C. Complex Numbers

1. Complex arithmetic.

Most people think that complex numbers arose from attempts to solve quadratic equations, but actually it was in connection with cubic equations they first appeared. Everyone knew that certain quadratic equations, like

$$x^2 + 1 = 0, \qquad \text{or} \qquad x^2 + 2x + 5 = 0,$$

had no solutions. The problem was with certain cubic equations, for example

$$x^3 - 6x + 2 = 0.$$

This equation was known to have three real roots, given by simple combinations of the expressions

$$A = \sqrt[3]{-1 + \sqrt{-7}}, \qquad B = \sqrt[3]{-1 - \sqrt{-7}}; \tag{1}$$

one of the roots for instance is $A + B$: it may not look like a real number, but it turns out to be one.

What was to be made of the expressions A and B? They were viewed as some sort of "imaginary numbers" which had no meaning in themselves, but which were useful as intermediate steps in calculations that would ultimately lead to the real numbers you were looking for (such as $A + B$).

This point of view persisted for several hundred years. But as more and more applications for these "imaginary numbers" were found, they gradually began to be accepted as valid "numbers" in their own right, even though they did not measure the length of any line segment. Nowadays we are fairly generous in the use of the word "number": numbers of one sort or another don't have to measure anything, but to merit the name they must belong to a system in which some type of addition, subtraction, multiplication, and division is possible, and where these operations obey those laws of arithmetic one learns in elementary school and has usually forgotten by high school — the commutative, associative, and distributive laws.

To describe the complex numbers, we use a formal symbol i representing $\sqrt{-1}$; then a **complex number** is an expression of the form

$$a + bi, \qquad a, b \quad \text{real numbers.} \tag{2}$$

If $a = 0$ or $b = 0$, they are omitted (unless both are 0); thus we write

$$a + 0i = a, \qquad 0 + bi = bi, \qquad 0 + 0i = 0\ .$$

The definition of *equality* between two complex numbers is

$$a + bi = c + di \quad \Leftrightarrow \quad a = c,\ b = d\ . \tag{3}$$

This shows that the numbers a and b are uniquely determined once the complex number $a + bi$ is given; we call them respectively the **real** and **imaginary** parts of $a + bi$. (It would be more logical to call bi the imaginary part, but this would be less convenient.) In symbols,

$$a = \mathrm{Re}\,(a + bi), \qquad b = \mathrm{Im}\,(a + bi) \tag{4}$$

Addition and multiplication of complex numbers are defined in the familiar way, making use of the fact that $i^2 = -1$:

(5a) **Addition** $(a+bi)+(c+di) \;=\; (a+c)+(b+d)i$

(5b) **Multiplication** $(a+bi)(c+di) \;=\; (ac-bd)+(ad+bc)i$

Division is a little more complicated; what is important is not so much the final formula but rather the procedure which produces it; assuming $c+di \neq 0$, it is:

(5c) **Division** $\dfrac{a+bi}{c+di} \;=\; \dfrac{a+bi}{c+di}\cdot\dfrac{c-di}{c-di} \;=\; \dfrac{ac+bd}{c^2+d^2}+\dfrac{bc-ad}{c^2+d^2}\,i$

This division procedure made use of *complex conjugation*: if $z = a+bi$, we define the **complex conjugate** of z to be the complex number

(6) $\bar{z} = a - bi$ (note that $z\bar{z} = a^2+b^2$).

The size of a complex number is measured by its **absolute value**, or *modulus*, defined by

(7) $|z| = |a+bi| = \sqrt{a^2+b^2};$ (thus : $z\bar{z} = |z|^2$).

Remarks. For the sake of computers, which do not understand what a "formal expression" is, one can define a complex number to be just an ordered pair (a,b) of real numbers, and define the arithmetic operations accordingly; using (5b), multiplication is defined by

$$(a,b)(c,d) \;=\; (ac-bd,\; ad+bc)\ .$$

Then if we let i represent the ordered pair $(0,1)$, and a the ordered pair $(a,0)$, it is easy to verify using the above definition of multiplication that

$$i^2 = (0,1)(0,1) = (-1,0) = -1 \qquad \text{and} \qquad (a,b) = (a,0)+(b,0)(0,1) = a+bi\ ,$$

and we recover the human way of writing complex numbers.

