A fast Chebyshev–Legendre transform using an asymptotic formula

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SIAM OPSFA, 5th June 2015
Introduction

What is the Chebyshev–Legendre transform?

- Suppose we have a degree $N - 1$ polynomial. It can be expressed in the Chebyshev or Legendre polynomial basis:

\[
p_{N-1}(x) = \sum_{k=0}^{N-1} c_{\text{leg}}^k P_k(x) \quad \iff \quad p_{N-1}(x) = \sum_{k=0}^{N-1} c_{\text{cheb}}^k T_k(x)
\]

- The Chebyshev–Legendre transform: forward transform

\[
c_{\text{leg}}^0, \ldots, c_{\text{leg}}^{N-1} \quad O(N(\log N)^2 / \log \log N) \quad c_{\text{cheb}}^0, \ldots, c_{\text{cheb}}^{N-1}
\]

- inverse transform

Applications in:
- Convolution [Hale & T., 14]
- Legendre-tau spectral methods
- QR of a quasimatrix [Trefethen, 08]
- best least squares approximation
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The Chebyshev–Legendre transform:

forward transform

$${c^\text{leg}_0, \ldots, c^\text{leg}_{N-1}} \xrightarrow{O(N(\log N)^2/\log \log N)} {c^\text{cheb}_0, \ldots, c^\text{cheb}_{N-1}}$$

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$$c^\text{leg}_0, \ldots, c^\text{leg}_{N-1} \quad \xrightarrow{\text{forward transform}} \quad O(N(\log N)^2 / \log \log N) \quad c^\text{cheb}_0, \ldots, c^\text{cheb}_{N-1}.$$ 

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New features of our algorithms

- FFT-based approach
- Analysis-based
- No precomputation
The FFT computes the DFT in $O(N \log N)$ operations:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi ink/N}, \quad 0 \leq k \leq N - 1.$$
Introduction

Many special functions are trigonometric-like

Trigonometric functions
\[ \cos(\omega x), \quad \sin(\omega x) \]

Chebyshev polynomials
\[ T_n(x) \]

Legendre polynomials
\[ P_n(x) \]

Bessel functions
\[ J_\nu(z) \]

Airy functions
\[ Ai(x) \]

Also, Jacobi polynomials, Hermite polynomials, cylinder functions, etc.
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- Airy functions: \( Ai(x) \)

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**Legendre polynomials**

\[ P_n(\cos \theta) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos \left( (m + n + \frac{1}{2}) \theta - (m + \frac{1}{2}) \frac{\pi}{2} \right)}{(2 \sin \theta)^{m+1/2}} + R_{n,M}(\theta) \]

\[ C_n = \sqrt{\frac{4}{\pi}} \frac{\Gamma(n + 1)}{\Gamma(n + 3/2)} \]

\[ h_{m,n} = \begin{cases} 1, & m = 0, \\ \prod_{j=1}^{m} \frac{(j-1/2)^2}{j(n+j+1/2)}, & m > 0. \end{cases} \]

Thomas Stieltjes

Boundary Region: Bessel–like

Interior Region: Trig–like
Introduction
Numerical pitfalls of an asymptotic expansionist

\[ P_n(\cos \theta) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos((m + n + \frac{1}{2})\theta - (m + \frac{1}{2})\frac{\pi}{2})}{(2 \sin \theta)^{m+1/2}} + R_{n,M}(\theta) \]

Fix \( M \). Where is the asymptotic expansion accurate?
Computing the Chebyshev–Legendre transform
The transform comes in two parts

The Chebyshev–Legendre transform naturally splits into two parts:

\[
C_{\text{leg}}^0, \ldots, C_{\text{leg}}^{N-1} \quad \longleftrightarrow \quad p_{N-1}(x_0^{\text{cheb}}), \ldots, p_{N-1}(x_{N-1}^{\text{cheb}}) \quad \longleftrightarrow \quad C_{\text{cheb}}^0, \ldots, C_{\text{cheb}}^{N-1}
\]

This bit

\[
\begin{pmatrix}
P_0(\cos \theta_0) & \cdots & P_{N-1}(\cos \theta_0) \\
\vdots & \ddots & \vdots \\
P_0(\cos \theta_{N-1}) & \cdots & P_{N-1}(\cos \theta_{N-1})
\end{pmatrix}
\begin{pmatrix}
C_{\text{leg}}^0 \\
\vdots \\
C_{\text{leg}}^{N-1}
\end{pmatrix}, \quad \theta_k = \frac{k\pi}{N-1}
\]

Task: Compute the following matrix-vector product in quasilinear time?
Computing the Chebyshev–Legendre transform

The transform comes in two parts

The Chebyshev–Legendre transform naturally splits into two parts:

\[ c_{\text{leg}}^0, \ldots, c_{\text{leg}}^{N-1} \rightarrow p_{N-1}(x_0^{\text{cheb}}), \ldots, p_{N-1}(x_N^{\text{cheb}}) \rightarrow c_{\text{cheb}}^0, \ldots, c_{\text{cheb}}^{N-1} \]

Task: Compute the following matrix-vector product in quasilinear time?

