Operads with Homological Stability detect Infinite Loop Spaces

Maria Basterra
University of New Hampshire

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joint with Irina Bobkova, Kate Ponto, Ulrike Tillmann
and Sarah Yeakel

Figure: Sarah, Kate and Ulrike (WIT II- BIRS 2016)
Introduction: Operads and infinite loop spaces.

Tillmann’s surface operad: Surprise Theorem.


OHS: Operads with homological stability.

Main Theorem: Group completions of algebras over OHS are infinite loop spaces.

Proof sketch.

Examples and applications
**Operads:** Useful way to collect multiple input operations and encode their interactions for varying \( n \).

\[ \mu_n : A^n \rightarrow A \]

In particular, useful to encode relations *up to homotopy* between operations.

**Example:** For a based topological space \((X, x_0)\), concatenation of loops defines operations on

\[ \Omega(X) = \text{maps}(((0,1], \partial), (X, x_0)) = \text{loops space on } (X, x_0) \]

that have inverses and are associative up to homotopy.
Example: For a based topological space \((X, x_0)\), and \(n \geq 2\) we obtain operations on

\[
\Omega^n(X) = \text{maps}(([0,1]^n, \partial), (X, x_0)) = \Omega(\Omega(\cdots \Omega(X, x_0))) = n\text{-th loop space on } (X, x_0)
\]

that have inverses, are associative and commutative up to homotopy. And, coherent homotopies of homotopies increasing with higher \(n\).
Definition
An \textbf{operad} is a collection of spaces

\[ \mathcal{O} = \{ \mathcal{O}(n) \}_{n \geq 0} \]

with base point \( \ast \in \mathcal{O}(0) \), \( 1 \in \mathcal{O}(1) \), a right action of the symmetric group \( \Sigma_n \) on \( \mathcal{O}(n) \) and structure maps

\[ \gamma : \mathcal{O}(k) \times [\mathcal{O}(j_1) \times \ldots \times \mathcal{O}(j_k)] \to \mathcal{O}(j_1 + \ldots + j_k) \]

that are required to be associative, unital, and equivariant.

A \textbf{map of operads} \( \mathcal{O} \to \mathcal{V} \) is a a collection of \( \Sigma_n \) equivariant maps \( \mathcal{O}(n) \to \mathcal{V}(n) \) which commute with the structure maps and preserve \( \ast \) and 1.

\textbf{Remark:} Note that above we do not insist that \( \mathcal{O}(0) = \ast \).
**Definition**
An \( O \)-algebra is a based space \((X, \ast)\) with equivariant structure maps
\[ O(j) \times X^j \longrightarrow X. \]
For a based space \((X, \ast)\), the free \( O \)-algebra on \( X \) is
\[ O(X) := \bigsqcup_{n \geq 0} (O(n) \times \Sigma_n X^n) / \sim \]
where \( \sim \) is a base point relation generated by
\[ (\gamma(c; 1^i, \ast, 1^{n-i-1}); x_1, \ldots, x_{n-1}) \sim (c; x_1, \ldots x_i, \ast, x_{i+1}, \ldots, x_{n-1}) \].

The class of \((1, \ast) \in O(1) \times X\) is the base point of \( O(X) \). Note that it coincides with the class of \( \ast \in O(0) \).

**Remark:** We will identify \( O(0) \) with \( O(\ast) \). In the cases of interest it will be a non-trivial \( O \)-algebra.
Example: The little $n$-disks operad $C_n$.

$$C_n(k) \subset \text{Emb}(\bigsqcup_k D^n, D^n) \simeq \text{Conf}_k(\mathbb{R}^n)$$

Figure: From Wikipidea

$$\gamma : C_2(3) \times [C_2(2) \times C_2(3) \times C_2(4)] \longrightarrow C_2(9)$$
Introduction: Operads

We have maps of operads:

$$C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_\infty$$

**Example:** $\Omega^n(X)$ is a $C_n$-algebra.

**Recognition Principle** (Stasheff, Boardman-Vogt, May, Barrett-Eccless, Milgram ... (1970’s):
Connected $C_n$-algebras are $\Omega^n$. More generally, the group completion of a $C_n$-algebra is an $\Omega^n$ space.
Tillmann’s surface operad $\mathcal{M}$

(Motivated by Segal’s cobordism category and definition of CFT)

Let $\Gamma_{g,n+1} = \pi_0(Diff^+(F_{g,n+1}; \partial))$ the mapping class group of an oriented surface of genus $g$ and $n+1$ boundary components.

