18.03 LA.11: Inhomogeneous Systems

[1] Planetarium

So now we have a symbol expressing a solution of

$$\dot{\mathbf{x}} = A\mathbf{x}$$

in terms of initial conditions:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0)$$

We saw for example that for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

[I used the opposite signs on Monday; the calculation is the same.] I want you to have the following vision: The differential equation causes a flow around the origin. As time increases, the whole plane rotates. The differential equation says: the velocity vector is always perpendicular to the position vector (and points off to the left). After a time t has elapsed, whatever starting vector you had has rotated by t radians, counterclockwise.

So let's see: If I start at $\mathbf{v}$ and let s seconds elapse, I'm at $e^{As}\mathbf{v}$. Then I let another t seconds elapse. The effect is that I rotate this new vector by t radians:

$$e^{At}(e^{As})\mathbf{v}$$

On the other hand, all that has happened is that $t + s$ seconds has elapsed since time 0, so

$$e^{At}e^{As}\mathbf{v} = e^{A(t+s)}\mathbf{v}$$

This is the exponential law. I had you thinking of the rotation matrix, but the same reasoning works in complete generality. (Note that I used associativity.)

It shows for example that

$$(e^{At})^{-1} = e^{-At}$$

I think you need another example.

[2] The Romance of matrices

The MIT Humanities people have analyzed the plot of Shakespeare's *Romeo and Juliet*, and determined that it is well modeled by the following system of equations. The two parameters are

$$R = \text{Romeo's affection for Juliet}$$

$$J = \text{Juliet's affection for Romeo}$$

These two young people are totally unaffected by any outside influences; this will be a *homogeneous* system. Romeo's affection for Juliet changes over time. The rate of change is based on how he currently feels and also on how she currently feels. Similarly for Juliet. The Humanities people have discovered that

$$\dot{R} = R - J \quad , \quad \dot{J} = R + J$$

that is

$$\frac{d}{dt}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix}$$

So Juliet is a straightforward girl: if she likes him, that causes her to start to like him better. If he likes her, same thing.

Romeo is more complicated. If he likes her, that causes him to start to like her even more. That's normal. But if he sees that she is starting to like him, well, that exerts a negative influence on his feelings towards her.

Question 20.1. This relationship

1. is stable and will settle down to calm old age along a spiral
2. is stable and will settle down along a node
3. is unstable and will blow up along a node
4. is unstable and will probably blow up along a saddle

5. is unstable and will spiral out of control

6. Who's Shakespeare?

Let's see how this plot develops. As the play opens, Romeo has noticed Juliet at a dance, and immediately $R = 1$. Juliet is otherwise occupied, though, and $J = 0$. But the deriviative is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: a promising sign.

Soon she does notice him and her affection rises. That very fact slows his growth of passion, though, so the plot trajectory curves down. Notice that $\dot{R} = 0$ when $R = J$: he peaks out when their affections towards each other are equal.

This continues. The more she likes him, the stronger a drag that is on his growth of affection for her, till he starts to actively dislike the girl. This continues till Juliet starts to lose heart. $\dot{J} = 0$ when $J = -R$. Then she starts to like him less; but she still does love him, so that causes Romeo to like her less, with predictable effects on her feelings towards him.

Presently her feelings become negative, and his continue to crater, but as she like him less he gets more interested. His feelings bottom out over here, but she continues to stay away. She starts to flirt with his friends. Things are very bad, but his complicated nature brings things around. Soon he starts to like her again; that decreases her rate of decline, and soon she bottoms out. As he gets more passionate, she does too, and after a while we're back to $J = 0$.

But what's R now? Well, you can work it out. For the present, let's just notice that $\operatorname{tr} A = 2$ and $\det A = 2$. This puts us in the unstable part of the spiral segment, so we could have predicted all this quite simply.

Shall we solve? $p_A(\lambda) = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1$ has roots $1 \pm i$. Positive real part, nonzero imaginary part: so unstable spiral. Eigenvectors for $1 + i$ are killed by $A - (1 + i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$; for example $\begin{bmatrix} 1 \\ i \end{bmatrix}$. So exponential solution $e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$, with real part $\mathbf{u_1}(t) = e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and imaginary part $\mathbf{u_2}(t) = e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. The general solution is a linear combination. Luckily the starting point we had above is here: our story line was $\mathbf{u_1}$.

