Frequency Response

1. Introduction

We will examine the response of a second order linear constant coefficient system to a sinusoidal input. We will pay special attention to the way the output changes as the frequency of the input changes. This is what we mean by the *frequency response* of the system. In particular, we will look at the *amplitude response* and the *phase response*; that is, the amplitude and phase lag of the system's output considered as functions of the input frequency.

In O.4 the Exponential Input Theorem was used to find a particular solution in the case of exponential or sinusoidal input. Here we will work out in detail the formulas for a second order system. We will then interpret these formulas as the frequency response of a mechanical system. In particular, we will look at damped-spring-mass systems. We will study carefully two cases: first, when the mass is driven by pushing on the spring and second, when the mass is driven by pushing on the dashpot.

Both these systems have the same form

$$p(D)x = q(t),$$

but their amplitude responses are very different. This is because, as we will see, it can make physical sense to designate something other than $q(t)$ as the input. For example, in the system

$$mx' + bx' + kx = by'$$

we will consider y to be the input. (Of course, y is related to the expression on the right-hand-side of the equation, but it is not exactly the same.)

2. Sinusoidally Driven Systems: Second Order Constant Coefficient DE's

We start with the second order linear constant coefficient (CC) DE, which as we've seen can be interpreted as modeling a **damped forced harmonic oscillator**. If we further specify the oscillator to be a mechanical system with mass m, damping coefficient b, spring constant k, and with a *sinusoidal* driving force $B\cos\omega t$ (with B constant), then the DE is

$$mx'' + bx' + kx = B\cos\omega t. \tag{1}$$

For many applications it is of interest to be able to predict the periodic response of the system to various values of ω. From this point of view we can picture having a *knob* you can turn to set the input frequency ω, and a screen where we can see how the shape of the system response changes as we turn the ω-knob.

The Exponential Input Theorem (O.4 (4), and see O.4 example 2) tells us how to find a particular solution to (1):

Characteristic polynomial: $p(r) = mr^2 + br + k$.

Complex replacement: $m\tilde{x}'' + b\tilde{x}' + k\tilde{x} = Be^{i\omega t}$, $x = \mathrm{Re}(\tilde{x})$.

Exponential Input Theorem: $\tilde{x}_p = \dfrac{Be^{i\omega t}}{p(i\omega)} = \dfrac{Be^{i\omega t}}{k - m\omega^2 + ib\omega}$

thus,

$$x_p = \text{Re}(\tilde{x}_p) = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi) = \frac{B}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi), \tag{2}$$

where $\phi = \text{Arg}(p(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$. (In this case ϕ must be between 0 and π. We say ϕ is in the first or second quadrants.)

Letting $A = \frac{B}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$, we can write the periodic response x_p as

$$x_p = A\cos(\omega t - \phi).$$

The *complex gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *complexified* equation, is

$$\tilde{g}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{k - m\omega^2 + ib\omega}.$$

The *gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *real* equation, is

$$g = g(\omega) = \frac{1}{|\ p(i\omega)\ |} = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}. \tag{3}$$

The *phase lag* is

$$\phi = \phi(\omega) = \text{Arg}(p(i\omega) = \tan^{-1}(\frac{b\omega}{k - m\omega^2}) \tag{4}$$

and we also have the *time lag* = ϕ/ω.

Terminology of Frequency Response
We call the gain $g(\omega)$ the **amplitude response** of the system. The phase lag $\phi(\omega)$ is called the **phase response** of the system. We refer to them collectively as the **frequency response** of the system.

Notes:
1. Observe that the whole DE scales by the input amplitude B.

2. All that is needed about the input for these formulas to be valid is that it is of the form (*constant*) $\times$ (a *sinusoidal* function). Here we have used the notation $B\cos\omega t$ but the amplitude factor in front of the cosine function can take any form, including having the constants depend on the system parameters and/or on ω. (And of course one could equally-well use $\sin\omega t$, or any other shift of cosine, for the sinusoid.) This point is very important in the physical applications of this DE and we will return to it again.

3. Along the same lines as the preceding: we always define the gain as the *the amplitude of the periodic output divided by the amplitude of the periodic input*. Later we will see examples where the gain is *not* just equal to $\frac{1}{p(i\omega)}$ (for complex gain) or $\frac{1}{|p(i\omega)|}$ (for real gain) – stay tuned!

