) Suppose that $f_n : [0,1] \to \mathbb{R}$ are a sequence of measurable functions such that $\int |f_n| = 1$ for each n. Suppose that $f_n \to f$ almost everywhere. What are the possible values of $\int |f|$? Explain your answer.

(6) Prove the following Poincare-type inequality called Wirtinger's inequality: Suppose that $f \in C^1(\mathbb{R})$ is 2π periodic, and satisfies

$$\int_0^{2\pi} f(x)\, dx = 0 \tag{3.2}$$

Then we have

$$\int_0^{2\pi} |f(x)|^2 \, dx \leq \int_0^{2\pi} |f'(x)|^2 \, dx \tag{3.3}$$

Why do we need to require (3.2)?

(7) In this exercise we show that good decay of the Fourier coefficients of a 2π periodic function implies regularity (smoothness). Suppose that $f \in C^0(\mathbb{R})$ is 2π-periodic and the Fourier coefficients $\hat{f}(n)$ satisfy

$$\left|\hat{f}(n)\right| \leq |n|^{-\frac{3}{2}}$$

for all $n \neq 0$. Show that there exists a constant $C > 0$ so that for all x, y,

$$|f(x) - f(y)| \leq C\, |x - y|^{\frac{1}{2}}$$

i.e., $f \in C^{\frac{1}{2}}(\mathbb{R})$. Hint: at some point you will want to estimate $\left|e^{inx} - e^{iny}\right|$ and you should consider distinguishing the cases $n > \frac{1}{|x-y|}$ and the case when $1 \leq n \leq \frac{1}{|x-y|}$.

(8) Use Egorov's theorem to prove the Bounded Convergence Theorem.

(9) Prove that the function given by $f(x) = \frac{1}{1+|x|^{d+1}}$ is integrable on $\mathbb{R}^d$. Prove that $g(x) = \frac{1}{1+|x|^d}$ is not integrable on $\mathbb{R}^d$.