LS.3 Complex and Repeated Eigenvalues

1. Complex eigenvalues.

In the previous chapter, we obtained the solutions to a homogeneous linear system with constant coefficients

$$A\mathbf{x} = \mathbf{0}$$

under the assumption that the roots of its characteristic equation $|A - \lambda I| = 0$ — i.e., the eigenvalues of A — were *real* and *distinct*.

In this section we consider what to do if there are complex eigenvalues. Since the chanracteristic equation has real coefficients, its complex roots must occur in conjugate pairs:

$$\lambda = a + bi, \qquad \bar{\lambda} = a - bi .$$

Let's start with the eigenvalue $a + bi$. According to the solution method described in Chapter LS.2, the next step would be to find the corresponding eigenvector $\vec{\alpha}$, by solving the equations (LS.2, (27))

$$(a - \lambda)a_1 + ba_2 = 0$$
$$ca_1 + (d - \lambda)a_2 = 0$$

for its components a_1 and a_2. Since λ is complex, the a_i will also be complex, and therefore the eigenvector $\vec{\alpha}$ corresponding to λ will have complex components.

Putting together the eigenvalue and eigenvector gives us formally the complex solution

(1) $$\mathbf{x} = \vec{\alpha}\, e^{(a+bi)t} .$$

Naturally, we want real solutions to the system, since it was real to start with. To get them, the following theorem tells us to just take the real and imaginary parts of (1).

Theorem 3.1 *Given a system* $\mathbf{x}' = A\mathbf{x}$, *where* A *is a real matrix. If* $\mathbf{x} = \mathbf{x}_1 + i\,\mathbf{x}_2$ *is a complex solution, then its real and imaginary parts* $\mathbf{x}_1, \mathbf{x}_2$ *are also solutions to the system.*

Proof. Since $\mathbf{x}_1 + i\,\mathbf{x}_2$ is a solution, we have

$$(\mathbf{x} + i\,\mathbf{x}_2)' = A(\mathbf{x} + i\,\mathbf{x}_2) ;$$

equating real and imaginary parts of this equation,

$$\mathbf{x}_1' = A\mathbf{x}_1 , \qquad \mathbf{x}_2' = A\mathbf{x}_2 ,$$

which shows that the real vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $\mathbf{x}' = A\mathbf{x}$. □

Example 1. Find the corresponding two real solutions to $\mathbf{x}' = A\mathbf{x}$ if a complex eigenvalue and corresponding eigenvector are

$$\lambda = -1 + 2i , \qquad \vec{\alpha} = \begin{pmatrix} i \\ 2 - 2i \end{pmatrix} .$$

Solution. First write $\vec{\alpha}$ in terms of its real and imaginary parts:

$$\vec{\alpha} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix} .$$

The corresponding complex solution $\mathbf{x} = \vec{\alpha}\, e^{\lambda t}$ to the system can then be written

$$\mathbf{x} \;=\; \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) e^{-t}\left(\cos 2t + i \sin 2t \right) ,$$

so that we get respectively for the real and imaginary parts of $\mathbf{x}$

$$x_1 \;=\; e^{-t}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin 2t \right) \;=\; e^{-t}\begin{pmatrix} -\sin 2t \\ 2\cos 2t + 2\sin 2t \end{pmatrix} ,$$

$$x_2 \;=\; e^{-t}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos 2t - i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin 2t \right) \;=\; e^{-t}\begin{pmatrix} -\cos 2t \\ -2\cos 2t + 2\sin 2t \end{pmatrix} ;$$

these are the two real solutions to the system.   □

In general, if the complex eigenvalue is $a + bi$, to get the real solutions to the system, we write the corresponding complex eigenvector $\vec{\alpha}$ in terms of its real and imaginary part:

$$\vec{\alpha} \;=\; \vec{\alpha}_1 + i\,\vec{\alpha}_2, \qquad \vec{\alpha}_i \quad \text{real vectors};$$

(study carefully in the above example how this is done in practice). Then we substitute into (1) and calculate as in the example:

$$\mathbf{x} \;=\; (\vec{\alpha}_1 + i\vec{\alpha}_2)\, e^{a}t(\cos bt + i \sin bt),$$

so that the real and imaginary parts of $\mathbf{x}$ give respectively the two real solutions

(3)
$$\mathbf{x}_1 \;=\; e^{at}(\vec{\alpha}_1 \cos bt - \vec{\alpha}_2 \sin bt) ,$$
$$\mathbf{x}_2 \;=\; e^{at}(\vec{\alpha}_1 \sin bt + \vec{\alpha}_2 \cos bt) .$$

These solutions are linearly independent if $n = 2$. If $n > 2$, that portion of the general solution corresonding to the eigenvalues $a \pm bi$ will be

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \ .$$

Note that, as for second-order ODE's, the complex conjugate eigenvalue $a - bi$ gives up to sign the same two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$.

The expression (3) was not written down for you to memorize, learn, or even use; the point was just for you to get some practice in seeing how a calculation like that in Example 1 looks when written out in general. To actually solve ODE systems having complex eigenvalues, imitate the procedure in Example 1.

Stop at this point, and practice on an example (try Example 3, p. 377).

2. Repeated eigenvalues.
Again we start with the real $n \times n$ system

(4) $$\mathbf{x}' \;=\; A\mathbf{x} \ .$$

We say an eigenvalue λ_1 of A is **repeated** if it is a multiple root of the characteristic equation of A—in other words, the characteristic polynomial $|A - \lambda I|$ has $(\lambda - \lambda_1)^2$ as a factor. Let's suppose that λ_1 is a double root; then we need to find *two* linearly independent solutions to the system (4) corresponding to λ_1.

One solution we can get: we find in the usual way an eigenvector $\vec{\alpha}_1$ corresponding to λ_1 by solving the system

(5) $$(A - \lambda_1 I)\,\mathbf{a} \;=\; \mathbf{0} \ .$$

This gives the solution $\mathbf{x}_1 \;=\; \vec{\alpha}_1 e^{\lambda_1 t}$ to the system (4). Our problem is to find a second solution. To do this we have to distinguish two cases. The first one is easy.

A. The complete case.

Still assuming λ_1 is a real double root of the characteristic equation of A, we say λ_1 is a **complete** eigenvalue if there are two linearly independent eigenvectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ corresponding to λ_1; i.e., if these two vectors are two linearly independent solutions to the system (5). Using them we get two independent solutions to (4), namely

$$(6) \qquad\qquad \mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \text{and} \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_1 t} .$$

Naturally we would like to see an example of this for $n = 2$, but the following theorem explains why there aren't any good examples: if the matrix A has a repeated eigenvalue, it is so simple that no one would solve the system (4) by fussing with eigenvectors!

Theorem 3.2 Let A be a 2×2 matrix and λ_1 an eigenvalue. Then
$$\lambda_1 \text{ is repeated and complete} \quad \Leftrightarrow \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Proof.   Let $\vec{\alpha}_1$ and $\vec{\alpha}_2$ be two independent solutions to (5). Then any 2-vector $\vec{\alpha}$ is a solution to (5), for by using the parallelogram law of addition, $\vec{\alpha}$ can be written in the form
$$\vec{\alpha} = c_1 \vec{\alpha}_1 + c_2 \vec{\alpha}_2 ,$$
and this shows it is a solution to (5), since $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are:
$$(A - \lambda_1 I)\vec{\alpha} = c_1(A - \lambda_1 I)\vec{\alpha}_1 + c_2(A - \lambda_1 I)\vec{\alpha}_2 = 0 + 0 .$$

In particular, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy (5); letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a = \lambda_1 \\ c = 0 \end{cases} ;$$

$$\begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} b = 0 \\ d = \lambda_1 \end{cases} .$$

This proves the theorem in the direction $\Rightarrow$; in the other direction, one sees immediately that the characteristic polynomial is $(\lambda - \lambda_1)^2$, so that λ_1 is a repeated eigenvalue; it is complete since the matrix $A - \lambda_1 I$ has 0 for all its entries, and therefore every 2-vector $\vec{\alpha}$ is a solution to (5).

For $n = 3$ the situation is more interesting. Still assuming λ_1 is a double root of the characteristic equation, it will be a complete eigenvalue when the system (5) has two independent solutions; this will happen when the system (5) has essentially only one equation: the other two equations are constant multiples of it (or identically 0). You can then find two independent solutions to the system just by inspection.

Example 2. If the system (5) turns out to be three equations, each of which is a constant multiple of say
$$2a_1 - a_2 + a_3 = 0 ,$$
we can give a_1 and a_2 arbitrary values, and then a_3 will be determined by the above equation. Hence two independent solutions (eigenvectors) would be the column 3-vectors
$$(1, 0, 2)^T \qquad \text{and} \qquad (0, 1, 1)^T .$$

In general, if an eigenvalue λ_1 of A is k-tuply repeated, meaning the polynomial $A - \lambda I$ has the power $(\lambda - \lambda_1)^k$ as a factor, but no higher power, the eigenvalue is called **complete** if it

has k independent associated eigenvectors, i.e., if the system (5) has k linearly independent solutions. These then produce k solutions to the ODE system (4).

There is an important theorem in linear algebra (it usually comes at the very end of a linear algebra course) which guarantees that all the eigenvalues of A will be complete, regardless of what their multiplicity is:

Theorem 3.3 *If the real square matrix A is symmetric, meaning $A^T = A$, then all its eigenvalues are real and complete.*

B. The defective case.

If the eigenvalue λ is a double root of the characteristic equation, but the system (5) has only one non-zero solution $\vec{\alpha}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective**, and no second eigenvector exists. In this case, the second solution to the system (4) has a different form. It is

$$(7) \qquad\qquad \mathbf{x} \; = \; (\vec{\beta} + \vec{\alpha}_1 t) e^{\lambda_1 t} \; ,$$

where $\vec{\beta}$ is an unknown vector which must be found. This may be done by substituting (7) into the system (4), and using the fact that $\vec{\alpha}_1$ is an eigenvector, i.e., a solution to (5) when $\lambda = \lambda_1$. When this is done, we find that $\vec{\beta}$ must be a solution to the system

$$(8) \qquad\qquad (A - \lambda_1 I)\, \vec{\beta} \; = \; \vec{\alpha}_1 \; .$$

This is an inhomogeneous system of equations for determining β. It is guaranteed to have a solution, provided that the eigenvalue λ_1 really is defective.

> Notice that (8) doesn't look very solvable, because the matrix of coefficients has determinant zero! So you won't solve it by finding the inverse matrix or by using Cramer's rule. It has to be solved by elimination.

Some people do not bother with (7) or (8); when they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (4) by elimination.

Example. Try Example 2 (section 5.6) in your book; do it by using (7) and (8) above to find β, then check your answer by instead using elimination to solve the ODE system.

Proof of (7) for $n = 2$**.** Let A be a 2×2 matrix.

Since λ_1 is to be a double root of the characteristic equation, that equation must be
$$(\lambda - \lambda_1)^2 \; = \; 0, \qquad i.e., \qquad \lambda^2 - 2\lambda_1\lambda + \lambda_1^2 \; = \; 0.$$

From the known form of the characteristic equation (LS.2, (25)), we see that
$$\text{trace } A \; = \; 2\lambda_1, \qquad \det A \; = \; \lambda_1^2.$$

A convenient way to write two numbers whose sum is $2\lambda_1$ is: $\lambda_1 \pm a$; doing this, we see that our matrix A takes the form

$$(9) \qquad A \; = \; \begin{pmatrix} \lambda_1 + a & b \\ c & \lambda_1 - a \end{pmatrix}, \qquad \text{where } bc = -a^2, \;\; (\text{since } \det A = \lambda_1^2).$$

Now calculate the eigenvectors of such a matrix A. Note that b and c are not both zero, for if they were, $a = 0$ by (9), and the eigenvalue would be complete. If say $b \neq 0$, we may choose as the eigenvector

$$\vec{\alpha}_1 \;=\; \begin{pmatrix} b \\ -a \end{pmatrix} \;,$$

and then by (8), we get

$$\beta \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;.$$

Exercises: Section 4D

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