Since it is easily verified from the definition that multiplication of complex numbers is commutative: $z_1z_2 = z_2z_1$, it does not matter whether the i comes before or after, i.e., whether we write $z = x+yi$ or $z = x+iy$. The former is used when x and y are simple numbers because it looks better; the latter is more usual when x and y represent functions (or values of functions), to make the i stand out clearly or to avoid having to use parentheses:

$$2+3i, \quad 5-2\pi i; \qquad \cos\tfrac{\pi}{2}+i\,\sin\tfrac{\pi}{2}, \quad x(t)+i\,y(t)\ .$$

2. Polar representation.

Complex numbers are represented geometrically by points in the plane: the number $a+ib$ is represented by the point (a,b) in Cartesian coordinates. When the points of the plane represent complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we get the **polar form** for a non-zero complex number: assuming $x+iy \neq 0$,

(8) $x+iy \;=\; r(\cos\theta + i\sin\theta)\ .$

When the complex number is written in polar form, we see from (7) that

$$r = |x+iy|. \qquad \textit{(absolute value, modulus)}$$

We call θ the *polar angle* or the *argument* of $x+iy$. In symbols, one sometimes sees

$$\theta = \arg\,(x+iy) \qquad \textit{(polar angle, argument)}\ .$$

The absolute value is uniquely determined by $x+iy$, but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \le \theta < 2\pi \qquad \textit{principal value} \text{ of the polar angle.}$$

This so-called principal value of the angle is sometimes indicated by writing $\operatorname{Arg}\,(x+iy)$. For example,

$$\operatorname{Arg}\,(-1) = \pi, \qquad\qquad \arg\,(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots\ .$$

Changing between Cartesian and polar representation of a complex number is essentially the same as changing between Cartesian and polar coordinates: the same equations are used.

Example 1. Give the polar form for: $-i$, $1+i$, $1-i$, $-1+i\sqrt{3}$.

Solution.

$$\begin{aligned} -i &= i\,\sin\tfrac{3\pi}{2} & 1+i &= \sqrt{2}\,(\cos\tfrac{\pi}{4} + i\,\sin\tfrac{\pi}{4}) \\ -1+i\sqrt{3} &= 2\,(\cos\tfrac{2\pi}{3} + i\,\sin\tfrac{2\pi}{3}) & 1-i &= \sqrt{2}\,(\cos\tfrac{-\pi}{4} + i\,\sin\tfrac{-\pi}{4}) \end{aligned}$$

The abbreviation $\operatorname{cis}\theta$ is sometimes used for $\cos\theta + i\sin\theta$; for students of science and engineering, however, it is important to get used to the exponential form for this expression:

$$(9) \qquad e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{Euler's formula}.$$

Equation (9) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it however is found in the infinite series

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ .$$

If we substitute $i\theta$ for t in the series, and collect the real and imaginary parts of the sum (remembering that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ldots\ ,$$

and so on, we get

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\ &= \cos\theta + i\sin\theta\ , \end{aligned}$$

in view of the infinite series representations for $\cos\theta$ and $\sin\theta$.

> Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (9). What it shows is that Euler's formula (9) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Using the complex exponential, the polar representation (8) is written

$$(10) \qquad x + iy = r\,e^{i\theta}$$

The most important reason for polar representation is that multiplication and division of complex numbers is particularly simple when they are written in polar form. Indeed, by using Euler's formula (9) and the trigonometric addition formulas, it is not hard to show

$$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')} . \tag{11}$$

This gives another justification for the definition (9) — it makes the complex exponential follow the same exponential addition law as the real exponential. The law (11) leads to the simple rules for multiplying and dividing complex numbers written in polar form:

(12a) **multiplication rule** $$r\,e^{i\theta}\cdot r'e^{i\theta'} = r\,r'\,e^{i(\theta+\theta')} ;$$

to multiply two complex numbers, you multiply the absolute values and add the angles.

(12b) **reciprocal rule** $$\frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta};$$

(12c) **division rule** $$\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}\,e^{i\,(\theta-\theta')};$$

to divide by a complex number, divide by its absolute value and subtract its angle.

The reciprocal rule (12b) follows from (12a), which shows that $\frac{1}{r}e^{-i\theta}\cdot re^{i\theta} = 1$.

The division rule follows by writing $\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{1}{r'e^{i\theta'}}\cdot r\,e^{i\theta}$ and using (12b) and then (12a).

Using (12a), we can raise $x+iy$ to a positive integer power by first using $x+iy = r\,e^{i\theta}$; the special case when $r = 1$ is called *DeMoivre's formula*:

(13) $(x+iy)^n = r^n e^{in\theta}$; **DeMoivre's formula:** $(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$.

Example 2. Express a) $(1+i)^6$ in Cartesian form; b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

Solution. a) Change to polar form, use (13), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}\,e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i\,6\pi/4} = 8\,e^{i\,3\pi/2} = -8i .$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (12c).

You can check the answer to (a) by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts; to (b) by doing the division in Cartesian form (5c), then converting the answer to polar form.