3. Frequency Response and Practical Resonance

The gain or amplitude response to the system (1) is a function of ω. It tells us the size of the system's response to the given input frequency. If the amplitude has a peak at ω_r we call this the **practical resonance frequency.** If the damping b gets too large then, for the system in equation (1), there is no peak and, hence, no practical resonance. The following figure shows two graphs of $g(\omega)$, one for small b and one for large b.

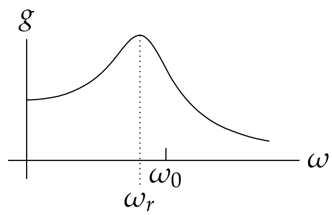


Fig 1a. Small b (has resonance).

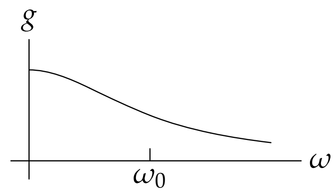


Fig 1b. Large b (no resonance)

In figure (1a) the damping constant b is small and there is practical resonance at the frequency ω_r. In figure (1b) b is large and there is no practical resonant frequency.

Finding the Practical Resonant Frequency.
We now turn our attention to finding a formula for the practical resonant frequency -if it exists- of the system in (1). Practical resonance occurs at the frequency ω_r where $g(w)$ has a maximum. For the system (1) with gain (3) it is clear that the maximum gain occurs when the expression under the radical has a minimum. Accordingly we look for the minimum of

$$f(\omega) = (k - m\omega^2)^2 + b^2\omega^2.$$

Setting $f'(\omega) = 0$ and solving gives

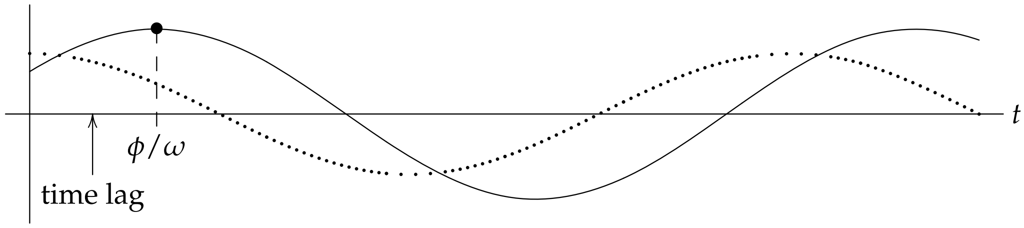
$$f'(\omega) = -4m\omega(k - m\omega^2) + 2b^2\omega = 0$$
$$\Rightarrow \omega = 0 \text{ or } m^2\omega^2 = mk - b^2/2.$$

We see that if $mk - b^2/2 > 0$ then there is a practical resonant frequency

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}. \tag{5}$$

Phase Lag:
In the picture below the dotted line is the input and the solid line is the response.

The damping causes a lag between when the input reaches its maximum and when the output does. In radians, the angle ϕ is called the *phase lag* and in units of time ϕ/ω is the *time lag*. The lag is important, but in this class we will be more interested in the amplitude response.



4. Mechanical Vibration System: Driving Through the Spring

The figure below shows a spring-mass-dashpot system that is driven through the spring.

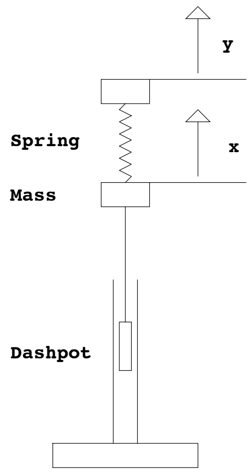


Figure 1. Spring-driven system

Suppose that y denotes the displacement of the plunger at the top of the spring and $x(t)$ denotes the position of the mass, arranged so that $x = y$ when the spring is unstretched and uncompressed. There are two forces acting on the mass: the spring exerts a force given by $k(y-x)$ (where k is the spring constant) and the dashpot exerts a force given by $-bx'$ (against the motion of the mass, with damping coefficient b). Newton's law gives

$$mx'' = k(y-x) - bx'$$

or, putting the system on the left and the driving term on the right,

$$mx'' + bx' + kx = ky\,. \tag{6}$$

In this example it is natural to regard y, rather than the right-hand side $q = ky$, as the input signal and the mass position x as the system response. Suppose that y is sinusoidal, that is,

$$y = B_1 \cos(\omega t).$$

Then we expect a sinusoidal solution of the form

$$x_p = A\cos(\omega t - \phi).$$

By definition the *gain* is the ratio of the amplitude of the system response to that of the input signal. Since B_1 is the amplitude of the input we have $g = A/B_1$.