We're in luck here again: these two solutions are normalized and we've

computed that

$$e^{At} = e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

The second factor is the rotation matrix we had above. So as time increases, the whole romance spins around while expanding exponentially. After one rotation, 2π time has elapsed, so now $R = e^{2\pi} \sim 535.49$! So my picture wasn't very accurate.

[3] Inhomogeneous

Of course this is somewhat oversimplified. The fact is that Romeo and Juliet were influenced by their families, the Montagues and Capulets. So this is not a homogeneous situation after all. A better model might be:

$$\frac{d}{dt}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

The two families in Verona agree on one thing: this romance is a terrible idea. Will they succeed in damping it down?

What's the general form of the answer going to be?

We've learned to look for

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$$

We know about $\mathbf{x_h}$. What might we take for a particular solution, with this constant forcing term (parental pressure)?

How about trying for a constant solution? It would have to be so that the left hand side is zero, so

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

This we can do; we have to compute

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

so

$$\mathbf{x_p} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

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So there is a stable point, an equilibrium, where $R = 10$ and $J = 0$, when everything is in balance. It must be quite uncomfortable for poor Romeo. But as soon as someone exhales, you add in a nonzero homogeneous equation and the couple quickly spirals out of control. This is an unstable equilibrium! The attempt at parental control works only if Romeo and Juliet happen to be in this state; and it's very unlikely to work for very long.

[4] **Variations** But what if the input signal here isn't constant? I want to solve
$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}(t) \quad \text{or} \quad \dot{\mathbf{x}} - A\mathbf{x} = \mathbf{q}$$
Again it suffices to find a single "particular solution."

Let's try what we did before: "variation of parameters." We know that the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is of the form

$$c_1 \mathbf{u_1}(\mathbf{t}) + \cdots + c_n \mathbf{u_n}(\mathbf{t})$$

where $\mathbf{u_1}, \ldots, \mathbf{u_n}$ are independent solutions, and we now have a slick way to write this, using the corresponding "fundamental matrix"

$$\Phi(t) = \begin{bmatrix} \mathbf{u_1}(t) & \mathbf{u_2}(t) & \cdots & \mathbf{u_n}(t) \end{bmatrix}$$

The fact that the columns are solutions is recorded by the matrix equation

$$\frac{d}{dt}\Phi(t) = A\Phi(t)$$

We can write the general solution of the homogeneous equation as $\Phi(t)\mathbf{c}$. The coefficient vector $\mathbf{c}$ is a vector of "parameters." Let's let it vary: try for

$$\mathbf{x}(t) = \Phi(t)\mathbf{u}(t)$$

Plug into the equation:

$$
\begin{array}{rcl}
\dot{\mathbf{x}}(t) & = & A\Phi(t)\mathbf{u}(t) \quad + \quad \Phi(t)\dot{\mathbf{u}}(t) \\
-A\mathbf{x}(t) & = & -A\Phi(t)\mathbf{u}(t) \\
\hline
\mathbf{q}(t) & = & \Phi(t)\dot{\mathbf{u}}(t)
\end{array}
$$

Solve for $\dot{\mathbf{u}}$:
$$\dot{\mathbf{u}}(t) = \Phi(t)^{-t}\mathbf{q}(t)$$

Then integrate and multiply by $\Phi(t)^{-1}$:

$$\mathbf{x}(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{q}(t) \, dt$$

Notice that the indefinite integral is only well defined up to adding a constant vector, which then gets multiplied by $\Phi(t)$: so this is the general solution in a nutshell.

Since we don't care about initial conditions at this point, there's no need to use the exponential matrix and it's generally more convenient not to. If you do, though, you can use the fact that $(e^{At})^{-1} = e^{-At}$ to write

$$\mathbf{x} = e^{At} \int e^{-At} \mathbf{q}(t) \, dt$$

[5] Exponential Input/Response

In this section we consider exponential input signals. Recall two things about ordinary constant coefficient equations. First, with exponential input we can use the method of optimism which leads to algebraic methods. Second, using complex replacement sinusoidal input signals can be handled in the same way as exponential signals.