\[
P_{NC} = \begin{pmatrix} P_0(\cos \theta_0) & \ldots & P_{N-1}(\cos \theta_0) \\ \vdots & \ddots & \vdots \\ P_0(\cos \theta_{N-1}) & \ldots & P_{N-1}(\cos \theta_{N-1}) \end{pmatrix} \begin{pmatrix} c_{\text{leg}}^0 \\ \vdots \\ c_{\text{leg}}^{N-1} \end{pmatrix}, \quad \theta_k = \frac{k\pi}{N - 1}
\]
The asymptotic expansion

\[ P_n(\cos \theta_k) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos((m + n + \frac{1}{2})\theta_k - (m + \frac{1}{2}) \frac{\pi}{2})}{(2 \sin \theta_k)^{m+1/2}} + R_{n,M}(\theta_k) \]

gives a matrix decomposition (sum of diagonally scaled DCTs and DSTs):

\[ P_N = \sum_{m=0}^{M-1} \left( D_{u_m} C_N D_{C_h_m} + D_{v_m} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{N-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} D_{C_h_m} \right) + R_{N,M}. \]
Computing the Chebyshev–Legendre transform

Asymptotic expansions as a matrix decomposition

The asymptotic expansion

\[ P_n(\cos \theta_k) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos((m + n + \frac{1}{2})\theta_k - (m + \frac{1}{2}) \frac{\pi}{2})}{(2 \sin \theta_k)^{m+1/2}} + R_{n,M}(\theta_k) \]

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\[ P_N = \sum_{m=0}^{M-1} \left( D_{um} C_N D_{Ch_m} + D_{vm} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{N-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} D_{Ch_m} \right) + R_{N,M}. \]
Computing the Chebyshev–Legendre transform
Asymptotic expansions as a matrix decomposition

The asymptotic expansion

\[ P_n(\cos \theta_k) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos((m + n + \frac{1}{2})\theta_k - (m + \frac{1}{2})\frac{\pi}{2})}{(2 \sin \theta_k)^{m+1/2}} + R_{n,M}(\theta_k) \]

gives a matrix decomposition (sum of diagonally scaled DCTs and DSTs):

\[
P_N = \sum_{m=0}^{M-1} \left( D_{u,m}C_ND_{Ch_m} + D_{v,m} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{N-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} D_{Ch_m} \right) + R_{N,M}.
\]
Computing the Chebyshev–Legendre transform

Be careful and stay safe

Error curve: \( |R_{n,M}(\theta_k)| = \epsilon \)

\[ R_{N,M} = \]

Fix \( M \). Where is the asymptotic expansion accurate?

\[ |R_{n,M}(\theta_k)| \leq \frac{2C_n h_{M,n}}{(2 \sin \theta_k)^{M+1/2}} \]
Theorem

The matrix-vector product $f = P_{NC} c$ can be computed in $O(N(\log N)^2/\log\log N)$ operations.

\( \alpha \) too small
Computing the Chebyshev–Legendre transform
Partitioning and balancing competing costs

\[ P_N = \begin{array}{cccc}
\alpha^3 N & \alpha^2 N & \alpha N & N \\
\end{array} \]

Theorem

The matrix-vector product \( f = P_{NC} \) can be computed in \( O(N (\log N)^2 / \log \log N) \) operations.

\[ P_{EVAL} \]

\[ O(N^2) \]

\[ O(N \log N) \]

\[ \alpha \text{ too small} \]
Theorem
The matrix-vector product \( f = P_N c \) can be computed in \( O(N(\log N)^2 / \log \log N) \) operations.

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The matrix-vector product $f = P_N c$ can be computed in $O(N(\log N)^2 / \log \log N)$ operations.
Theorem
The matrix-vector product $f = P_N c$ can be computed in $O(N(\log N)^2 / \log \log N)$ operations.

$$\alpha = O(1/\log N)$$
Computing the Chebyshev–Legendre transform

Numerical results

No precomputation.

\[ \int_{-1}^{1} P_n(x) e^{-i\omega x} \, dx = i^m \sqrt{\frac{2\pi}{-\omega}} J_{m+1/2}(-\omega) \]
The integral formula for Legendre coefficients gives the following relation:

\[
C_{N}^{\text{leg}} = \left[ I_{N+1} | 0_N \right] D_{S_{2N}} P_{2N}(x_{2N}^{\text{cheb}})^T D_{W_{2N}} T_{2N}(x_{2N}^{\text{cheb}}) \left[ I_{N+1} | 0_N \right] C_{N}^{\text{cheb}},
\]

Execution time:

- \(O(N^2)\)
- \(O(N(\log N)^2 / \log \log N)\)
Spherical harmonic transform:

\[ f(\theta, \phi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \alpha_l^m P_l^m(\cos \theta) e^{im\phi} \]

[Mohlenkamp, 1999], [Rokhlin & Tygert, 2006], [Tygert, 2008]

A new generation of fast transforms with no precomputation.
Thank you


Iteration-free computation of Gauss–Legendre quadrature nodes and weights, SISC, 2014.


References