$$\mathcal{M}(n) \simeq \bigsqcup_{g \geq 0} B\Gamma_{g,n+1}$$

A version of the little 2-disk operad is a sub-operad of $\mathcal{M}$ so that a grouplike $\mathcal{M}$-algebra is in particular a double loop space. But the surprising part is that

**Theorem (Tillmann, 2000)**

*Group like $\mathcal{M}$-algebras are infinite-loop spaces with an infinite loop space action by $\mathcal{M}^+ (= the\ group\ completion\ of\ the\ free \ \mathcal{M}$-algebra\ on\ a\ point).*
Tillmann’s surface operad theorem

Main ingredient on the proof: Harer’s homology stability theorem: $H_*B\Gamma_{g,n+1}$ is independent of $g$ and $n$ for $g$ large enough.

Inconvenient feature of the proof: Requires strict multiplication: some surfaces had to be identified and diffeomorphisms are replaced by mapping class groups.
Lemma (May (GILS))

For a monad $\mathcal{T}$, a $\mathcal{T}$-functor $F$ and a $\mathcal{T}$-algebra $X$, define

$$B_\bullet(F, \mathcal{T}, X) := \{ q \mapsto F(\mathcal{T}^q X) \}$$

1. For any functor $G$, $|B_\bullet(GF, \mathcal{T}, X)| \simeq |GB_\bullet(F, \mathcal{T}, X)|$.
2. $|B_\bullet(\mathcal{T}, \mathcal{T}, X)| \simeq X$.
3. $|B_\bullet(F, \mathcal{T}, \mathcal{T}(X))| \simeq F(X)$.
4. If $\delta : \mathcal{T} \rightarrow \mathcal{T}'$ is a natural transformation of monads, then $\mathcal{T}'$ is an $\mathcal{T}$-functor and $B_\bullet(\mathcal{T}', \mathcal{T}, X)$ is a simplicial $\mathcal{T}'$-algebra.
Corollary

Let $\mathcal{A}$ be an $A_\infty$-operad and let $\delta: \mathcal{A} \rightarrow \mathcal{A}s$ be the map of monads associated to the augmentation of operads $\mathcal{A} \rightarrow \mathcal{A}s$. For an $\mathcal{A}$-algebra $X$, there is a topological monoid $M_{\mathcal{A}}(X) := |B_\bullet(\mathcal{A}s, \mathcal{A}, X)|$ and a strong deformation retract

$$\rho: X \rightarrow M_{\mathcal{A}}(X)$$

that is natural in $X$ and induces an isomorphism of homology Pontryagin rings.
Group Completion

Algebraic monoids: $M \rightarrow \mathcal{G}M$ the Grothendieck group of $M$.

Topological monoids: $M \rightarrow \mathcal{G}M = \Omega BM$ where $BM = |N \cdot M|$

$A_\infty$ algebras: $X \rightarrow \mathcal{G}X = \Omega BM_A(X)$ the composite

$$X \rightarrow M_A(X) \rightarrow \Omega BM_A(X)$$

Theorem (Quillen, McDuff-Segal)

Let $M = \bigsqcup_{n\geq 0} M_n$ be a topological monoid such that the multiplication on $H_*(M)$ is commutative. Then

$$H_*(\Omega BM) = \mathbb{Z} \times \lim_{n \rightarrow \infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty).$$
Definition
Let $I$ be a commutative, finitely generated monoid. An $I$-grading on an operad $O$ is a decomposition

$$O(n) = \bigsqcup_{g \in I} O_g(n)$$

for each $n$ so that:

1. the basepoint $*$ lies in $O_0(0)$;
2. the $\Sigma_n$ action on $O(n)$ restricts to an action on each $O_g(n)$;
3. the structure maps restrict to maps

$$\gamma : O_g(k) \times [O_{g_1}(j_1) \times \ldots \times O_{g_k}(j_k)] \longrightarrow O_{g+g_1+\ldots+g_k}(j_1+\ldots+j_k).$$
For an $I$-graded operad $O$ let $s$ be the product of a set of generators for $I$, and choose a **propagator** $\tilde{s} \in O_s(1)$. Let $D = \gamma(-; *, \cdots, *)$ and $\bar{s} := \gamma(\tilde{s}, -)$. The diagram

\[
\begin{array}{ccc}
O_g(n) & \xrightarrow{\tilde{s}} & O_{g+s}(n) \\
D \downarrow & & \downarrow D \\
O_g(0) & \xrightarrow{\bar{s}} & O_{g+s}(0)
\end{array}
\]

commutes and defines a map $D_\infty : O_\infty(n) \longrightarrow O_\infty(0)$ where

\[O_\infty(n) =: \hocolim_{\tilde{s}} O_g(n)\]
Definition

An operad $O$ is an **operad with homological stability (OHS)** if

1. it is $I$-graded;
2. there is an $A_{\infty}$-operad $\mathcal{A}$ and a map of graded operads
   \[
   \mu : \mathcal{A} \longrightarrow O
   \]
   (multiplication map)
   with $\mu(\mathcal{A}(2)) \subset O_0(2)$ path connected; and
3. the maps
   \[
   D_\infty : O_\infty(n) \longrightarrow O_\infty(0)
   \]
   induce homology isomorphisms.
Operads with homological stability

Examples:

1. $C_\infty$ is and OHS concentrated in degree zero and multiplication $\mu : C_1 \rightarrow C_\infty$. Since $C_\infty(n)$ is contractible, conditions 2 and 3 are trivially satisfied.

2. The Riemann surfaces operad $\mathcal{M}$ with $\mathcal{M}(n) = \coprod_{g \geq 0} \mathcal{M}_{g,n+1} \simeq \coprod_{g \geq 0} \mathcal{B}\Gamma_{g,n+1}$

Figure: $\gamma : \mathcal{M}_{0,2+1} \times [\mathcal{M}_{1,2+1} \times \mathcal{M}_{0,0+1}] \rightarrow \mathcal{M}_{1,2+1}$

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OHS detect infinite loops spaces
Theorem (B., Bobkova, Ponto, Tillmann, Yeakel)

Suppose $O$ is an OHS. Then,

$$G : O \text{– algebras} \longrightarrow \Omega^\infty \text{– spaces}$$

is a functor with image $\Omega^\infty$-spaces with a compatible $\Omega^\infty$-map

$$G O(*) \times GX \longrightarrow GX,$$

where the source is given the product $\Omega^\infty$-space structure.
Proof sketch: Step 1 - Operad replacement

Let $\mathcal{O}$ be an OHS. Then the product operad $\tilde{\mathcal{O}} := \mathcal{O} \times C_\infty$ is an OHS with compatible maps of operads

$$\mathcal{O} \leftarrow \tilde{\mathcal{O}} \rightarrow C_\infty.$$ 

Then, any $\mathcal{O}$-algebra is an $\tilde{\mathcal{O}}$-algebra. W.L.O.G we assume a compatible map $\pi : \mathcal{O} \rightarrow C_\infty$. For any space $X$, there is a map of $\mathcal{O}$-algebras

$$\tau \times \pi : \mathcal{O}(X) \rightarrow \mathcal{O}(\ast) \times C_\infty(X),$$

where $\tau$ is induced by $X \rightarrow \ast$ and the target has the diagonal action of $\mathcal{O}$. 

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OHS detect infinite loops spaces
Claim: For any based space $X$,

$$G(\tau) \times G(\pi) : G(O(X)) \rightarrow G(O(*)) \times G(C_{\infty}(X))$$

is a weak homotopy equivalence.

“Proof”: By Whitehead theorem e.t.s isomorphism in homology.

By the group completion theorem e.t.s

$$\tau_{\infty} \times \pi_{\infty} : O_{\infty}(X) \rightarrow O_{\infty}(* \times C_{\infty}(X))$$

induces isomorphism in homology.

Filtering by arity in the operad and taking filtration quotients reduces to show that for each $n$ and $\Sigma_n$ space $Y$

$$\tilde{D}_{\infty} \times (\pi_{\infty} \times 1_Y) : O_{\infty}(n) \times_{\Sigma_n} Y \rightarrow O_{\infty}(0) \times (C_{\infty}(n) \times_{\Sigma_n} Y)$$

is a homology isomorphism.

This follows by homological stability of $O$. 

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Step 3: A functor from $O$-algebras to $\Omega^\infty$ spaces

**Claim:** The assignment $X \mapsto |GB_\bullet(C_\infty, O, X)|$ defines a functor from $O$-algebras to $\Omega^\infty$-spaces.

**Proof:** Recall (May): there is a map of monads

$$\alpha: C_\infty \longrightarrow \Omega^\infty \Sigma^\infty;$$

and for every based space $Z$, the map

$$\alpha: C_\infty Z \longrightarrow \Omega^\infty \Sigma^\infty Z$$

is a group completion.

