ALGEBRAIC SURFACES, LECTURE 2

KARTIK VENKATRAM

Remark. In the definition of (L, M) we wrote $M = O_X(A - B)$ where A and B are irreducible curves. We can think of this as a moving lemma.

1. Linear Equivalence, Algebraic Equivalence, numerical equivalence of divisors

Two divisors C and D are linearly equivalent on $X \Leftrightarrow$ there is an $f \in k(X) s.t. C = D + (f)$. This is the same as saying there is a sheaf isom $O_X(C) \cong O_X(D), 1 \mapsto f$.

Two divisors C and D are algebraically equivalent if $O_X(C)$ is algebraically equivalent to $O_X(D)$. We say two line bundles L_1 and L_2 on X are algebraically equivalent if there is a connected scheme T, two closed points $t_1, t_2 \in T$ and a line bundle L on $X \times T$ such that $L_{X \times \{t_1\}} \cong L_1$ and $L_{X \times \{t_2\}} \cong L_2$, with the obvious abuse of notation.

Alternately, two divisors C and D are alg. equivalent if there is a divisor E on $X \times T$, flat on T, s.t. $E|_{t_1} = C$ and $E|_{t_2} = D$. We say $C \sim_{alg} D$.

We say C is numerically equivalent to D, $C \equiv D$, if $C \cdot E = D \cdot E$ for every divisor E on X.

We have an intersection pairing $\text{Div}X \times \text{Div}X \to \mathbb{Z}$ which factors through $\text{Pic}X \times \text{Pic}X \to \mathbb{Z}$, which shows that linear equivalence $\implies$ num equivalence. In fact, lin equivalence $\implies$ alg equivalence (map to $\mathbb{P}^1$ defined by (f)) and alg equivalence $\implies$ numerical equivalence ($\chi()$ is locally constant for a flat morphism, T connected).

Notation. $\text{Pic}(X)$ is the space of divisors modulo linear equivalence, $P^\tau(X)$ is the set of divisor classes numerically equivalent to 0, $\text{Pic}^o(X)$ is the space of divisor classes algebraically equivalent to 0. $\text{Num}(X) = \text{Pic}(X)/\text{Pic}^0(X)$. ...

1.1. Adjunction Formula.

Let C be a curve on X with ideal sheaf $\mathcal{I}$.

$$O \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_C \to \Omega_{C/k} \to 0 \tag{1}$$

with dual exact sequence

$$0 \to T_C \to T_X \otimes \mathcal{O}_C \to \mathcal{N}_{C/X} = (\mathcal{I}/\mathcal{I}^2)^* \to 0 \tag{2}$$

Taking $\wedge^2$ gives $\omega_X \otimes \mathcal{O}_C = \mathcal{O}_X(-C)|_C \otimes \Omega_C$ or $K_C = (K_X + C)|_C$ so deg $K_C = 2g(C) - 2 = C.(C + K)$ (genus formula). Note: $C^2 = \deg\ (\mathcal{O}_X(C) \otimes \mathcal{O}_C)$ by definition. $\mathcal{I}/\mathcal{I}^2$ is the conormal bundle, and is $\cong \mathcal{O}(-C) \otimes \mathcal{O}_C$, while $\mathcal{N}_{C/X}$ is the normal bundle $\cong \mathcal{O}(C) \otimes \mathcal{O}_C$.

Theorem 1 (Riemann-Roch)**.** $\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - L \cdot \omega_X)$.

Proof. $\mathcal{L}^{-1} \cdot \mathcal{L} \otimes \omega_X^{-1} = \chi(\mathcal{O}_X) - \chi(\mathcal{L}) - \chi(\omega_X \otimes \mathcal{L}^{-1}) + \chi(\omega_X)$. By Serre duality, $\chi(\mathcal{O}_X) = \chi(\omega_X)$ and $\chi(\omega_X \otimes \mathcal{L}^{-1}) = \chi(\mathcal{L})$. So we get that the RHS is $2(\chi(\mathcal{O}_X) - \chi(\mathcal{L}))$ and thus the desired formula. □

As a consequence of the generalized Grothendieck-Riemann-Roch, we get

Theorem 2 (Noether's Formula)**.** *$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{12}(K^2 + c_2)$ where c_1, c_2 are the Chern classes of T_X, K is the class of ω_X, $c_2 = b_0 - b_1 + b_2 - b_3 + b_4 = e(X)$ is the Euler characteristic of X. See [Borel-Serre], [Grothendieck: Chern classes], [Igusa: Betti and Picard numbers], [SGA 4.5], [Hartshorne].*

Remark. If H is ample on X, then for any curve C on X, we have $C \cdot H > 0$ (equals $\frac{1}{n} \cdots$ (degree of C in embedding by nH) for larger n).

1.2. **Hodge Index Theorem.**

Lemma 1. *Let D_1, D_2 be two divisors on X s.t. $h^0(X, D_2) \neq 0$. Then $h^0(X, D_1) \leq h^0(X, D_1 + D_2)$.*

Proof. Let $a \neq 0 \in H^0(X, D_2)$. Then $H^0(X, D_1) \xrightarrow{a} H^0(X, D_1) \otimes_k H^0(X, D_2) \to H_0(X, D_1 + D_2)$ is injective. □

Proposition 1. *If D is a divisor on X with $D^2 > 0$ and H is a hyperplane section of X, then exactly one of the following holds: (a) $(D \cdot H) > 0$ and $h^0(nD) \to \infty$ as $n \to \infty$. (b) $(D \cdot H) < 0$ and $h^0(nD) \to \infty$ as $n \to -\infty$.*

Proof. $h^0(nD) + h^0(K - nD) \geq \frac{1}{2}n^2D^2 - \frac{1}{2}n(D \cdot K) + \chi(\mathcal{O}_X) \to \infty$ as $n \to \pm\infty$ because $D^2 > 0$. Can't have $h^0(nD)$ and $h^0(K - nD)$ both going to ∞ as $n \to \infty$ or $n \to -\infty$ (otherwise $h^0(nD) \neq 0$ gives $h^0(K - nD) \leq h^0(K)$, a contradiction). Similarly, $h^0(nD)$ can't go to ∞ both as $n \to \infty$ and as $n \to -\infty$. Similarly for $h^0(K - nD)$. Finally, note that $h^0(nD) \neq 0$ implies $(nD \cdot H) > 0$ and so $D \cdot H > 0$. □

Corollary 1. *If D is a divisor on X and H is a hyperplane section on X s.t. $(D \cdot H) = 0$ then $D^2 \leq 0$ and $D^2 = 0 \Leftrightarrow D \equiv 0$.*

Proof. Only the last statement is left to be proven. If $D \not\equiv 0$ but $D^2 = 0$, then $\exists E$ on X s.t. $D.E \neq 0$. Let $E' = (H^2)E - (E \cdot H)H$, and get $D \cdot E' = (H^2)D \cdot E \neq 0$ and $H \cdot E' = 0$. Thus, replacing E with E', we can assume $H \cdot E = 0$. Next, let $D' = nD + E$, so $D' \cdot H = 0$ and $D'^2 = 2nD \cdot E + E^2$. Taking $n >> 0$ if $D \cdot E > 0$ and $n << 0$ if $D \cdot E < 0$, we get $D'^2 > 0$ and $D' \cdot H = 0$, contradicting the above proposition. □

Theorem 3. *(HIT): Let $\mathrm{Num}X = \mathrm{Pic}X/\mathrm{Pic}^\tau X$. Then we get a pairing $\mathrm{Num}X \times NumX \to \mathbb{Z}$. Let $M = \mathrm{Num}X \otimes_\mathbb{Z} \mathbb{R}$. This is a finite dimensional vector space over $\mathbb{R}$ of dimension ρ (the Picard number) and signature $(1, \rho - 1)$.*

Proof. Embed this in ℓ-adic cohomology $H^2(X, \mathbb{Q}_\ell(1))$ which is finite dimensional, and $C.D$ equals $C \cup D$ under $H^2(X, \mathbb{Q}_\ell(1)) \times H^2(X, \mathbb{Q}_\ell(1)) \to H^4(X, \mathbb{Q}_\ell(2)) \cong \mathbb{Q}_\ell$. The map $\mathrm{Num}X \ni C \to [C] \in H^2$ is an injective map. The intersection numbers define a symmetric bilinear nondegenerate form on $M (= \mathrm{Num}X \otimes_\mathbb{Z} \mathbb{R})$. Let h be the class in M of a hyperplane section on X. We can complete to a basis for M, say $h = H_1, h_2, \ldots, h_\rho$ s.t. $(h, h_i) = 0$ for $i \geq 2, (h_i, h_j) = 0$ for $i \neq j$. By the above, $(\cdot, \cdot)$ has signature $(1, \rho - 1)$ in this basis. Therefore, if E is any divisor on X s.t. $E^2 > 0$, then for every divisor D on X s.t. $D \cdot E = 0$, we have $D^2 \equiv 0$. □

1.3. **Nakai-Moishezon.** Let X/k be a proper nonsingular surface over k. Then $\mathcal{L}$ is ample $\Leftrightarrow$ for $(\mathcal{L} \cdot \mathcal{L}) > 0$ and for every curve C on X, $(\mathcal{L} \cdot \mathcal{O}_X(C)) > 0$. Note: we define the intersection number of $\mathcal{L} \cdot \mathcal{M}$ to be the coefficient of $n_1 \cdot n_2$ in $\chi(\mathcal{L}^{n_1} \otimes \mathcal{M}^{n_2})$ (check that this is bilinear, etc., and that it coincides with the previous definition).

Proof. Sketch when X is projective. $\Longrightarrow$ is easy. For the converse, $\chi(\mathcal{L}^n) \to \infty$ as $n \to \infty$ (Riemann-Roch, or by defn). Replace $\mathcal{L}$ by $\mathcal{L}^n$ to assume $\mathcal{L} = \mathcal{O}_X(D), D$ effective.

$$0 \to \mathcal{L}^{n-1} \xrightarrow{s_0} \mathcal{L}^n \to \mathcal{L}^n \otimes \mathcal{O}_D \to 0 \tag{3}$$

$\mathcal{L}^n \otimes \mathcal{O}_D = \mathcal{L}^n|_D$ is ample on D (since $\mathcal{L} \cdot D = \mathcal{L}^2 > 0$) so $H^1(\mathcal{L}^n|_D) = 0$ for $n >> 0$. $H^0(\mathcal{L}^n) \to H^0(\mathcal{L}^n|D) \to H^1(\mathcal{L}^{n-1}) \to H^1(\mathcal{L}^n) \to 0$ for $n >> 0 \implies h^1(\mathcal{L}^n) \leq h^1(\mathcal{L}^{n-1})$ so $h^1(\mathcal{L}^n)$ stabilizes and the map $H^1(\mathcal{L}^{n-1}) \to H^1(\mathcal{L}^n)$ is an isomorphism. So $H^0(\mathcal{L}|_D) \to H^0(\mathcal{L}^n|_D)$ is surjective for $n >> 0$. Taking global sections $\overline{s_1}, \ldots, \overline{s_k}$ generating $\mathcal{L}^n|_D$ and pulling back to $H^0(\mathcal{L}^n)$, we get generators $s_0, \ldots, s_k$. Get $f : X \to \mathbb{P}^k, f^*(\mathcal{O}_{\mathbb{P}^k}(1)) \cong \mathcal{L}^n$. f is a finite morphism (or else $\exists C \subset X$ with $f(C) = \star \implies C \cdot \mathcal{L} = 0$, a contradiction). $\mathcal{O}_{\mathbb{P}^k}(1)$ is ample $\implies \mathcal{L}^n$ is ample $\implies \mathcal{L}$ is ample. □

1.4. **Blowups.** Let X be a smooth surface, p a point on X. The blowup $\tilde{X} \xrightarrow{\pi} X$ at p is a smooth surface s.t. $\tilde{X} \setminus \pi^{-1}(p) \to X \setminus \{p\}$ is an isomorphism and $\pi^{-1}(p)$ is a curve $\cong \mathbb{P}^1$ (called the exceptional curve). We explicitly construct this as follows: take local coordinates at p, i.e $x, y \in \mathfrak{m}_p\mathcal{O}_{X,p}$ defined in some neighborhood U of p. Shrink U if necessary so that p is the only point in U where x, y both vanish. Let $\tilde{U} \subset U \times \mathbb{P}^1$ be defined by $xY - yX = 0$. $\tilde{U} \to U, x, y, x : y \to x, y$ is an isomorphism on $\tilde{U} \setminus (x = y = 0)$ to $U \setminus \{p\}$ and the preimage of p is $\cong \mathbb{P}^1$. Patch/glue with $X \setminus \{p\}$ to get $\tilde{X}$. Easy check: $\tilde{X}$ is nonsingular, $E = \mathbb{P}^1$ is the projective space bundle over p corresponding to $\mathfrak{m}_p/\mathfrak{m}_p^2$. The normal bundle $N_{E/\tilde{X}}$ is $\mathcal{O}_E(-1)$.

Note: this is a specific case of a more general fact (Hartshorne 8.24). For $Y \subset X$ a closed subscheme with corresponding ideal sheaf $\mathcal{I}$, blow up X along Y to get the projective bundle $Y' \to Y$ given by $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, and overall blowup $\tilde{X} = \text{Proj } \bigoplus \mathcal{I}^d$, $\mathcal{O}_{\tilde{X}}(1) = \tilde{\mathcal{I}} = \pi^{-1}\mathcal{I}\mathcal{O}_{\tilde{X}}$. $\tilde{\mathcal{I}}/\tilde{\mathcal{I}}^2 = \mathcal{O}_{X'}(1)$ so $N_{Y',\tilde{X}} = \mathcal{O}_{Y'}(-1)$.

If C is an irreducible curve on X passing through P with multiplicity m, then the closure of $\pi^{-1}(C \smallsetminus \{p\})$ in $\tilde{X}$ is an irreducible curve $\tilde{C}$ called the strict transform of C. π^*C defined in the obvious way: think of C as a Cartier divisor, defined locally by some equation, and pull back up $\pi^\# : \mathcal{O}_X \to \mathcal{O}_{\tilde{X}}$, which will cut out π^*C on $\tilde{X}$.

Lemma 2. *$\pi^*C = \tilde{C} + mE$.*

Proof. Assume C is cut out at p by some f, expand f as the completion in the local ring at p. □