ALGEBRAIC SURFACES, LECTURE 1

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1. Castelnuovo's Criterion for Rationality

Theorem 1. *Let X be a surface with $q = h^1(X, \mathcal{O}_X) = 0$, and $p_2 = h^0(X, \omega_X^{\otimes 2}) = 0$. Then X is rational.*

Note. Every rational surface satisfies these: they are birational invariants which vanish for $\mathbb{P}^2$.

Proposition 1. *Reduction 1: Let X be a minimal surface with $q = p_2 = 0$. It is enough to show there is a smooth rational curve C on X with $C^2 \geq 0$.*

Proof. First, observe that $2g(C) - 2 = -2 = C \cdot C(C+K)$ and $\chi(\mathcal{O}_X(C)) = \chi(\mathcal{O}_X) + \frac{1}{2}C(C-K)$. Since $p_2 = 0$, $p_1 = h^0(X, \omega) = h^2(X, \mathcal{O}_X) = 0$ and $\chi(\mathcal{O}_X) = 1$. Since $h^2(C) = h^0(K - C) \leq h^0(K) = 0$, $h^0(C) \geq 1 + \frac{1}{2}C(C-K)$, so $h^0(C) \geq 2 + C^2 \geq 2$. Choose a pencil inside this system containing C, i.e. a subspace of dimension 2. The pencil has no fixed component (the only possibility is C, but C moves in the pencil): after blowing up finitely many base points, we get a morphism $\tilde{X} \to \mathbb{P}^1$ with a fiber isomorphic to $C \cong \mathbb{P}^1$. Therefore $\tilde{X}$ is ruled over $\mathbb{P}^1$ and $\tilde{X}$ is rational (as is X). ☐

Proposition 2. *Reduction 2: Let X be a minimal surface with $q = p_2 = 0$. It is enough to show that $\exists$ an effective divisor D on X s.t. $|K + D| = \varnothing$ and $K \cdot D < 0$.*

Proof. This implies that some irreducible component C of D satisfies $K \cdot C < 0$. Clearly, $|K + C| \subset |K + D|$. Using Riemann-Roch for $K + C$ gives $0 = h^0(K+C) + h^2(K+C) \geq 1 + \frac{1}{2}(K+C)\cdot C = g(C)$. We thus obtain a smooth, rational curve C on X: $-2 = 2g - 2 = C(C+K)$ and $C \cdot K < 0 \implies C^2 \geq -1$. Since X is minimal, $C^2 \neq -1$, so $C^2 \geq 0$ as desired. ☐

We now prove our second reduction: there are three cases.

$K^2 = 0$: Riemann-Roch applied to $-K$ gives

$$h^0(-K) = h^0(-K) + h^2(-K) \geq 1 + \frac{1}{2}K \cdot 2K = 1 + K^2 = 1 \implies |-K| \neq \varnothing \tag{1}$$

Take a hyperplane section H of X. Then there is an $n \geq 0$ s.t. $|H + nK| \neq \varnothing$ but $|H + (n+1)K| = \varnothing$. Let $D \in |H + nK|$: then $|D + K| = \varnothing$ and $K \cdot D = K(H + nK) = K \cdot H < 0$ since $-K$ is effective, H very ample.

$K^2 < 0$: It is enough to find an effective divisor E on X s.t. $K \cdot E < 0$. Then some component C of E will have $K \cdot C$. The genus formula gives $-2 \leq 2g - 2 = C(C+K) \implies C^2 \geq -1$. $C^2 = -1$ is impossible since X is minimal, so $C^2 \geq 0$. Now $(C + nK) \cdot C$ is negative for $n >> 0$, so $C + nK$ is not effective for $n >> 0$ by previous method. $\exists n$ s.t. $|C + nK| \neq \varnothing$ but $|C + (n+1)K| = \varnothing$. Choosing $D \in |C + nK|$ gives the desired divisor.

We show find the claimed E. Again, let H be a hyperplane section: if $K \cdot H < 0$, we can take $E = H$; if $K \cdot H = 0$, we can take $K + nH$ for $n >> 0$; so assume $K \cdot H > 0$. Let $\gamma = \frac{-K \cdot H}{K^2} > 0$ so that $(H + \gamma K) \cdot K = 0$. Also,

$$(H + \gamma K)^2 > H^2 + 2\gamma(H \cdot K) + \gamma^2 K = H^2 + \frac{(K \cdot H)^2}{(-K^2)} > 0 \tag{2}$$

So take β rational and slightly larger than γ to get $(H + \beta K) \cdot K < (H + \gamma K) \cdot K = 0$ ($K^2 < 0$) and $(H + \beta K)^2 > 0$. Therefore, $(H + \beta K) \cdot H > 0$. Write $\beta = \frac{r}{s}$. Then

$$(rH + sK)^2 > 0, (rH + sK) \cdot K < 0, (rH + sK) \cdot H > 0 \tag{3}$$

by equivalent facts for β. Let $D = rH + sK$. For $m >> 0$, by Riemann-Roch we get $h^0(mD) + h^0(K - mD) \geq \frac{1}{2}mD(mD - K) + 1 \to \infty$. Moreover, $K - mD$ is not effectover for $m >> 0$ since $(K - mD) \cdot H = (K \cdot H) - m(D \cdot H)$. Thus, mD is effective for large m, and we gan take $E \in |mD|$.

$K^2 > 0$ Assume that there is no such D, i.e. $K \cdot D \geq 0$ for every effective divisor D s.t. $|K + D| = \varnothing$.

Lemma 1. *Under this assumption:*

1. $\mathrm{Pic}(X)$ *is generated by* $\omega_X = \mathcal{O}_X(K)$, *and the anticanonical bundle is ample. In particular,* X *doesn't have any rational curves.*
2. *Every divisor of* $|-K|$ *is an integral curve of arithmetic genus* 1.
3. $(K^2) \leq 5, b_2 \geq 5$.

Proof. First, let us see that every element D of $|-K|$ is an irreducible curve. If not, let C be a component of D s.t. $K \cdot C < 0$. If $D = C + C', |K + C| = |-D + C| = |-C'| = \varnothing$ since C' is effective. Thus, $C \cdot K < 0$, contradicting the hypothesis. So D is irreducible, and similarly D is not a multiple. Furthermore, $p_a(D) = \frac{1}{2}D(D + K) + 1 = 1$, showing (2).

Next, we claim that the only effective divisor s.t. $|D + K| = \varnothing$ is the zero divisor. Assume not, i.e. $\exists D > 0$ s.t. $|K + D| = \varnothing$. Let $x \in D$: then since $h^0(-K) \geq 1 + K^2 \geq 2$, there is a $C \in |-K|$ passing through x. C is an integral curve, and cannot be a component of D since then $|K + D| \supset |K + C| = |0| \neq \varnothing$. So $C \cdot D > 0$ since they meet at least in x. Then $K \cdot D = -C \cdot D < 0$, contradicting the hypothesis.

As an aside, we claim that $p_n = 0$ for all $n \geq 1$: we know that $p_2 = 0 \implies p_1 = 0$; if $3K$ were effective then $2K$ would be too since $-K$ is effective, which contradicts $p_2 = 0 \implies p_3 = 0$ and by induction $p_n = 0$ for all $n \geq 1$.

We claim that adjuction terminates: if D' is any divisor on X, then there is an integer n_D s.t. $|D + nK| = \varnothing$ for $n \geq n_D$. To see this, note that $(D + nK) \cdot (-K)$ will eventually become negative. $-K$ is represented by an irreducible curve of positive self-intersection, so by the useful lemma $D + nK$ is not effective for $n >> 0$. Now, let Δ be an arbitrary effective divisor. Then $\exists n \geq 0$ s.t. $|\Delta + nK| \neq 0$ but $|\Delta + (n + 1)K| = \varnothing$. Take $D \in |\Delta + nK|$ effective. $|D + K| = \varnothing \implies D = 0$ from above. Since any divisor is a difference of effective divisors, $\mathrm{Pic}(X)$ is generated by K. If H is a hyperplane section on X, then $H \sim -nK$ with $k > 0$, implying that $-K$ is ample. Let C be any integral curve on X: then $C \sim -mK$ for some $m \geq 1$. $p_a(C) = \frac{1}{2}(-mK)(-mK + K) + 1 = \frac{1}{2}m(m - 1)K^2 + 1 \geq 1$ so there is no smooth rational curve on X, completing (1).

We are left to prove (3). Assume that $(K^2) \geq 6$. Then $h^0(-K) \geq 1 + K^2 \geq 7$. Fix points x and y on X: we claim that $\exists C \in |-K|$ with x and y singular points of C. This would be a contradiction, since $p_a(C) = 1 \implies p_a(\tilde{C}) < 0$ which is absurd. So $K^2 \leq 5$. To see the existence of this C, let

$$I_x = \mathrm{Ker}\ (\mathcal{O}_X \to \mathcal{O}_{X,x}/\mathfrak{m}_x^2), I_y = \mathrm{Ker}\ (\mathcal{O}_X \to \mathcal{O}_{X,y}/\mathfrak{m}_y^2) \tag{4}$$

Then we get

$$0 \to \mathcal{O}_X(-K) \otimes I_x \otimes I_y \to \mathcal{O}_X(-K) \to k^6 \to 0 \tag{5}$$

since $\mathcal{O}_{X,x}/\mathfrak{m}_x^2, \mathcal{O}_{X,y}/\mathfrak{m}_y^2$ have dimension 3 over k. Taking the long exact sequence, we find that $h^0(\mathcal{O}_X(-K) \otimes I_x \otimes I_y) \neq 0$, and get a nonzero section of that sheaf. It is a divisor of zero passing through x and y with multiplicity at least 2, giving us the claimed curve.

Finally, by Noether's formula, $1 = \chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + e(X))$, where $e(X) = 2 - 2b_1 + b_2$. $b_1 = 2q$ by hodge theory, so $10 = K^2 + b_2 \implies b_2 \geq 5$. $\square$

We now show that no surface has these properties. In characteristic 0, the Lefshetz principle allows us to reduce to $k = \mathbb{C}$. Taking the cohomology of the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X^{an} \to (\mathcal{O}_X^{an})^* \to 1$ gives

$$H^0(\mathcal{O}_X^{an}) \to H^1((\mathcal{O}_X^{an})^*) \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X^{an}) \to \cdots \tag{6}$$

By Serre's GAGA, $H^i(X, \mathcal{F}) \cong H^i(X^{an}, \mathcal{F}^{an})$ for an $\mathcal{O}_X$-module $\mathcal{F}$. Since $q = p_g = 0, H^1(\mathcal{O}_X^{an}) = h^2(\mathcal{O}_X^{an}) = 0$, and $H^1((\mathcal{O}_X^{an})^*) \cong H^1(\mathcal{O}_X^*) = \mathrm{Pic}X \cong H^2(X, \mathbb{Z})$ This implies that $b_2 = \mathrm{rk}H^2(X, \mathbb{Z}) = rk\mathrm{Pic}X = 1$ contradicting $b_2 \geq 5$. For positive characteristic, we will sketch a proof: the first proof was given by Zariski, and the second using etale cohomology by Artin and by Kurke. Our proof will be by reduction to characteristic 0.