18.03 PDE.1: Fourier's Theory of Heat

1. Temperature Profile.

2. The Heat Equation.

3. Separation of Variables (the birth of Fourier series)

4. Superposition.

In this note we meet our first partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

This is the equation satisfied by the temperature $u(x,t)$ at position x and time t of a bar depicted as a segment,

$$0 \le x \le L, \quad t \ge 0$$

The constant k is the conductivity of the material the bar is made out of.

We will focus on one physical experiment. Suppose that the initial temperature is 1, and then the ends of the bar are put in ice. We write this as

$$\boxed{u(x,0) = 1, \quad 0 \le x \le L} \qquad \boxed{u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0}.$$

The value(s) of $u = 1$ at $t = 0$ are called *initial conditions*. The values at the ends are called *endpoint or boundary conditions*. We think of the initial and endpoint values of u as the input, and the temperature $u(x,t)$ for $t > 0$, $0 < x < L$ as the response. (For simplicity, we assume that only the ends are exposed to the lower temperature. The rest of the bar is insulated, not subject to any external change in temperature. Fourier's techniques also yield answers even when there is heat input over time at other points along the bar.)

As time passes, the temperature decreases as cooling from the ends spreads toward the middle. At the midpoint, $L/2$, one finds Newton's law of cooling,

$$u(L/2,t) \approx ce^{-t/\tau}, \quad t > \tau$$

The so-called characteristic time τ is inversely proportional to the conductivity of the material. If we choose units so that $\tau = 1$ for copper, then according to Wikipedia,

$$\tau \sim 7 \quad \text{(cast iron)}; \quad \tau \sim 7000 \quad \text{(dry snow)}$$

The constant c, on the other hand, is **universal**:

$$c \approx 1.3$$

It depends only on the fact that the shape is a bar (modeled as a line segment).

Fourier figured out not only how to explain c using differential equations, but the whole

temperature profile: $u(x,t) \approx e^{-t/\tau}h(x); \quad h(x) = \frac{4}{\pi}\sin\left(\frac{\pi}{L}x\right), \quad t > \tau.$

The shape of h reflects how much faster the temperature drops near the ends than in the middle. It's natural that h should be some kind of hump, symmetric around $L/2$.

We looked at the heat equation applet to see this profile emerge as t increases. It's remarkable that a sine function emerges out of the input $u(x,0) = 1$. There is no evident mechanism creating a sine function, no spring, no circle, no periodic input. The sine function and the number $4/\pi$ arise naturally out of differential equations alone.

Deriving the heat equation. To explain the heat equation, we start with a thought experiment. If we fix the temperature at the ends, $u(0,t) = 0$ and $u(L,t) = T$, what will happen in the long term as $t \to \infty$? The answer is that

$$u(x,t) \to U_{\text{steady}}(x), \quad t \to \infty$$

where U_{steady} is the steady, or equilbrium, temperature, and

$$U_{\text{steady}}(x) = \frac{T}{L}x \quad \text{(linear)}$$

The temperature $u(L/2,t)$ at the midpoint $L/2$ tends to the average of 0 and T, namely $T/2$. At the point $L/4$, half way between 0 and $L/2$, the temperature tends to the average of the temperature at 0 and $T/2$, and so forth.

At a very small scale, this same mechanism, the tendency of the temperature profile toward a straight line equilibrium means that if u is concave down then the temperature in the middle should decrease (so the profile becomes closer to being straight). If u is concave up, then the temperature in the middle should increase (so that, once again, the profile becomes closer to being straight). We write this as

$$\frac{\partial^2 u}{\partial x^2} < 0 \implies \frac{\partial u}{\partial t} < 0$$
$$\frac{\partial^2 u}{\partial x^2} > 0 \implies \frac{\partial u}{\partial t} > 0$$

The simplest relationship that reflects this is a linear (proportional) relationship,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad k > 0$$

Fourier's reasoning. Fourier introduced the heat equation, solved it, and confirmed in many cases that it predicts correctly the behavior of temperature in experiments like the one with the metal bar.

Actually, Fourier crushed the problem, figuring out the whole formula for $u(x,t)$ and not just when the initial value is $u(x,0) = 1$, but also when the initial temperature varies with x. His formula even predicts accurately what happens when $0 < t < \tau$.