For any map of $O$-algebras $f: X \longrightarrow Y$ the following diagram commutes. (The vertical arrows are equivalences and the horizontal ones are induced by $f$.)
Step 3: A functor from $O$-algebras to $\Omega^\infty$ spaces

\[
\begin{array}{ccc}
|GB_\bullet(C_\infty, O, X)| & \longrightarrow & |GB_\bullet(C_\infty, O, Y)| \\
\downarrow & & \downarrow \\
|GB_\bullet(\Omega^\infty \Sigma_\infty, O, X)| & \longrightarrow & |GB_\bullet(\Omega^\infty \Sigma_\infty, O, Y)| \\
\uparrow & & \uparrow \\
|G\Omega^\infty B_\bullet(\Sigma_\infty, O, X)| & \longrightarrow & |G\Omega^\infty B_\bullet(\Sigma_\infty, O, Y)| \\
\uparrow & & \uparrow \\
|\Omega^\infty B_\bullet(\Sigma_\infty, O, X)| & \longrightarrow & |\Omega^\infty B_\bullet(\Sigma_\infty, O, Y)| \\
\uparrow & & \uparrow \\
\Omega^\infty |B_\bullet(\Sigma_\infty, O, X)| & \longrightarrow & \Omega^\infty |B_\bullet(\Sigma_\infty, O, Y)|
\end{array}
\]
We have seen that for any based space $X$

$\mathcal{G}(\mathcal{O}(X)) \simeq \mathcal{G}(\mathcal{O}(\ast)) \times \mathcal{G}(\mathcal{C}_\infty(X))$

For an $\mathcal{O}$-algebra $X$ we have a homotopy fibration sequence

$$\mathcal{G}\mathcal{O}(\ast) \rightarrow |\mathcal{G}B\ast(\mathcal{O}, \mathcal{O}, X)| \rightarrow |\mathcal{G}B\ast(\mathcal{C}_\infty, \mathcal{O}, X)|$$

Applying it to the product $\mathcal{O}$-algebra $\mathcal{O}(\ast) \times X$ allows to conclude that

$$\mathcal{G}X \simeq |\mathcal{G}B\ast(\mathcal{C}_\infty, \mathcal{O}, \mathcal{O}(\ast) \times X)|.$$

which we saw to be an $\Omega^\infty$-space.
Examples and applications: Surface operads

Oriented surfaces and diffeomorphisms: $S$

$S(n) = \bigsqcup_{g \geq 0} BS_{g,n+1}$.

By Madsen-Weiss $\mathcal{GS}(0) \simeq \mathbb{Z} \times B\Gamma_{\infty}^+ \simeq \Omega^{\infty} \mathbb{M} \mathbb{T} \mathbb{SO}(2)$.
Nonorientable surfaces and diffeomorphisms: \( \mathcal{N} \)

Let \( N = \mathbb{R}P^2 \setminus (D^2 \coprod D^2) \). Let \( N_{k,n+1} \) be a surface of nonorientable genus \( k \) with one outgoing and \( n \) incoming boundary components built out of \( D, P, S^1 \) and \( N \).

\[
\mathcal{N}(n) \approx \coprod_{k \geq 0} BN_{k,n+1}.
\]

Homology stability results of Wahl give that \( \mathcal{N} \) is an OHS and

\[
\mathcal{G}\mathcal{N}(0) \approx \mathbb{Z} \times BN_{\infty}^+ \approx \Omega^\infty MTO(2),
\]

where \( \mathcal{N}_{\infty} = \lim_{k \to \infty} \pi_0 \text{Diff}(N_{k,1}, \partial) \) denotes the infinite mapping class group.
Let $W_{g,j+1}$ be the connected sum of $g$ copies of $S^k \times S^k$ with $j + 1$ open disks removed.

Let $\theta : B \to BO(2k)$ be the $k$-th connected cover and fix a bundle map $\ell_W : TW \to \theta^*\gamma_{2k}$.

We construct a graded operad with

$$W_{g}^{2k}(j) \simeq M_k^\theta(W_{g,j+1}, \ell_{W_{g,j+1}})$$

By homological stability results of Galatius and Randal-Williams we have that for $2k \geq 2$ the operad $W^{2k}$ is an OHS and

$$\Omega B_0 W^{2k}(0) \simeq \left( \operatorname{hocolim}_{g \to \infty} M_k^\theta(W_{g,1}, \ell_{W_{g,1}}) \right)^+ \simeq \Omega_0^{\infty} \mathbf{MT}\theta.$$