3. Complex exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them.

Euler's formula (9) defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b). \tag{14}$$

We stress that the equation (14) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (14) we see that

$$\mathrm{Re}\,(e^{a+ib}) = e^a\cos b, \qquad \mathrm{Im}\,(e^{a+ib}) = e^a\sin b . \tag{15}$$

The complex exponential obeys the usual law of exponents:

$$e^{z+z'} = e^z e^{z'} , \tag{16}$$

as is easily seen by combining (14) and (11).

The complex exponential is expressed in terms of the sine and cosine by Euler's formula (9). Conversely, the sin and cos functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

$$\cos x = \mathrm{Re}\,(e^{ix}), \qquad \sin x = \mathrm{Im}\,(e^{ix}) ; \tag{17}$$

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}) . \tag{18}$$

The equations in (18) follow easily from Euler's formula (9); their derivation is left for the exercises. Here are some examples of their use.

Example 3. Express $\cos^3 x$ in terms of the functions $\cos nx$, for suitable n.

Solution. We use (18) and the binomial theorem, then (18) again:

$$\begin{aligned} \cos^3 x &= \tfrac{1}{8}(e^{ix} + e^{-ix})^3 \\ &= \tfrac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x . \qquad \square \end{aligned}$$

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos x + i \sin x$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + i\,v(x), \qquad u,\ v \text{ real-valued}$$

and its derivative and integral with respect to x are defined to be

$$a)\ D(u+iv) = Du + iDv, \qquad b)\ \int (u+iv)\,dx = \int u\,dx + i\int v\,dx . \tag{19a,b}$$

From this it follows by a calculation that

$$D(e^{(a+ib)x} = (a+ib)e^{(a+ib)x}, \qquad \text{and therefore} \qquad \int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} . \tag{20}$$

Example 4. Calculate $\int e^x \cos 2x\,dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$\begin{aligned} e^x \cos 2x &= \mathrm{Re}\left(e^{(1+2i)x}\right), && \text{by (14) or (15); therefore} \\ \int e^x \cos 2x\,dx &= \mathrm{Re}\left(\int e^{(1+2i)x}\,dx\right), && \text{by (19b).} \end{aligned}$$

Calculating the integral,

$$\begin{aligned} \int e^{(1+2i)x}\,dx &= \frac{1}{1+2i}\,e^{(1+2i)x} && \text{by (20);} \\ &= \left(\frac{1}{5} - \frac{2}{5}i\right)\left(e^x \cos 2x + i\,e^x \sin 2x\right), \end{aligned}$$

using (14) and complex division (5c). According to the second line above, we want the real part of this last expression. Multiply using (5b) and take the real part; you get

$$\tfrac{1}{5}e^x \cos 2x + \tfrac{2}{5}e^x \sin 2x. \qquad \square$$

In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and you need to get used to it.

4. Finding n-th roots.

To solve linear differential equations with constant coefficients, you need to be able find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where α is a complex number, i.e., finding the n-th roots of α. Polar representation will be a big help in this.

Let's begin with a special case: the **n-th roots of unity**: the solutions to

$$z^n = 1 \ .$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

$$r^n e^{in\theta} \ = \ 1 \cdot e^{(2k\pi i)}, \qquad k = 0, \pm 1, \pm 2, \dots \ .$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \qquad n\theta = 2k\pi \ , \qquad k = 0, \pm 1, \pm 2, \dots \ ,$$

from which we conclude that

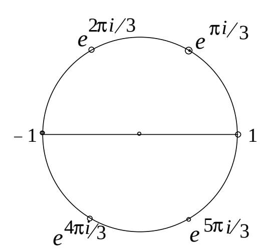
$$(*) \qquad r = 1, \qquad \theta \ = \ \frac{2k\pi}{n} \ , \qquad k = 0, 1, \dots, n-1 \ .$$

In the above, we get only the value $r = 1$, since r must be real and non-negative. We don't need any integer values of k other than $0, \dots, n-1$ since they would not produce a complex number different from the above n numbers. That is, if we add an, an integer multiple of n, to k, we get the same complex number:

$$\theta' \ = \ \frac{2(k+an)\pi}{n} \ = \ \theta + 2a\pi; \qquad \text{and} \qquad e^{i\theta'} = e^{i\theta}, \quad \text{since} \ \ e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from $(*)$ therefore that

(21) *the n-th roots of 1 are the numbers* $e^{2k\pi i/n}, \quad k = 0, \dots, n-1.$



This shows there are n complex n-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with 1; the angle between two consecutive ones is $2\pi/n$. These facts are illustrated on the right for the case $n = 6$.