In equations (3) and (4) we gave the formulas for g and ϕ for the system (1). We can now use them with the following small change. The k on the right-hand-side of equation (6) needs to be included in the gain (since we don't include it as part of the input). We get

$$g(\omega) = \frac{A}{B_1} = \frac{k}{|p(i\omega)|} = \frac{k}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$$
$$\phi(\omega) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right).$$

Note that the gain is a function of ω, i.e. $g = g(\omega)$. Similarly, the *phase lag* $\phi = \phi(\omega)$ is a function of ω. The entire story of the steady state system response $x_p = A\cos(\omega t - \phi)$ to sinusoidal input signals is encoded in these two functions of ω, the gain and the phase lag.

We see that choosing the input to be y instead of ky scales the gain by k and does not affect the phase lag.

The factor of k in the gain does not affect the frequency where the gain is greatest, i.e. the practical resonant frequency. From (5) we know this is

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

Note: Another system leading to the same equation is a series RLC circuit. We will favor the mechanical system notation, but it is interesting to note the mathematics is exactly the same for both systems.

5. Mechanical Vibration System: Driving Through the Dashpot

Now suppose instead that we fix the top of the spring and drive the system by moving the bottom of the dashpot instead.

Suppose that the position of the bottom of the dashpot is given by $y(t)$ and the position of the mass is given by $x(t)$, arranged so that $x = 0$ when the spring is relaxed. Then the force on the mass is given by

$$mx'' = -kx + b\,\frac{d}{dt}(y - x)$$

since the force exerted by a dashpot is supposed to be proportional to the speed of the piston moving through it. This can be rewritten as

$$mx'' + bx' + kx = by'\,. \tag{7}$$

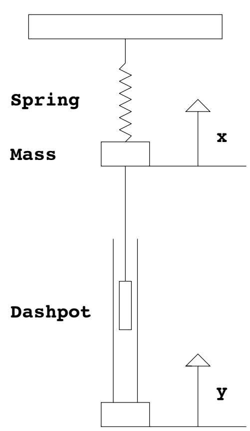


Figure 2. Dashpot-driven system

We will consider x as the system response, and again on physical grounds we specify as the input signal the position y of the back end of the dashpot. Note that the *derivative* of the input signal (multiplied by b) occurs on the right hand side of the equation.

Again we suppose that the input signal is of sinusoidal form

$$y = B_1 \cos(\omega t).$$

We will now work out the frequency response analysis of this problem.

First, $y = B_1 \cos(\omega t) \Rightarrow y' = -\omega B_1 \sin(\omega t)$, so our equation is

$$mx'' + bx' + kx = -b\omega B_1 \sin(\omega t)\,. \tag{8}$$

We know that the periodic system response will be sinusoidal, and as usual we choose the amplitude-phase form with the cosine function

$$x_p = A\cos(\omega t - \phi)\,.$$

Since $y = B_1 \cos(\omega t)$ was chosen as the input, the gain g is given by $g = \frac{A}{B_1}$.

As usual, we compute the gain and phase lag ϕ by making a complex replacement.

One natural choice would be to regard $q(t) = -b\omega B_1 \sin(\omega t)$ as the imaginary part of a complex equation. This would work, but we must keep in mind that the input signal is $B_1 \cos(\omega t)$ and also that we want to express the solution x_p as $x_p = A\cos(\omega t - \phi)$.