Separation of Variables. For simplicity, take $L = \pi$ and $k = 1$. The idea is not to try to solve for what looks like the simplest initial condition namely $u(x,0) = 1$, but instead to look for solutions of the form

$$u(x,t) = v(x)w(t)$$

Plugging into the equation, we find

$$\frac{\partial u}{\partial t} = v(x)\dot{w}(t), \quad \frac{\partial^2 u}{\partial^2 x} = v''(x)w(t)$$

Therefore, since $k = 1$,

$$v(x)\dot{w}(t) = v''(x)w(t) \implies \frac{\dot{w}(t)}{w(t)} = \frac{v''(x)}{v(x)} = c \quad \text{(constant).}$$

This is the first key step. We divided by $v(x)$ and $w(t)$ to "separate" the variables. But the function $\dot{w}(t)/w(t)$ is independent of x, whereas $v''(x)/v(x)$ is indendent of t. And since these are equal, this function depends neither on x nor on t, and must be a constant. Notice also that the constant, which we are calling c for the time being, is the same constant in two separate ordinary differential equations:

$$\dot{w}(t) = cw(t), \quad v''(x) = cv(x).$$

The best way to proceed is to remember the endpoint conditions

$$u(0,t) = u(\pi, 0) = 0 \implies v(0) = v(\pi) = 0.$$

We know what the solutions to $v''(x) = cv(x)$, $v(0) = v(\pi) = 0$ look like. They are

$$v_n(x) = \sin nx, \quad n = 1, 2, 3, \ldots$$

Moreover, $v_n''(x) = -n^2 \sin nx = -n^2 v_n(x)$, so that $c = -n^2$. We now turn to the equation for w, which becomes

$$\dot{w}_n(t) = -n^2 w(t) \implies w_n(t) = e^{-n^2 t}.$$

(We may as well take $w(0) = 1$. We will be taking multiples later.) In summary, we have found a large collection of solutions to the equation, namely,

$$u_n(x,t) = v_n(x) w_n(t) = e^{-n^2 t} \sin nx$$

For these solutions, the endpoint condition $u_n(0,t) = u_n(\pi, t) = 0$ is satisfied, but the initial condition is

$$u_n(x, 0) = v_n(x) = \sin nx\,.$$

This is where Fourier made an inspired step. What if we try to write the function $u(x,0) = 1$ as a linear combination of $v_n(x) = \sin nx$?

On the face of it, expressing 1 as a sum of terms like $\sin nx$ makes no sense. We know that $\sin nx$ is zero at the ends $x = 0$ and $x = \pi$. But something tricky should be happening at the ends because the boundary conditions are discontinuous in time. At $t = 0$ we had temperature 1 at the ends, then suddenly when we plunged the ends in ice, we had temperature 0. So it's not crazy that the endpoints should behave in a peculiar way.

If there is any chance to write

$$u(x,0) = 1 = \sum b_n \sin nx, \quad 0 < x < \pi,$$

then it must be that the function is odd. In other words, we need to look at

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Moreover, the function has to be periodic of period 2π. This is none other than the square wave $f(x) = Sq(x)$, the very first Fourier series we computed.

$$1 = Sq(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \right), \quad 0 < x < \pi.$$

Now since initial conditions $v_n(x)$ yield the solution $u_n(x,t)$, we can apply the

Principle of Superposition $\quad u(x,0) = \sum b_n \sin nx \implies u(x,t) = \sum b_n e^{-n^2 t} \sin nx$

In other words, if $u(x,0)=1$, $0<x<\pi$, then

$$u(x,t)=\frac{4}{\pi}\left(e^{-t}\sin x+\frac{1}{3}e^{-3^2t}\sin 3x+\frac{1}{5}e^{-5^2t}\sin 5x+\cdots\right)\quad 0\le x\le\pi,\quad t>0.$$

The exact formula for the solution u to the heat equation is this series; it **cannot be expressed in any simpler form**. But often one or two terms already give a good approximation. Fourier series work as well, both numerically and conceptually, as any finite sum of terms involving functions like e^{-t} and $\sin x$. Look at the Heat Equation applet to see the first term (main hump) emerge, while the next term $b_3e^{-9t}\sin 3x$ tends to zero much more quickly. (The other terms are negligible after an even shorter time.)

For this example, the characteristic time is $\tau=1$, $e^{-t/\tau}=e^{-t}$, and

$$u(x,t)=\frac{4}{\pi}e^{-t}\sin x+\text{smaller terms as } t\to\infty.$$

To get an idea how small the smaller terms are, take an example.

Example. Fix $t_1=\ln 2$, then $e^{-t_1}=1/2$, and

$$u(x,t_1)=\frac{4}{\pi}\left(\frac{1}{2}\sin x+\frac{1}{3\cdot 2^9}\sin 3x+\cdots\right)=\frac{2}{\pi}\sin x\pm 10^{-3}$$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises