18.103 HOMEWORK #1

DUE FRIDAY, SEPTEMBER 16TH AT 5PM IN 2-267

1. Reading and Practice

Read Chapter 1 of Stein and Shakarchi, *Fourier Analysis.* Go through the Exercises at the end of Chapter 1 to remind yourself of some basic facts about complex numbers, convergence of sequences and series, and a few ODEs. You don't have to hand any of these in.

Some of the problems below will be related to the following notion of uniform convergence that you learned in your previous analysis courses. Let $X \subset \mathbb{R}$. Recall that a sequence of functions $\{f_n\}$, $f_n : X \to \mathbb{C}$ is said to converge uniformly to a function $f : X \to \mathbb{C}$ if for all $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ so that for all $n \geq N$ and for all $x \in X$, $|f_n(x) - f(x)| < \epsilon$.

In your 18.100 class you proved the following important proposition:

Proposition 1.1. *Let $f_n : \mathbb{R} \to \mathbb{C}$ be a sequence of continuous functions converging uniformly to f. Then f is also continuous.*

Look up and read over the proof of this proposition from your favorite introductory analysis book.

2. Exercises to be handed in

(1) Let f be defined as follows: on the interval $[-\pi, \pi]$ f, is given by

$$f(x) = \begin{cases} 0 & \text{if } \ -\pi < x \leq 0 \\ 1 & \text{if } \ 0 < x \leq \pi \end{cases}$$

and extend f to all of $\mathbb{R}$ by making it 2π periodic. Compute the Fourier coefficients $\hat{f}(n)$ of f. (Recall that

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

as we defined them in class.)

(2) Consider the trigonometric series $\sum_{n \in \mathbb{Z}} a_n e^{inx}$. Suppose that $\sum_{n \in \mathbb{Z}} |a_n|$ converges, i.e., $\sum_{n \in \mathbb{Z}} |a_n| < \infty$. This implies that for each $x \in \mathbb{R}$ the trigonometric series converges and thus defines a function

$$f(x) := \sum_{n \in \mathbb{Z}} a_n e^{inx}.$$

Prove that f is continuous.

Thanks to Larry Guth for the use of several problems from his 18.103 class from previous years.

(3) A major theme in the class this year will be proving quantitative estimates. In your previous class, there was a big focus on proving *qualitative* statements, for example Proposition 1.1 above establishes a qualitative statement about the limit function f – that it's continous. To begin to get a feel for quantitative estimates, below is a problem related to Proposition 1.1.

Let $f_n : [0,1] \to \mathbb{C}$ be a sequence of continuous functions and let $f : [0,1] \to \mathbb{C}$ be continuous. Suppose that f_n, f obey the following inequalities: For each $x \in [0,1]$,

$$|f_n(x) - f(x)| < \frac{1}{n}. \tag{2.1}$$

And for each $n \in \mathbb{N}$, f_n satisfies the following estimate: for all $x, x' \in [0,1]$ we have

$$|f_n(x) - f_n(x')| \leq n\,|x - x'| \tag{2.2}$$

Estimate (2.1) ensures that $f_n \to f$ uniformly on $[0,1]$ and estimate (2.2) shows us that each f_n is continuous. Prove the following quantitative estimate about the continuity of the limiting function f: For all $x, x' \in [0,1]$ we have

$$|f(x) - f(x')| \leq 1000\,|x - x'|^{\frac{1}{2}}\,. \tag{2.3}$$