Instead we will go back to equation (7) and complexify before taking the derivative of the right-hand-side. Our input $y = B_1 \cos(\omega t)$ becomes $\tilde{y} = B_1 e^{i\omega t}$ and the DE becomes

$$mz'' + bz' + kz = b\tilde{y}' = i\omega b B_1 e^{i\omega t}. \tag{9}$$

Since $y = \text{Re}(\tilde{y})$ we have $x = \text{Re}(z)$; that is, the sinusoidal system response x_p of (8) is the real part of the exponential system response z_p of (9). The Exponential Input Theorem gives

$$z_p = \frac{i\omega b B_1}{p(i\omega)}\, e^{i\omega t}$$

where

$$p(s) = ms^2 + bs + k$$

is the characteristic polynomial.

The complex gain (scale factor that multiplies the input signal to get the output signal) is

$$\tilde{g}(\omega) = \frac{i\omega b}{p(i\omega)}.$$

Thus, $z_p = B_1 \tilde{g}(\omega) e^{i\omega t}$.

We can write $\tilde{g} = |\tilde{g}| e^{-i\phi}$, where $\phi = -\text{Arg}(\tilde{g})$. (We use the minus sign so ϕ will come out as the phase lag.) Substitute this expression into the formula for z_p to get

$$z_p = B_1 |\tilde{g}|\, e^{i(\omega t - \phi)}.$$

Taking the real part we have

$$x_p = B_1 |\tilde{g}|\, \cos(\omega t - \phi).$$

All that's left is to compute the gain $g = |\tilde{g}|$ and the phase lag $\phi = -\text{Arg}(\tilde{g})$. We have

$$p(i\omega) = m(i\omega)^2 + bi\omega + k = (k - m\omega^2) + bi\omega\,,$$

so,

$$\tilde{g} = \frac{i\omega b}{p(i\omega)} = \frac{i\omega b}{(k - m\omega^2) + bi\omega}. \tag{10}$$

This gives

$$g(\omega) = |\tilde{g}| = \frac{\omega b}{|p(i\omega)|} = \frac{\omega b}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

In computing the phase ϕ we have to be careful not to forget the factor of i in the numerator of $\tilde{g}$. After a little algebra we get

$$\phi(\omega) = -\text{Arg}(\tilde{g}) = \tan^{-1}(-(k - m\omega^2)/(b\omega)).$$

As with the system driven through the spring, we try to find the input frequency $\omega = \omega_r$ which gives the largest system response. In this case we can find ω_r without any calculus by using the following shortcut: divide the numerator and denominator in (10) by $bi\omega$ and rearrange to get

$$\tilde{g} = \frac{1}{1 + (k - m\omega^2)/(i\omega b)} = \frac{1}{1 - i(k - m\omega^2)/(\omega b)}\,.$$

Now the gain $g = |\tilde{g}|$ can be written as

$$g \;=\; \frac{1}{\sqrt{1 + (k - m\omega^2)^2/(\omega b)^2}}.$$

Because squares are always positive, this is clearly largest when the term $k - m\omega^2 = 0$. At this point $g = 1$ and $\omega_r = \sqrt{k/m} = \omega_0$, i.e. the resonant frequency is the natural frequency.

Since $\tilde{g}(\omega_0) = 1$, we also see that the phase lag $\phi = \text{Arg}(\tilde{g})$ is 0 at ω_r Thus the input and output sinusoids are in phase at resonance.

We have found interesting and rather surprising results for this dashpot-driven mechanical system, namely, that the resonant frequency occurs at the system's natural undamped frequency ω_0; that this resonance is independent of the damping coefficient b; and that the maximum gain which can be obtained is $g = 1$. We can contrast this with the spring-side driven system worked out in the previous note, where the resonant frequency certainly *did* depend on the damping coefficient. In fact, there was no resonance at all if the system is too heavily damped. In addition, the gain could, in principle, be arbitarily large.

Comparing these two mechanical systems side-by-side, we can see the importance of the choice of the specification for the input in terms of understanding the resulting behavior of the physical system. In both cases the right-hand side of the DE is a sinusoidal function of the form $B\cos\omega t$ or $B\sin\omega t$, and the resulting mathematical formulas are essentially the same. The key difference lies in the dependence of the constant B on either the system parameters m, b, k and/or the input frequency ω. It is in fact the dependence of B on ω and b in the dashpot-driven case that results in the radically different result for the resonant input frequency ω_r.

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises