ALGEBRAIC MODELS, DUALITY, AND RESONANCE

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Poincaré duality algebras

- Let A be a graded, graded-commutative algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i\geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - $\cdot\colon A^i \otimes A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in B^j$.
- We will assume that A is connected ($A^0 = \Bbbk \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\Bbbk} A^i$ are finite).
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension n if there is a $\Bbbk$-linear map $\varepsilon\colon A^n \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{n-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- Consequently,
  - $b_i(A) = b_{n-i}(A)$, and $A^i = 0$ for $i > n$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{n-i})^*, \mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
  - Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{n-i}$, such that $\varepsilon(aa^\vee) = 1$.
  - The *orientation class* is defined as $\omega_A = 1^\vee$, so that $\varepsilon(\omega_A) = 1$.

THE ASSOCIATED ALTERNATING FORM

- Associated to a $\Bbbk$-PD$_n$ algebra there is an alternating n-form,
$$\mu_A\colon \bigwedge^n A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_n) = \varepsilon(a_1 \cdots a_n).$$
- Assume now that $n = 3$, and set $r = b_1(A)$. Fix a basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{e_1^\vee, \dots, e_r^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by
$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$
where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Alternatively, let $A_i = (A^i)^*$, and let $e^i \in A_1$ be the (Kronecker) dual of e_i. We may then view μ dually as a trivector,
$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$
which encodes the algebra structure of A.

POINCARÉ DUALITY IN ORIENTABLE MANIFOLDS

- If M is a compact, connected, orientable, n-dimensional manifold, then the cohomology ring $A = H^{\cdot}(M, \Bbbk)$ is a PD_n algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."

- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

DUALITY SPACES

A more general notion of duality is due to Bieri and Eckmann (1978).

Let X be a connected, finite-type CW-complex, and set $\pi = \pi_1(X, x_0)$.

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

ABELIAN DUALITY SPACES

We introduce in [Denham–S.–Yuzvinsky 2016/17] an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent:

EXAMPLE

- Surface groups of genus at least 2 are not abelian duality groups, though they are (Poincaré) duality groups.
- Let $\pi = \mathbb{Z}^2 * G$, where
$$G = \langle x_1, \dots, x_4 \mid x_1^{-2}x_2x_1x_2^{-1}, \dots, x_4^{-2}x_1x_4x_1^{-1} \rangle$$
is Higman's acyclic group. Then π is an abelian duality group (of dimension 2), but not a duality group.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.

THEOREM (DENHAM–S. 2017)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \setminus U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

LINEAR, ELLIPTIC, AND TORIC ARRANGEMENTS

THEOREM (DS17)

Suppose that $\mathcal{A}$ is one of the following:

1. *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
2. *A non-empty elliptic arrangement in E^n;*
3. *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2014).

COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

- Let $A = (A^\bullet, \mathrm{d})$ be a commutative, differential graded algebra over a field $\Bbbk$ of characteristic 0. That is:
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot \colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $\mathrm{d}\colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b)$.
- A CDGA A is of *finite-type* (or *q-finite*) if it is connected (i.e., $A^0 = \Bbbk \cdot 1$) and $\dim A^i < \infty$ for all $i \leqslant q$.
- $H^\bullet(A)$ inherits an algebra structure from A.
- A cdga morphism $\varphi\colon A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$.

- A map $\varphi\colon A \to B$ is a *quasi-isomorphism* if φ^* is an isomorphism. Likewise, φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two cdgas, A and B, are *(q-)equivalent* ($\simeq_q$) if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.
- A cdga A is *formal* (or just *q-formal*) if it is (q-)equivalent to $(H^\bullet(A), d=0)$.
- A CDGA is *q-minimal* if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$.
- Every CDGA A with $H^0(A) = \Bbbk$ admits a *q-minimal model*, $\mathcal{M}_q(A)$ (i.e., a q-equivalence $\mathcal{M}_q(A) \to A$ with $\mathcal{M}_q(A) = (\bigwedge V, d)$ a q-minimal cdga), unique up to iso.

ALGEBRAIC MODELS FOR SPACES

- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$, such that $H^{\bullet}(A_{\mathrm{PL}}(X)) = H^{\bullet}(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over $\Bbbk$) for X is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- If M is a smooth manifold, then $\Omega_{\mathrm{dR}}(M)$ is a model for M (over $\mathbb{R}$).
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

RESONANCE VARIETIES OF A CDGA

- Let $A = (A^\bullet, \mathrm{d})$ be a connected, finite-type CDGA over $\Bbbk = \mathbb{C}$.
- For each $a \in Z^1(A) \cong H^1(A)$, we have a cochain complex,

$$(A^\bullet, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(A) = \{a \in H^1(A) \mid \dim H^i(A^\bullet, \delta_a) \geqslant s\}.$$

- If A is a CGA (that is, $\mathrm{d} = 0$), the resonance varieties $\mathcal{R}_s^i(A)$ are *homogeneous* subvarieties of A^1.
- If X is a connected, finite-type CW-complex, we set $\mathcal{R}_s^i(X) := \mathcal{R}_s^i(H^\bullet(X, \mathbb{C}))$.

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_r\}$ for A^1, and let $\{x_1, \dots, x_r\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_r]$, the coordinate ring of the affine space A^1.
- Define a cochain complex of free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes f) = \sum_{j=1}^n e_j u \otimes f x_j + \mathrm{d}\, u \otimes f$.
- The specialization of $(A \otimes S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).
- Hence, $\mathcal{R}^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.
- In particular, $\mathcal{R}^1_s(A) = V(I_{r-s}(\delta^1))$, the zero-set of the ideal of codimension s minors of δ^1.

RESONANCE VARIETIES OF PD-ALGEBRAS

- Let A be a PD_n algebra.
- For all $0 \leqslant i \leqslant n$ and all $a \in A^1$, the square

$$\begin{tikzcd}
(A^{n-i})^* \arrow[r, "(\delta_a^{n-i-1})^*"] & (A^{n-i-1})^* \\
A^i \arrow[u, "\mathrm{PD}", "\cong"'] \arrow[r, "\delta_a^i"] & A^{i+1} \arrow[u, "\mathrm{PD}", "\cong"']
\end{tikzcd}$$

  commutes up to a sign of $(-1)^i$.
- Consequently,

$$\big(H^i(A, \delta_a)\big)^* \cong H^{n-i}(A, \delta_{-a}).$$

- Hence, for all i and s,

$$\mathcal{R}_s^i(A) = \mathcal{R}_s^{n-i}(A).$$

- In particular, $\mathcal{R}_1^n(A) = \{0\}$.

3-DIMENSIONAL POINCARÉ DUALITY ALGEBRAS

- Let A be a PD_3-algebra with $b_1(A) = r > 0$. Then
  - $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$.
  - $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ for $1 \leqslant s \leqslant r$.
  - $\mathcal{R}^i_s(A) = \emptyset$, otherwise.
- Write $\mathcal{R}_s(A) = \mathcal{R}^1_s(A)$. Then
  - $\mathcal{R}_{2k}(A) = \mathcal{R}_{2k+1}(A)$ if r is even.
  - $\mathcal{R}_{2k-1}(A) = \mathcal{R}_{2k}(A)$ if r is odd.
- If μ_A has rank $r \geqslant 3$, then $\mathcal{R}_{r-2}(A) = \mathcal{R}_{r-1}(A) = \mathcal{R}_r(A) = \{0\}$.
- If $r \geqslant 4$, then $\dim \mathcal{R}_1(A) \geqslant \mathrm{null}(\mu_A) \geqslant 2$.
  - Here, the *rank* of a form $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.
  - The *nullity* of μ is the maximum dimension of a subspace $U \subset V$ such that $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

- If r is even, then $\mathcal{R}_1(A) = \mathcal{R}_0(A) = A^1$.
- If r is odd > 1, then $\mathcal{R}_1(A) \neq A^1$ if and only if μ_A is "generic," that is, there is a $c \in A^1$ such that the 2-form $\gamma_c \in \bigwedge^2 A_1$ given by $\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c)$ has rank $2g$, i.e., $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$.
- In that case, $\mathcal{R}_1(A)$ is the hypersurface $\operatorname{Pf}(\mu_A) = 0$, where $\operatorname{pf}(\delta^1(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$.

EXAMPLE

Let $M = S^1 \times \Sigma_g$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^g a_i b_i c$ is generic, and $\operatorname{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1 = \cdots = \mathcal{R}_{2g-2} = \{x_{2g+1} = 0\}$ and $\mathcal{R}_{2g-1} = \mathcal{R}_{2g} = \mathcal{R}_{2g+1} = \{0\}$.

