18.103 MIDTERM

PLEASE WRITE CLEARLY AND SHOW ALL YOUR WORK. CLEARLY LABEL WHICH QUESTION YOU ARE ANSWERING IN YOUR WRITE-UPS. PLEASE HAND IN THE WRITE UP SECTION AND WRITE YOUR NAME CLEARLY ON THE FIRST PAGE.

1. Questions

(1) (a) Let g be a 2π periodic function that is Lebesgue square-integrable on the interval $[0, 2\pi]$, i.e, $g \in L^2([0, 2\pi])$. Let $\hat{g}(n)$ denote the Fourier coefficients of g. State Parseval's identity for g.

(b) Let $f \in C^2(\mathbb{R})$ be a 2π periodic function. Recall that $f \in C^2(\mathbb{R})$ means that f, f', and f'' are all continuous functions. Suppose that f' satisfies

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(x)|^2 \, dx = 1. \tag{1.1}$$

Let $S_N f$ denote the Nth partial Fourier sum of f. Prove that there exists a constant $C > 0$ so that

$$|S_N f(x) - f(x)| \leq C N^{-\frac{1}{2}}.$$

(2) Let $f \in L^1(\mathbb{R}^d)$. Given $E \subset \mathbb{R}^d$ measurable, let $\mu(E)$ denote the Lebesgue measure of E. Prove that

$$\mu(\{x \in \mathbb{R}^d \mid |f(x)| > \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f(x)| \, \mu(dx) \tag{1.2}$$

(3) Let $f \in L^1(\mathbb{R})$. Define

$$F(x) := \int_{(-\infty, x]} f(y) \, dy \tag{1.3}$$

Show that F is uniformly continuous on $\mathbb{R}$.

(4) In this problem we show that good decay of the Fourier coefficients of a 2π periodic function implies regularity (smoothness). Suppose that $f \in C^0(\mathbb{R})$ is 2π-periodic and the Fourier coefficients $\hat{f}(n)$ satisfy

$$\left|\hat{f}(n)\right| \leq |n|^{-\frac{3}{2}}$$

for all $n \neq 0$. Show that there exists a constant $C > 0$ so that for all x, y,

$$|f(x) - f(y)| \leq C |x - y|^{\frac{1}{2}}$$

i.e., $f \in C^{\frac{1}{2}}(\mathbb{R})$.

Please write your name here: ______________________________

2. Answers