The Johnson filtration is finitely generated

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MIT Topology Seminar
Mapping class group

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Basic object in topology:
- Gluing data for 3-manifolds.
- Monodromies of \( \Sigma_g \)-bundles (e.g. families of algebraic curves).
- \( \pi_1 \) of moduli space of algebraic curves.
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\( \text{Mod}_g \) has many other finiteness properties: finitely presentable (McCool, Hatcher–Thurston), all \( H_k \) finitely generated (Harer?), etc.
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\( \text{Mod}_g \cap H_1(\Sigma_g) \), preserves alg. isect. pairing.

\[ \leadsto \text{Mod}_g \rightarrow \text{Sp}_{2g}(\mathbb{Z}). \]
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\[
1 \longrightarrow \mathcal{I}_g \longrightarrow \text{Mod}_g \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1
\]
Examples of elts of Torelli

\[ I_g = \ker(\text{Mod} g \mapsto H_1(\Sigma_g)) \]

Action of Dehn twist \( T_x \) on \( H_1(\Sigma_g) \) determined by \([x] \in H_1(\Sigma_g)\).

Separating twists: \( T_x \) with \([x] = 0\), i.e. \( x \) separating.

Bounding pair: \( T_x T_{-1} y \) with \( x \cap y = \emptyset \) and \([x] = [y]\), i.e. \( x \cup y \) bounds.

Theorem (Birman, Powell): \( I_g \) is generated by separating twists and bounding pairs.
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![Diagram of separating twists]
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$I_g$ is gen. by sep twists and bounding pairs.
Finiteness properties of Torelli

Theorem (Classical)
$I_1 = 1$, so $\text{Mod}_1 \sim = \text{SL}_2(\mathbb{Z})$.

Theorem (Mess)
$I_2$ is an $\infty$-rank free group.

Theorem (Johnson)
$I_g$ is fin gen for $g \geq 3$.

Open question
Is $I_g$ fin pres?
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For \( g \geq 3 \), following are same subgroup \( K^g \) of \( I^g \):  

- \( K^g = \ker(I^g \to H^1(I^g)/\text{torsion}) \)
- \( K^g \) subgroup of \( I^g \) gen. by sep twists.
- \( K^g \) is \( f \in I^g \) s.t. map. torus \( M_f \) has same cup products as \( \Sigma^g \times S^1 \).

Remark

For all \( f \in I^g \), \( M_f \) has homology of \( \Sigma^g \times S^1 \).

Observation

\( K^g = \ker(I^g \to H^1(I^g)/\text{torsion}) \) \( \Rightarrow \) \( K^g \) is commensurable with \( [I^g, I^g] \).

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$\Rightarrow$ naively, expects finiteness of $\gamma_k(I_g)$ to get worse as $k \uparrow \infty$. 
Deeper finiteness properties

Theorem (Dimca–Papadima, 2007)
$H_1([I_g, I_g]; Q)$ is fin. dim. for $g \geq 4$.

Theorem (Ershov–He, 2017)
$[I_g, I_g]$ is fin. gen. for $g \geq 12$.

Theorem (Church–Ershov–P, 2017)
$\gamma_k([I_g])$ is fin. gen. for $g \geq \max(2k - 1, 4)$.

Goal for rest of talk
Prove that $[I_g, I_g]$ (and hence Johnson kernel) is fin gen. for $g \geq 4$. 
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Bieri–Neumann–Strebel (BNS) invariants

The BNS invariant $\Sigma(G) \subset G^*$ is the set of all $f \in G^*$ such that

$$\{ g \in G | f(g) \geq 0 \}$$

is a connected subgraph of the Cayley graph $\text{Cay}(G, S)$. Nonobvious fact: independent of genset $S$. 
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$G$ grp w/ fin genset $S$. 

Characters:

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**Cayley graph**: $\text{Cay}(G, S) =$ vertices $G$, edges $g \cdot gs$ for $g \in G$, $s \in S$. 

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The BNS invariant $\Sigma(G) \subset G^* = \text{Hom}(G, \mathbb{R})$ is set of all $f \in G^*$ s.t. $\{g \in G \mid f(g) \geq 0\}$ is connected subgraph of Cay($G, S$).

Example: $\Sigma(F_2) = 0$

$F_2 = \langle a, b \rangle$
Consider $f : F_2 \rightarrow \mathbb{R}, f(a) = 1$ and $f(b) = 0.$
BNS Properties

Basic facts:
▶ Cone on open subset of sphere.
▶ For 3-manifold group, is cone on interiors of fibered faces.

Fundamental Theorem (Bieri–Neumann-Strebel)
\[ G \text{ fin gen grp}, \quad H < G \text{ w/ } [G, G] \subset H \subset G. \]

\[ H \text{ is fin gen } \iff \{ f \in G^* | f|_H = 0 \} \subset \Sigma(G). \]

Previous examples reflect that all \( 0 \subset H \subset \mathbb{Z}_n \) are fin gen, but no \( [F_n, F_n] \subset H \subset F_n \) are fin gen except \( H \) finite-index in \( F_n \).

Special Case \( [G, G] \) is fin gen iff \( \Sigma(G) = G^* \).
BNS Properties

Basic facts:

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\text{Cone on open subset of sphere.} \\
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BNS Properties

Basic facts:
- Cone on open subset of sphere.
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Fundamental Theorem (Bieri–Neumann-Strebel)

Given a finitely generated group $G$ and a subgroup $H$ of $G$ such that $[G, G] \subset H \subset G$,

$$ H \text{ is fin gen } \iff \{ f \in G^* \mid f|_H = 0 \} \subset \Sigma(G). $$

Previous examples reflect that all $0 \subset H \subset \mathbb{Z}^n$ are fin gen, but no $[F_n, F_n] \subset H \subset F_n$ are fin gen except $H$ finite-index in $F_n$.