RESONANCE VARIETIES OF 3-FORMS OF LOW RANK

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3$
5	125+345	$\{x_5 = 0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4$
6	123+456	$\mathbb{C}^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3 = x_5 = x_6 = 0\}$	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3 = \mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367	$\{x_7 = 0\}$	$\{x_7 = 0\}$	0
	456+147+257+367	$\{x_7 = 0\}$	$\{x_4 = x_5 = x_6 = x_7 = 0\}$	0
	123+456+147	$\{x_1 = 0\} \cup \{x_4 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = 0\}$	0
	123+456+147+257	$\{x_1 x_4 + x_2 x_5 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_7^2 - x_3 x_6 = 0\}$	0
	123+456+147+257+367	$\{x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4 = \mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7 = 0\}$	$\{x_3 = x_5 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_4 = x_5 = x_7 = 0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5 = x_7 = 0\}$	$\{x_3 = x_4 = x_5 = x_7 = x_1 x_8 + x_6^2 = 0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = 0\}$	$\{x_1 = x_3 = x_4 = x_5 = x_2 x_6 + x_7 x_8 = 0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3 = x_5 = x_1 x_4 - x_7^2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$
	147+268+358	$\mathbb{C}^8$	$\{x_1 = x_4 = x_7 = 0\} \cup \{x_8 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_8 = 0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3$	$L_1 \cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	$L_1 \cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3 \cup L_4$	$L_1' \cup L_2' \cup L_3'$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1 \cup C_2 \cup C_3$	$L_1 \cup L_2 \cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1 = \cdots = f_{20} = 0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1 = \cdots = g_{20} = 0\}$	0

PROPAGATION OF RESONANCE

- We say that the resonance varieties of a graded algebra $A = \bigoplus_{i=0}^{n} A^i$ *propagate* if $\mathcal{R}_1^1(A) \subseteq \cdots \subseteq \mathcal{R}_1^n(A)$.
- (Eisenbud–Popescu–Yuzvinsky 2003) If X is the complement of a hyperplane arrangement, then its resonance varieties propagate.

THEOREM (DSY 2016/17)

- *Suppose the $\Bbbk$-dual of A has a linear free resolution over $E = \bigwedge A^1$. Then the resonance varieties of A propagate.*
- *Let X be a formal, abelian duality space. Then the resonance varieties of X propagate.*
- *Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do* not *propagate.*

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X, x_0)$ is a finitely presented group, with $\pi_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{\mathrm{ab}}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{\mathrm{ab}})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.
- Set $\mathcal{V}_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

EXAMPLE

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group π with $\pi_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}_1(\pi) = \mathbf{V}(f)$.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{C}^*$ satisfies $H^i(X, \mathbb{C}_\rho) \neq 0$, then $H^j(X, \mathbb{C}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

COROLLARY

Let X be an abelian duality space of dimension n. Then The characteristic varieties propagate, i.e., $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.

INFINITESIMAL FINITENESS OBSTRUCTIONS

QUESTION

Let X be a connected CW-complex with finite q-skeleton. Does X admit a q-finite q-model A?

THEOREM

If X is as above, then, for all $i \leqslant q$ and all s:

- *(Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
  In particular, if X is q-formal, then $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(X)_{(0)}$.
- *(Macinic, Papadima, Popescu, S. 2017)* $\mathrm{TC}_0(\mathcal{R}^i_s(A)) \subseteq \mathcal{R}^i_s(X)$.
- *(Budur–Wang 2017) All the irreducible components of $\mathcal{V}^i(X)$ passing through the origin of $H^1(X, \mathbb{C}^*)$ are* algebraic subtori.

EXAMPLE

Let π be a finitely presented group with $\pi_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}_1(\pi) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then π admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Suppose X is $(q+1)$ finite, or X admits a q-finite q-model. Then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

COROLLARY

Let π be a finitely generated group. Assume that either π is finitely presented, or π has a 1-finite 1-model. Then $b_2(\mathcal{M}_1(\pi)) < \infty$.

EXAMPLE

- Consider the free metabelian group $\pi = \mathrm{F}_n / \mathrm{F}_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}_1(\pi) = \mathcal{V}_1(\mathrm{F}_n) = (\mathbb{C}^*)^n$, and so π passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(\pi)) = \infty$, and so π admits no 1-finite 1-model (and is not finitely presented).

A TANGENT CONE THEOREM FOR 3-MANIFOLDS

THEOREM

Let M be a closed, orientable, 3-dimensional manifold. Suppose $b_1(M)$ is odd and μ_M is generic. Then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$.

- If $b_1(M)$ is even, the conclusion may or may not hold:
  - Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\mathcal{V}^1_1(M) = \mathrm{Char}(M) = (\mathbb{C}^*)^2$, and so $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) = \mathbb{C}^2$.
  - Let M be the Heisenberg nilmanifold; then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \{0\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$.
- Let M be a closed, orientable 3-manifold with $b_1 = 7$ and $\mu = e_1e_3e_5 + e_1e_4e_7 + e_2e_5e_7 + e_3e_6e_7 + e_4e_5e_6$. Then μ is generic and $\mathrm{Pf}(\mu) = (x_5^2 + x_7^2)^2$. Hence, $\mathcal{R}^1_1(M) = \{x_5^2 + x_7^2 = 0\}$ splits as a union of two hyperplanes over $\mathbb{C}$, but not over $\mathbb{Q}$.

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