Multiscale Analysis on bounded domains with restricted interpolation points

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Supervised by Holger Wendland

29th June 2011
Overview

1. Introduction to the Multilevel algorithm
2. Motivation
3. Previous Results
4. Extended Convergence Results
5. Numerical Simulations
6. Future Work

Details are in: Multilevel analysis in Sobolev spaces on bounded domains with restricted data points. (In Preparation) by Townsend & Wendland
Interpolating a target function

**INPUT**

- \( \Omega \subseteq \mathbb{R}^d \) bounded Lipschitz domain.
- Continuous target function \( f \in \mathcal{H}^r(\Omega) \)

**Multilevel Algorithm**

\[
\begin{align*}
&f_0 = 0, \quad e_0 = f \\
&\text{for } j = 1, 2, \ldots, n \\
&\quad \text{Compute } s_j \text{ such that } s_j(x) = e_{j-1}(x) \quad \forall x \in X_j \\
&\quad f_j = f_{j-1} + s_j \\
&\quad e_j = e_{j-1} - s_j \\
&\text{end}
\end{align*}
\]

**OUTPUT**

- Interpolant to \( f \) and an idea of how well we did.
Notation

For any data set $X = \{x_1, \ldots, x_N\} \subset \Omega$ then the **fill distance** is

$$h_{X,\Omega} := \sup_{x \in \Omega} \min_{1 \leq i \leq n} \|x - x_i\|_2$$

This is the parameter we use to state all convergence orders.

Choose sequence of quasi-uniform data sets $X_1, X_2, \ldots, X_n \subset \Omega$ to have decreasing fill distance.

**Definition (Compactly Supported Radial Function)**

A function $\Phi : \mathbb{R}^d \mapsto \mathbb{R}$ such that $\Phi(x) = \phi(\|x\|_2)$ for $\forall x \in \mathbb{R}$ with a continuous $\phi : [0, \infty) \mapsto \mathbb{R}$ and $\phi(t) = 0$ for $t \geq 1$.

$s_j$ is formed by linear combination of scaled and translated compactly supported radial basis functions.
Further Notation

That means,

\[ s_j(x) = \sum_{i=1}^{\mid X_j \mid} \alpha_i^{(j)} \Phi_{\delta_j}(x - x_i) \]

where

\[ \Phi_{\delta_j}(\cdot) = \delta_j^{-d} \Phi \left( \frac{\cdot}{\delta_j} \right) \]

By interpolation conditions we get \( s_j(x_k) = e_{j-1}(x_k) \ \forall x_k \in X_j \).
That means,

\[
s_j(x) = \sum_{i=1}^{|X_j|} \alpha_i^{(j)} \phi_{\delta_j}(x - x_i)
\]

where

\[
\phi_{\delta_j}(\cdot) = \delta_j^{-d} \Phi \left( \frac{\cdot}{\delta_j} \right)
\]

By interpolation conditions we get \( s_j(x_k) = e_{j-1}(x_k) \ \forall x_k \in X_j \).

This corresponds to the symmetric, positive definite linear system

\[
A_{X,\Phi} \alpha^{(j)} = e_{j-1}|x_j
\]

Compactly supported radial basis functions \( \Rightarrow A_{X_j,\Phi} \) is sparse.
Multilevel Motivation

For compactly supported RBFs [Schaback, 1997]:

High accuracy $\leftrightarrow$ low efficiency

Each level is fast but low accuracy. Get high accuracy by working on finer levels!

Interpolation matrices have good conditioning.

Capture large scale variation on coarse level and details on finer levels.
Multiscale analysis in Sobolev spaces on bounded domains

Theorem (Wendland, 2010)

Let $\Omega \subset \mathbb{R}^d$ be Lipschitz bounded domain and a sequence of denser data sets $X_1, X_2 \cdots \subset \Omega$. Further, let $\Phi$ be a compactly supported radial basis function with reproducing Hilbert space equivalent to $\mathcal{H}^\tau(\mathbb{R}^d)$. Then if the target function $f \in \mathcal{H}^\tau(\Omega)$, $\tau > d/2$ then the multilevel algorithm converges with

$$\|e_n\|_{L_2(\Omega)} \leq C h_n^{\tau-\epsilon} \|f\|_{\mathcal{H}^\tau(\Omega)}$$

for some constants $C$ and $\epsilon < \tau$.