Special Case

$[G, G]$ is fin gen iff $\Sigma(G) = G^*$. 
Main goal
Main goal

Goal

\([\mathcal{I}_g, \mathcal{I}_g]\) is fin gen for \(g \geq 4\)
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$[\mathcal{I}_g, \mathcal{I}_g]$ is fin gen for $g \geq 4$, i.e. $\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*$. 

Step 1: Find large piece of BNS invariant
Exists fin gen set $S \subset \mathcal{I}_g$ of genus 1 bounding pairs s.t.

$$\{ f \in (\mathcal{I}_g)^* \mid f(s) \neq 0 \text{ for all } s \in S \} \subset \Sigma(\mathcal{I}_g).$$
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Lemma (Folklore)

$G$ grp w/ fin genset $S$. Assume graph w/ vertices $S$ and edge between $s, s' \in S$ when $[s, s'] = 1$ connected. Then

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For $g \geq 4$, graph w/ vertices genus 1 bounding pairs and edges between disjoint bounding pairs is connected.
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Lemma (Folklore)
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For $g \geq 4$, graph w/ vertices genus 1 bounding pairs and edges between disjoint bounding pairs is connected.

Take $S$ finite subgraph containing genset for $\mathcal{I}_g$. 
Main goal

Goal
\([\mathcal{I}_g, \mathcal{I}_g]\) is fin gen for \(g \geq 4\), i.e. \(\Sigma(\mathcal{I}_g) = (\mathcal{I}_g)^*\).

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$\text{Mod}_g \trianglelefteq I_g$ by conjugation
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$\text{Mod}_g \circ \mathcal{I}_g$ by conjugation $\sim \rightarrow$ $\text{Mod}_g \circ (\mathcal{I}_g)^*$ preserving $\Sigma(\mathcal{I}_g)$. 

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For nonzero $f \in (I_g)^{*}$, exists $\phi \in$ Mod$_g$ s.t. $(\phi \cdot f)(s) \neq 0$ for all $s \in S$.

$$\implies \phi \cdot f \in \Sigma(I_g)$$
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Give $\text{Mod}_g$ pullback of Zariski topology under

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Claim 1 (proved next slide)
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Desired \( \phi \) is any elt of \( \text{Mod}_g \) not in this union.
Mod$_g$ has pullback of Zariski topology under

$$\text{Mod}_g \to \text{Aut}((\mathcal{I}_g)^*) \cong \text{GL}_n(\mathbb{R}).$$

**Claim 1**

Mod$_g$ irreducible space (not finite union of proper closed subspaces).
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**Theorem (Johnson)**

For $g \geq 3$, $H_1(\mathcal{I}_g; \mathbb{R}) \cong (\wedge^3 H)/H$ w/ $H = H_1(\Sigma_g; \mathbb{R})$. 
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$\Rightarrow$ Have factorization

$$\text{Mod}_g \to \text{Sp}_{2g}(\mathbb{R}) \hookrightarrow \text{Aut} \left( (\mathcal{I}_g)^* \right).$$
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\[ \text{Mod}\_g \rightarrow \text{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \text{Sp}_{2g}(\mathbb{R}) \hookrightarrow \text{Aut}\left((I_g)^*\right). \]

Induced topology on \( \text{Sp}_{2g}(\mathbb{R}) \) is usual Zariski topology.
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$\text{Sp}_{2g}(\mathbb{R})$ connected alg. group, so $\text{Sp}_{2g}(\mathbb{R})$ irreducible.
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Mod$_g$ has pullback of Zariski topology under
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$\text{Sp}_{2g}(\mathbb{Z})$ Zariski dense in $\text{Sp}_{2g}(\mathbb{R})$, so $\text{Sp}_{2g}(\mathbb{Z})$ irreducible.

Mod$_g \twoheadrightarrow \text{Sp}_{2g}(\mathbb{Z})$ surjective, so Mod$_g$ irreducible.
For $s \in S$, set $Z_s = \{ \phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0 \}$.

Claim 2: Push everything into that piece of BNS $Z_s$ proper subspace of $\text{Mod}_g$. 

For $s \in S$, set $Z_s = \{ \phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0 \}$.

**Claim 2: Push everything into that piece of BNS**

$Z_s$ proper subspace of $\text{Mod}_g$.

Assume $Z_s = \text{Mod}_g$. Write $s = T_x T_y^{-1}$.

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![Diagram](image)
For \( s \in S \), set \( Z_s = \{ \phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0 \} \).

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![Diagram of genus 1 bounding pairs](image)

For $\phi \in \text{Mod}_g$,  

$$0 = (\phi \cdot f)(T_x T_y^{-1})$$
For \( s \in S \), set \( Z_s = \{ \phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0 \} \).

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For \( \phi \in \text{Mod}_g \),

\[
0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1})
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$$0 = (\phi \cdot f)(T_x T_y^{-1}) = f(\phi T_x T_y^{-1} \phi^{-1}) = f(T_{\phi(x)} T_{\phi(y)}^{-1}).$$
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All genus 1 bounding pairs of form \( T_{\phi(x)} T_{\phi(y)}^{-1} \) for some \( \phi \in \text{Mod}_g \).
For \( s \in S \), set \( Z_s = \{ \phi \in \text{Mod}_g \mid (\phi \cdot f)(s) = 0 \} \).

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All genus 1 bounding pairs of form \( T_{\phi(x)} T_{\phi(y)}^{-1} \) for some \( \phi \in \text{Mod}_g \).

These generate \( \mathcal{I}_g \), so \( f = 0 \), contradiction.