18.103 MIDTERM REVIEW

MIDTERM WILL TAKE PLACE FRIDAY NOV 4TH IN CLASS

1. Reading

The final will cover there material from Chapters 2, 3, 5, 6 of *Fourier Analysis* by Stein and Shakarchi, Chapters 1, 2 of *Real Analysis* by Stein and Shakarchi, the material we covered in class from Chapter 3.1 and 3.2 from of *Real Analysis.* And the material on Hilbert spaces and L^2 from Chapter 4 and Chapter 5.1 of *Real Analysis* that we covered in class.

2. HW Exercises to Review

Some of the questions on the Final may be drawn from your HW problems including the recommended problems.

3. MIdterm Review

please review the material on the Midterm review as well.

4. additional practice problems

(1) The goal of this problem is to prove a version of Bernstein's inequality for 2π periodic functions on the line.

Proposition 4.1 (Bernstein's Lemma)**.** *Let f be a trigonometric polynomial with $\hat{f}(k) = 0$ for all $|k| > n$. Then, there exists an absolute constants $C_1 > 0$ and $C_2 > 0$ so that*

$$\|f'\|_{L^\infty(\mathbb{S}^1)} \leq C_1 n \|f\|_{L^\infty(\mathbb{S}^1)} \tag{4.1}$$

and

$$\|f'\|_{L^1([0,2\pi])} \leq C_2 n \|f\|_{L^1([0,2\pi])} \tag{4.2}$$

Outline of proof. For the L^1 estimate in (4.2) use the following hint: Let K_n denote the Fejer kernels and define

$$V_n(x) := (1 + e(nx) + e(-nx))K_n(x)$$

where we have used the notation $e(nx) := e^{inx}$. V_n is called de la Vallée Poussin's kernel. Show that

$$\hat{V}_n(j) = 1 \quad \forall\, |j| \leq n$$

and

$$\|V_n'\|_{L^1(\mathbb{S}^1)} \leq Cn$$

Next show that we can write $f = V_n * f$ and go from there. There is an easier argument for the L^∞ estimate in (4.1). $\square$

(2) In this problem we'll prove a more general version of Bernstein's inequality on $\mathbb{R}^d$ than the L^2 version we proved in class. However, we'll have to assume that we know the following inequality about convolutions called Young's inequality. You can take Young's inequality for granted or look up the proof. Recall the $L^p(\mathbb{R}^d)$ norm of a function f is defined by

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Lemma 4.2 (Young's inequality). *Let $1 \leq r, p, q \leq \infty$ so that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

Then if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ the convolution $f * g$ is defined for a.e. $x \in \mathbb{R}^d$ and we have

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$$

Prove the following version of Bernstein's inequality:

Proposition 4.3 (Bernstein's inequality). *Let $f \in \mathcal{S}(\mathbb{R}^d)$ and suppose that its Fourier transform $\hat{f}$ is supported in the disc of radius R, i.e,*

$$\operatorname{supp}(\hat{f}) \subset B(0, R) := \{\xi \in \mathbb{R}^d \mid |\xi| < R\}$$

Then, there exists an absolute constant $C > 0$ so that for any $1 \leq p \leq r \leq \infty$ we have

$$\|f\|_{L^r(\mathbb{R}^d)} \leq C R^{d(\frac{1}{p} - \frac{1}{r})} \|f\|_{L^p(\mathbb{R}^d)}$$

(Hint: let $\phi \in \mathcal{S}$ be such that $\hat{\phi}(\xi) = 1$ on $B(0, 1)$. Set $\hat{\phi}_{R^{-1}}(\xi) := \hat{\phi}(R^{-1}\xi)$ and note that this is equal to 1 on $B(0, R)$. Show that $f = \phi_{R^{-1}} * f$. Then use Young's inequality.)

(3) Consider the Cauchy problem for the Schrödinger equation on $\mathbb{R}^d$ with Schwartz data

$$\begin{aligned} i\partial_t \psi + \Delta \psi &= 0 \\ \psi(0, x) &= f \in \mathcal{S}(\mathbb{R}^d) \end{aligned} \tag{4.3}$$

Use the Fourier transform to show that a solution can be represented by the formula

$$\psi(t, x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-4it\pi^2 |\xi|^2} \hat{f}(\xi) \, d\xi$$

Show that this solution conserves mass in the sense that

$$\|\psi(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)} \quad \forall t \in \mathbb{R}$$

Show in addition that we can represent the solution by the formula,

$$\psi(t, x) = c_d t^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) \, dy$$

Conclude that solutions decay like $t^{-\frac{d}{2}}$ in time, by showing the bound

$$\|\psi(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq c_d t^{-\frac{d}{2}} \|f\|_{L^1}$$

where above $c_d > 0$ is a constant that depends only on the dimension d.

(4) Let $E \subset \mathbb{R}$ be a measurable set with $\mu(E) = 1$. Prove that that there exists an open interval $I \subset \mathbb{R}$ so that

$$\mu(E \cap I) \geq \frac{9}{10}\mu(I)$$

Hint: recall that any finite measure subset of $\mathbb{R}$ can be well approximated by a finite union of intervals. And any finite union of intervals can be expressed as a disjoint finite union of intervals in a unique way.

(5) Suppose that $f \in \mathcal{S}(\mathbb{R})$ so that $\int_{\mathbb{R}} |f|^2 = 1$. Suppose in addition that $\hat{f}$ is supported in the interval $[-1, 1]$, i.e., $\operatorname{supp}(\hat{f}) \subset [-1, 1]$. Prove that there exists an absolute constant $C > 0$ so that

$$|f(x) - f(y)| \leq C\|f\|_{L^2} |x - y|$$

for all $x, y \in \mathbb{R}$.

(6) Let $f \in L^1(\mathbb{R})$. Define $g(x)$ via the convolution

$$g = f * e^{-|x|^2} = \int_{\mathbb{R}} f(y)e^{-|x-y|^2} \, dy$$

Prove that

$$\lim_{|x|\to\infty} g(x) = 0$$

(7) Read and study section 5.2.1 in *Fourier Analysis* on the time-dependent heat equation on $\mathbb{R}$.

(8) Prove that if $\{K_\delta\}_{\delta>0}$ is an approximation to the identity in the sense of Chapter 3.2 in *Real Analysis*, then

$$\sup_{\delta>0} |K_\delta * f(x)| \leq cf^*(x)$$

for some absolute constant $c > 0$. Above $f^*(x)$ is the Hardy-Littlewood maximal function defined by

$$f^*(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, dy$$

where the supremum above is over all balls B containing the point x.