From (21), we get another notation for the roots of unity (ζ is the Greek letter "zeta"):

$$\text{(22)} \qquad \textit{the } n\textit{-th roots of 1 are} \quad 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \quad \text{where} \;\; \zeta = e^{2\pi i/n}.$$

We now generalize the above to find the n-th roots of an arbitrary complex number w. We begin by writing w in polar form:

$$w \;=\; r\,e^{i\theta}\;; \qquad \theta = \operatorname{Arg} w,\; 0 \le \theta < 2\pi,$$

i.e., θ is the principal value of the polar angle of w. Then the same reasoning as we used above shows that if z is an n-th root of w, then

$$\text{(23)} \qquad z^n = w = r\,e^{i\theta}, \qquad \text{so} \qquad z = \sqrt[n]{r}\,e^{i(\theta+2k\pi)/n}, \qquad k = 0, 1, \ldots, n-1.$$

Comparing this with (22), we see that these n roots can be written in the suggestive form

$$\text{(24)} \qquad \sqrt[n]{w} \;=\; z_0,\; z_0\zeta,\; z_0\zeta^2,\; \ldots,\; z_0\zeta^{n-1}, \qquad \text{where} \;\; z_0 = \sqrt[n]{r}\,e^{i\theta/n}\;.$$

As a check, we see that all of the n complex numbers in (24) satisfy $z^n = w$:

$$\begin{aligned} (z_0\zeta^i)^n \;&=\; z_0^n\zeta^{ni} \;=\; z_0^n \cdot 1^i, &&\text{since } \zeta^n = 1, \text{ by (22);}\\ &=\; w, &&\text{by the definition (24) of } z_0 \text{ and (23).} \end{aligned}$$

Example 5. Find in Cartesian form all values of a) $\sqrt[3]{1}$ b) $\sqrt[4]{i}$.

Solution. a) According to (22), the cube roots of 1 are $1, \omega$, and ω^2, where

$$\begin{aligned} \omega \;&=\; e^{2\pi i/3} &&= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} &&= -\frac{1}{2} + i\frac{\sqrt{3}}{2}\\ \omega^2 \;&=\; e^{-2\pi i/3} &&= \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} &&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}\;. \end{aligned}$$

The greek letter ω ("omega") is traditionally used for this cube root. Note that for the polar angle of ω^2 we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x\;.$$

Note that $\omega^2 = \bar{\omega}$. Another way to do this problem would be to draw the position of ω^2 and ω on the unit circle, and use geometry to figure out their coordinates.

b) To find $\sqrt[4]{i}$, we can use (24). We know that $\sqrt[4]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture, or by using (22)). Therefore by (24), we get

$$\begin{aligned} \sqrt[4]{i} \;&= z_0,\; z_0 i,\; -z_0,\; -z_0 i, &&\text{where } z_0 = e^{\pi i/8} \;=\; \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\;;\\ &= a+ib,\; -b+ia,\; -a-ib,\; b-ia, &&\text{where } z_0 = a+ib \;=\; \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\;. \end{aligned}$$

Example 6. Solve the equation $x^6 - 2x^3 + 2 = 0$.

Solution. Treating this as a quadratic equation in x^3, we solve the quadratic by using the quadratic formula, the two roots are $1+i$ and $1-i$ (check this!), so the roots of the original equation satisfy either

$$x^3 = 1+i, \qquad \text{or} \qquad x^3 = 1-i\ .$$

This reduces the problem to finding the cube roots of the two complex numbers $1 \pm i$. We begin by writing them in polar form:

$$1+i = \sqrt{2}\, e^{\pi i/4}, \qquad 1-i = \sqrt{2}\, e^{-\pi i/4}\ .$$

(Once again, note the use of the negative polar angle for $1-i$, which is more convenient for calculations.) The three cube roots of the first of these are (by (23)),

$$\begin{aligned}
\sqrt[6]{2}\; e^{\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\
\sqrt[6]{2}\; e^{3\pi i/4} &= \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \qquad \text{since}\ \ \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4}; \\
\sqrt[6]{2}\; e^{-7\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right), \qquad \text{since}\ \ \frac{\pi}{12} - \frac{2\pi}{3} = -\frac{7\pi}{12}.
\end{aligned}$$

The second cube root can also be written as $\sqrt[6]{2}\left(\dfrac{-1+i}{\sqrt{2}}\right) = \dfrac{-1+i}{\sqrt[3]{2}}$.

This gives three of the cube roots. The other three are the cube roots of $1-i$, which may be found by replacing i by $-i$ everywhere above (i.e., taking the complex conjugate).

The cube roots can also according to (24) be described as

$$z_1,\ z_1\omega,\ z_1\omega^2 \quad \text{and} \quad z_2,\ z_2\omega,\ z_2\omega^2, \qquad \text{where}\ \ z_1 = \sqrt[6]{2}\; e^{\pi i/12},\ \ z_2 = \sqrt[6]{2}\; e^{-\pi i/12}.$$

Exercises: Section 2E

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18.03 Notes and Exercises