Problem: Support of interpolant overlaps the domain boundary.
Support is not contained on domain

Support overlapping the boundary causes three main problems:

- Large point-wise error when on cracked domains.
- Unable to enforce boundary conditions of the interpolant.
- Makes Galerkin methods for solving PDEs hard to analyse.
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Multilevel with restricted interpolation points in action

Target function

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Multilevel with restricted interpolation points in action

Interpolant after three levels

$\ell_2$ error = $1.7 \times 10^{-1}$
Multilevel with restricted interpolation points in action

Interpolant after five levels

$l^2$ error = 2.8e−02

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Interpolant after ten levels

\[ l_2 \text{ error } = 2.8 \times 10^{-7} \]
Restricting Interpolation points

Definition ($\delta$-interior of a domain)

Given $\delta > 0$ and a bounded domain $\Omega \subset \mathbb{R}^d$ the $\delta$-interior of $\Omega$ is

$$\Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}$$

- Want to do
  $$\| e_j \|_{L^2(\Omega)} \leq \| e_j \|_{L^2(\Omega_\delta)} + \| e_j \|_{L^2(\Omega \setminus \Omega_\delta)}$$
- For the region near the boundary, no existing theory.
- If the $\delta$-interior domain has smooth boundary we can use previous ideas.
Problem

Ω is a Lipschitz domain ⇆ Ω_δ is a Lipschitz domain.

Things that can go wrong – Cusp forms

Interweave δ-interior domains with interior cone domains:

\[ K_1 \subseteq \Omega_{\delta_1} \subseteq K_2 \subseteq \Omega_{\delta_2} \subseteq \ldots \subseteq K_n \subseteq \Omega_{\delta_n} \]

For each level \( j \),

\[ \| e_j \|_{L^2(\Omega)} \leq \| e_j \|_{L^2(K_j)} + \| e_j \|_{L^2(\Omega \setminus K_j)} \]
Make the interpolation matrix banded.
Estimate convexity of $p \mapsto \| A^{-1} \|_p$
With quasi-uniform data set $X_j \subset \Omega_{\delta_j}$ and $\delta_j = \nu h_{X_j, \Omega_{\delta_j}}$ we have

$$\left\| A_{X_j, \Phi}^{-1} \right\|_{\infty} \leq C_{\delta_j} \delta_j^d$$
Getting Convergence

- Allows us to bound interpolants,
  \[ \| s_f \|_{L_\infty(\Omega)} \leq C \| f \|_{L_\infty(\Omega)} \]
  and error at each level,
  \[ \| e_n \|_{L_\infty(\Omega)} \leq D(1 + C)^n \| f \|_{H^\tau(\Omega)} \]

- Using **Conditional Brownian Motion in rapidly exhaustible domains**
  [Falkner, 1987] gives
  \[ \text{Vol}(\Omega \setminus K_n) \leq C\delta_n \]
  Hence,
  \[ \| e_n \|_{L_2(\Omega \setminus K_n)} \leq C\delta_n^{1/2} \| e_n \|_{L_\infty(\Omega)} \leq C(1 + C)^n\delta_n^{1/2} \| f \|_{H^\tau(\Omega)} \]
New convergence result

Theorem

Let \( \Omega \subset \mathbb{R}^d \) be Lipschitz bounded domain and a sequence of denser quasi-uniform data sets \( X_1 \subset \Omega_{\delta_1}, X_2 \subset \Omega_{\delta_2} \ldots \). Further, let \( \Phi \) be a compactly supported radial basis function with reproducing Hilbert space equivalent to \( \mathcal{H}^\tau(\mathbb{R}^d) \). Then if the target function \( f \in \mathcal{H}^\tau(\Omega), \tau > d/2 \) then the multilevel algorithm converges with

\[
\| e_n \|_{L_2(\Omega)} \leq \left( C_1 h_n^{\tau-\epsilon_1} + C_2 h_n^{1/2-\epsilon_2} \right) \| f \|_{\mathcal{H}^\tau(\Omega)}
\]

for some constants \( C_1, C_2, \epsilon_1 \) and \( \epsilon_2 \).

If we have \( l \in \mathbb{N}, l < \tau - d/2 \) vanishing derivatives of the function on the boundary. Then,

\[
\| e_n \|_{L_2(\Omega)} \leq \left( D_1 h_n^{\tau-\epsilon_3} + D_2 h_n^{l+1/2-\epsilon_4} \right) \| f \|_{\mathcal{H}^\tau(\Omega)}
\]
Apply the multilevel algorithm with Wendland’s radial basis function $\phi_{2,1} \in \mathcal{H}^{2.5}(\Omega)$ to

$$f_k(x) = (\sin(\pi x) \sin(\pi y))^k$$

on the domain $\Omega = [-1, 1]^2$. 

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</table>
Apply the multilevel algorithm with Wendland’s radial basis function $\phi_{2,1} \in \mathcal{H}^{2.5}(\Omega)$ to

$$ f_k = (1 - x^2)^k(1 - y^2)^k(tanh(100(x - y)) + 1)/9 $$

on the domain $\Omega = [-1, 1]^2$. 

![Convergence rates for increasing vanishing derivatives](image)

<table>
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<th>$l_2$</th>
<th>$l_2$ order</th>
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Future Directions

- Convergence Results for variations on the multilevel algorithm.
- Multilevel algorithm backwards.
- Galerkin Methods for solving PDEs.
- Optimising constants.