Let A be a constant matrix, a a constant and $\mathbf{K}$ a constant column vector. We consider here the system

$$\dot{\mathbf{x}} = A\mathbf{x} + e^{at}\mathbf{K}$$

To solve this equation we use the method of optimism and try a particular solution of the form $\mathbf{x_p} = e^{at}\mathbf{v}$, where $\mathbf{v}$ is an unknown constant vector. Substituting $\mathbf{x_p}$ into the equation and doing some simple algebra we find

$$
\begin{aligned}
\mathbf{x_p} &= A\mathbf{x} + e^{at}\mathbf{K} \\
\Leftrightarrow \quad ae^{at}\mathbf{v} &= e^{at}A\mathbf{v} + e^{at}\mathbf{K} \\
\Leftrightarrow \quad (aI - A)\mathbf{v} &= \mathbf{K} \\
\Leftrightarrow \quad \mathbf{v} &= (aI - A)^{-1}\mathbf{K} \\
\Leftrightarrow \quad \mathbf{x_p} &= e^{at}\mathbf{v} \\
&= e^{at}(aI - A)^{1}\mathbf{K}
\end{aligned}
$$

Notes: 1. In the third equation we replaced $a\mathbf{v}$ by $aI\mathbf{v}$. The identity matrix changes nothing, but allows us to do the algebra of matrix subtraction.

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2. This is only valid if $(aI - A)^{-1}$ exists, that is if $\det(aI - A) \neq 0$. (Note, this is equivalent to saying a is *not* an eigenvalue of A.)

Following our usage for ordinary differential equations we call the formula $\mathbf{x} = e^{at}(aI - A)^{-}\mathbf{K}$ the exponential response formula or the exponential input theorem.

Examples

1. Solve
$$
\begin{aligned}
\dot{x} &= 3x - y + e^{2t} \\
\dot{y} &= 4x - y - e^{2t}
\end{aligned}
$$

Solution. Here $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $a = 2$, $\mathbf{K} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$
(2I - A)^{-1} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \Rightarrow \boxed{\mathbf{x_p} = e^{2t} \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}
$$

2. (Same matrix as in example 1.) Solve

$$
\begin{aligned}
\dot{x} &= 3x - y + 3 \\
\dot{y} &= 4x - y + 2
\end{aligned}
$$

Solution: $\quad a = 0$, $\mathbf{K} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad -A^{-1} = -\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}.$

$$
\boxed{\mathbf{x_p} = -A^{-1} \cdot \mathbf{K} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.}
$$

Note: Here the input is a constant so our method of optimism is equivalent to guessing a constant solution.

3. Solve
$$
\begin{aligned}
\dot{x} &= x + 2y + \cos t \\
\dot{y} &= 2x + y
\end{aligned}
$$

Solution: We use complex replacement:

$$
\dot{\mathbf{z}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{z} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{x} = \operatorname{Re}(\mathbf{z}).
$$

The exponential response formula gives

$$
\mathbf{z_p}(t) = e^{it}(iI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

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We have

$$iI - A = \begin{bmatrix} i - 1 & -2 \\ -2 & i - 1 \end{bmatrix} \Rightarrow (iI - A)^{-1} = \frac{1}{-4 - 2i} \begin{bmatrix} i - 1 & 2 \\ 2 & i - 1 \end{bmatrix}$$

So (not showing all of the complex arithmetic),

$$
\begin{aligned}
\mathbf{z_p}(t) &= \frac{1}{-4-2i} e^{it} \begin{bmatrix} i - 1 & 2 \\ 2 & i - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= (\cos t + i \sin t) \frac{1}{10} \begin{bmatrix} 1 - 3i \\ -4 + 2i \end{bmatrix} \\
&= \frac{1}{10} \left(\begin{bmatrix} \cos t + 3\sin t \\ -4\cos t - 2\sin t \end{bmatrix} + i \begin{bmatrix} -3\cos t + \sin t \\ 2\cos t - 4\sin t \end{bmatrix} \right)
\end{aligned}
$$

$$\boxed{\mathbf{x_p}(t) = \mathrm{Re}\,(\mathbf{z_p})(t) = \frac{1}{10} \begin{bmatrix} \cos t + 3\sin t \\ -4\cos t - 2\sin t \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \end{bmatrix}}$$

Superposition

For linear constant coefficient equations the principle of superposition allows us to use the exponential input method for input functions like $\mathbf{f} = \begin{bmatrix} 3e^{2t} \\ -e^t \end{bmatrix}$.
That is we can split $\mathbf{f}$ into a sum:

$$\mathbf{f} = \begin{bmatrix} 3e^{2t} \\ -e^t \end{bmatrix} = e^{2t} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

and solve with each piece separately and then sum the two solutions.

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M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises