18.103 HOMEWORK #2

DUE FRIDAY, SEPTEMBER 23RD AT 5PM IN 2-267

1. Reading and Practice

Read Chapter 2 of Stein and Shakarchi, *Fourier Analysis*.

2. Exercises to be handed in

(1) Do Exercises $6, 8, 10$ in in Chapter 2 of Fourier Analysis by Stein and Shakarchi

(2) Recall that for each $N \in \mathbb{N}$ the Dirichlet kernel is defined by

$$D_N(x) := \sum_{n=-N}^{N} e^{inx}$$

Prove the following:

(a)

$$D_N(x) = \frac{\sin[(N+\frac{1}{2})x]}{\sin x/2}$$

(b) There exists a constant $C > 0$ (independent of N) so that

$$|D_N(x)| \leq C \min(N, \frac{1}{|x|}) \tag{2.1}$$

(c) and in fact one can find constants $0 < b < B$ (independent of N) so that

$$b \log(N) \leq \int_{-\pi}^{\pi} |D_N(x)|\, dx \leq B \log(N) \tag{2.2}$$

(Hint: part (c) requires a few tricks. See Problem 2 in Chapter 2 of Fourier Analysis by Stein and Shakarchi for a step by step outline)

(3) This problem ties up a few loose ends from class on Friday in our proof that if f is periodic with period 1, and $f \in C^{\alpha}([-1/2, 1/2])$ with $\alpha \in (0,1)$ then we have $\|S_N f - f\|_{\infty} \to 0$ as $n \to \infty$.

Let $f \in C^{\alpha}([-1/2, 1/2])$ for $\alpha \in (0,1)$ be a 1-periodic function, i.e., f is α-Hölder continuous on $[-1/2, 1/2]$ and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Recall the definitions

$$\|f\|_{\infty} := \sup_{x \in [-1/2,1/2]} |f(x)|$$

$$\|f\|_{C^{\alpha}} := \sup_{x,y \in [-1/2,1/2]} \frac{|f(x) - f(x-y)|}{|y|^{\alpha}}$$

Define

$$h_x(y) = \frac{f(x-y) - f(x)}{\sin \pi y}$$

Fix $\delta > 0$ with $0 < \delta < 1/2$. Show that

(a) There exists a constant $C > 0$ so that

$$|h_x(y)| \leq C \frac{\|f\|_\infty}{\delta}$$

as long as $|y| > \delta$.

(b) Let $\tau > 0$ so that $2\tau < \delta < |y|$. Then there exists a constant $C > 0$ so that

$$|h_x(y) - h_x(y+\tau)| \leq C \left(\frac{|\tau|^\alpha \|f\|_{C^\alpha}}{\delta} + \frac{\|f\|_\infty}{\delta^2} |\tau| \right)$$