18.03 LA.7: Two dimensional dynamics

[**1**] Rabbits
[**2**] Springs

[1] Rabbits

Farmer Jones and Farmer McGregor have adjacent farms, both afflicted with rabbits. Let's model this. Write $x(t)$ for the number of rabbits in Jones's farm, and $y(t)$ for the number in McGregor's.

Rabbits breed fast: growth rate of 5 per year: $\dot{x} = 5x$, $\dot{y} = 5y$.

But wait, these two systems are coupled. The rabbits can jump over the hedge between the farms. McGregor's grass is greener, so it happens twice as often into his than out if his, per unit population. So we have

$$\begin{cases} \dot{x} & = & 3x + y \\ \dot{y} & = & 2x + 4y \end{cases}$$

The equation is *homogeneous*, at least till McGregor gets his gun. In matrices, with $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$\dot{\mathbf{u}} = A\mathbf{u} \quad , \quad A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

We could eliminate, but now we know better: we look for solutions of the form

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{v} \quad , \mathbf{v} \neq \mathbf{0}$$

That is, you separate the time dependence from the high dimensionality. You look for *ray solutions*. He pointed out what happens when you substitute this into the equation:

$$\dot{\mathbf{u}} = \lambda e^{\lambda t}\mathbf{v}$$

while

$$Ax = Ae^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$$

and the only way these can be equal is if

$$A\mathbf{v} = \lambda\mathbf{v}$$

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That is, λ is an eigenvalue of A, and $\mathbf{v}$ is a nonzero eigenvector.

Let's try it: The characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$$

and the roots of this polynomial are $\lambda_1 = 2$, $\lambda_2 = 5$.

So we have two "normal mode" solutions, one growing like e^{2t} and the other much faster, like e^{5t}. (They both get large as t grows, but when $e^{2t} = 100$, $e^{5t} = 100,000$.)

Then find nonzero eigenvectors by finding nonzero vectors killed by $A - \lambda I$. With $\lambda = 2$,

$$A - (2I) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad : \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or any nonzero multiple. In general one has the row reduction algorithm, but for 2×2 cases you can just eyeball it. I like to look at one of the rows, reverse the order and change one sign. Then check your work using the other row. Remember, $A - \lambda I$ is supposed to be a *singular* matrix, zero determinant, so the rows should say the same things.

The other eigenvalue gives

$$A - (5I) = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \quad : \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The two *normal mode* solutions are thus

$$e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad , \quad e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution is a linear combination of these two.

This way of solving is much more perspicacious than the elimination we did back in September: the variables are equally important and are put on equal footing.

Remember the phase diagram: Plot the trajectory of $\mathbf{x}(t)$. There are two sets of ray solutions, along the two eigenvectors. All solutions except the constant one at $\mathbf{0}$ go off exponentially to infinity. Other solutions are linear combinations of these two. As $t \to -\infty$, both exponentials get small, but e^{5t} gets smaller much faster, so the solutions become asymptotic to the other eigenline.

The picture is this shown well on the Mathlet "Linear Phase Portraits: Matrix Entry." This phase portrait is called a "Node."

[2] **Springs again.** Another source of systems is the companion matrix: The companion matrix of

$$\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x$$

for example is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

In the harmonic oscillator $\ddot{x} + \omega^2 x = 0$ for example the companion matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

We know the solutions of the harmonic oscillator, but let's solve using eigenvectors.

The characteristic polynomial is $p_A(\lambda) = \lambda^2 + \omega^2$. This is a general fact, true in any dimension:

The characteristic polynomial of an LTI operator is the same as that of its companion matrix.

The eigenvalues here are $\pm\omega i$. Plunge on and find corresponding eigenvectors: For $\lambda_1 = i\omega$,

$$A - \lambda I = \begin{bmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{bmatrix} \quad : \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$$

(Check the second row!) Complex eigenvalues give rise to complex eigenvectors. The other eigenvalue is $-i\omega = \overline{i\omega}$, and the corresponding eigenvector is the complex conjugate of $\mathbf{v_1}$.

These are the normal modes: $e^{\pm i\omega t}\begin{bmatrix} 1 \\ \pm i\omega \end{bmatrix}$. We can extract real solutions in the usual way, by taking real and imaginary parts:

$$\mathbf{x_1} = \begin{bmatrix} \cos(\omega t) \\ -\omega\sin(\omega t) \end{bmatrix} \quad , \quad \mathbf{x_2} = \begin{bmatrix} \sin(\omega t) \\ \omega\cos(\omega t) \end{bmatrix}$$

Now the trajectories are ellipses. This phase portrait is called a *center*.

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