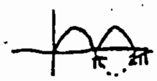
# FOURIER SERIES

**7A-1**

a) For $\sin kt$, $\cos kt$ the frequency is $k$, and $\boxed{(\text{frequency})(\text{period}) = 2\pi}$.

$\therefore \quad \frac{\pi}{3} \cdot P = 2\pi, \quad P = \boxed{6}$

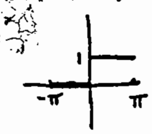
b)  Period is $\boxed{\pi}$: $|\sin(t+\pi)| = |-\sin t| = |\sin t|$

c) $\cos 3t$ has period $= \frac{2\pi}{3}$ (see problem 4)

$\cos^2 3t$ has period $\frac{1}{2}\cdot\frac{2\pi}{3}$ (as in prob. 9):

$$\left(\cos 3\left(t+\tfrac{\pi}{3}\right)\right)^2 = (\cos(3t+\pi))^2 = (-\cos(3t))^2 = (\cos(3t))^2$$

**7A-2** a)



$$a_n = \frac{1}{\pi}\int_0^\pi \cos nt\,dt = \frac{\sin nt}{\pi n}\Big]_0^\pi = 0$$

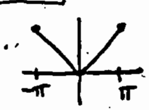
$$\left(a_0 = \frac{1}{\pi}\int_0^\pi dt = 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nt\,dt = \frac{-\cos nt}{n\pi}\Big]_0^\pi = \frac{-(-1)^n-(-1)}{n\pi}$$

$$= \frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd}\end{cases}$$

$$\therefore f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

**7A-2 b**



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\,dt = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \boxed{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\cos nt}_{\text{even function}}\,dt = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$$

$$= \frac{2}{\pi}\left[t\,\frac{\sin nt}{n} - \int \frac{\sin nt}{n}\,dt\right]_0^\pi$$

$$= \frac{2}{\pi}\left(0 + \left[\frac{\cos nt}{n^2}\right]_0^\pi\right) = \frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ -4/\pi n^2, & n \text{ odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\sin nt}_{\text{odd function}}\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \dots\right)$$

**7A-3**

$$\int_{-\pi}^{\pi}\cos mt\cos nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m+n)t + \cos(m-n)t)\,dt$$

$$\begin{cases} = \frac{1}{2}\left[\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n}\right]_{-\pi}^{\pi} = 0 \text{ if } m \ne n \\ = \frac{1}{2}\left[\frac{\sin 2mt}{2m} + t\right]_{-\pi}^{\pi} = \frac{\pi-(-\pi)}{2} = \pi, \text{ if } m = n\end{cases}$$

**7A-4**

a)
$$\int_P^{a+P} f(t)\,dt = \int_0^a f(u+P)\,du = \int_0^a f(u)\,du$$

$u = t-P$ so $t = u+P$ (since $f(u+P) = f(u)$)

Then: (b)

$$\int_a^{a+P} f(t)\,dt = \int_a^P f(t)\,dt + \int_P^{a+P} f(t)\,dt$$

$$= \int_a^P f(t)\,dt + \int_0^a f(t)\,dt \quad \text{by the first part}$$

$$= \int_0^P f(t)\,dt.$$

**7B-1.** a) $a_0 = 2\int_0^1 (1-t)\,dt = 2t - t^2\Big]_0^1 = 1$

$a_n = 2\int_0^1 (1-t)\cos n\pi t\,dt$  Integ. by parts:

$$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} - \int(-1)\frac{\sin n\pi t}{n\pi}\,dt\right]_0^1$$

$$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} + -\frac{\cos n\pi t}{(n\pi)^2}\right]_0^1$$

$$= \frac{-2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ 4/n^2\pi^2, & n \text{ odd}\end{cases}$$

$$f(t) \sim \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \dots\right)$$

Fourier cosine series (picture below)

b) $b_n = 2\int_0^1 (1-t)\sin n\pi t\,dt$  Integ. by parts

$$= 2\left[(1-t)\left(\frac{-\cos n\pi t}{n\pi}\right) - \int(-1)\left(\frac{-\cos n\pi t}{n\pi}\right)\right]_0^1$$

(this part is 0)

$$= 2\left[0 + \frac{1}{n\pi}\right]$$

$$\therefore f(t) \sim \frac{2}{\pi}\left[\sin\pi t + \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} + \dots\right]$$

Fourier sine series (picture below)

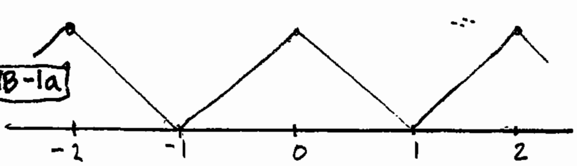
$\boxed{\text{7B-3}}$ a) $\int_{-a}^{0} f(t)\,dt = \int_a^0 f(-u)(-du) = \int_0^a f(u)\,du$

f even  (put $t = -u$)  ($f(-u) = f(u)$)

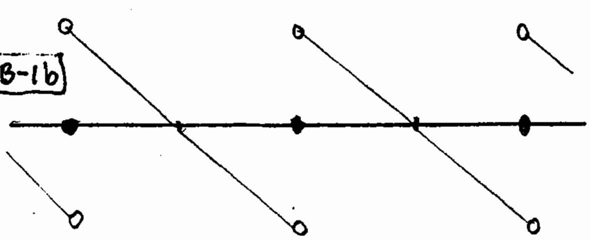
b) $\int_{-a}^{0} f(t)\,dt = \int_a^0 -f(u)(-du) = -\int_0^a f(u)\,du$

f odd  $t = -u$, $f(-u) = -f(u)$

$\boxed{\text{7B-1a}}$

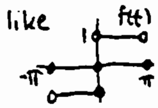



$\boxed{\text{7B-1b}}$



$\boxed{\text{7B-2a}}$ $x'' + 2x = 1$, $x(0) = x(\pi) = 0$

) First expand 1 in a Fourier sine series. This means the (odd) periodic extension ($f(t)$) looks like




. We can then get a F. sine series for $x(t)$, + it will fit the bdry. conditions.

By (21), 8.1,

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \dots\right) \qquad (*)$$

) Look for a series $x(t) = \sum b_n \sin nt$ (this satisfies $x(0) = x(\pi) = 0$).

$$x'' = \sum -b_n \cdot n^2 \sin nt$$
$$+\ 2x = \sum 2b_n \sin nt \qquad \text{Adding}$$
$$f(x) = \sum b_n(2-n^2)\sin nt$$
$$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \dots\right)$$

$\therefore b_n = 0$, $n$ even

$b_n = \frac{4}{\pi}\cdot\frac{1}{2-n^2}\cdot\frac{1}{n}$, if $n$ is odd

$= \frac{-4}{n(n^2-2)\pi}$, $n$ odd.

$$\therefore x(t) = \frac{-4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n(n^2-2)}, \qquad 0 \le t \le \pi$$

$\boxed{\text{7B-2b}}$

$x'' + 2x = t$, $x'(0) = x'(\pi) = 0$

a) Expand $t$ in a Fourier cosine series; (we will then get a F. cosine series for $x(t)$, + it will satisfy the 2 endpoint conditions).

Get ~~=~~ $a_n = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$  Integ by parts

$$= \frac{2}{\pi}\left[t\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^\pi = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}$$

$$a_n = \begin{cases} = \frac{-4}{n^2\pi} & \text{if } n \text{ odd} \\ = 0 & \text{if } n \text{ even.}\end{cases} \qquad a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$\therefore t \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \dots\right)$$

b) $x = \frac{A_0}{2} + \sum A_n \cos nt$  (×2

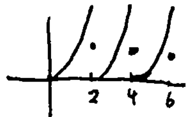
$$x'' = -\sum n^2 A_n \cos nt \qquad \text{Adding,}$$
$$t = A_0 + \sum A_n(2-n^2)\cos nt$$

$\therefore A_0 = \frac{\pi}{2}$, $A_n = 0$ if $n$ even  $A_n = -4$

$A_n = -\frac{4}{\pi}\cdot\frac{1}{n^2(2-n^2)}$ if $n$ odd

## 7B-4

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n}$$




$f(t)$

$$f'(t) \overset{?}{=} -\frac{4}{\pi^2}\sum \frac{\sin n\pi t}{n} - \frac{4}{\pi}\sum_1^\infty \cos n\pi t$$

This series doesn't converge (the cosine terms don't add up – for example, when $t=0$). So it certainly can't converge to $f'(t)$

## 7C-1

**Preliminary remarks**

$$mx'' + kx = F(t)$$

The natural frequency of the spring-mass system is $\omega_0 = \sqrt{k/m}$

The typical term of the Fourier expansion of $F(t)$ is $\cos \frac{n\pi}{L}t$, $\sin \frac{n\pi}{L}t$; thus we get pure resonance if and only if the Fourier series has a $\cos\frac{n\pi}{L}t$ or $\sin\frac{n\pi}{L}t$ term where $\boxed{\frac{n\pi}{L} = \omega_0}$

a) $\omega_0 = \sqrt{5}$ for spring-mass system
$L = 1$

Fourier series is $\sum b_n \sin n\pi t$

$n\pi \neq \sqrt{5}$ $\therefore$ no resonance

b) $\omega_0 = 2\pi$ $L=1$

Fourier series is $\sum b_n \sin n\pi t$, and $n\pi = 2\pi$ if $n=2$

Example 1, 8.4 shows that this term actually occurs in the Fourier series for $2t$ (just change scale). $\therefore$ get resonance

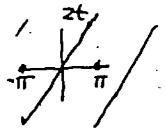
c) $\omega_0 = 3$ Fourier series is a sine series ($F(t)$ is odd):

$F(t) = \sum b_n \sin nt$ all odd $n$ occur (see Problem 8.3/11, or ex. 1, **8.1**)

$\therefore n=3$ occurs, + we get resonance.

## 7C-2

Fourier series for $F(t)$




will be same (up to factor 2) as the Fourier sine series in Example 1, **8.3** $(L=\pi)$

$$F(t) = 4\left(\sin t - \tfrac{1}{2}\sin 2t + \tfrac{1}{3}\sin 3t - \dots\right)$$

$$\begin{aligned} x' &= \sum B_n \sin nt \quad \lfloor \times 3 \\ x'' &= \sum -B_n \cdot n^2 \sin nt \quad \text{Adding:} \\ \hline F(t) &= \sum B_n(3-n^2)\sin nt \end{aligned}$$

$$\therefore B_n = (-1)^{n+1}\cdot\frac{4}{n}\cdot\frac{1}{(3-n^2)} = \frac{(-1)^n\cdot 4}{n(n^2-3)}$$

## 7C-3a

The natural frequency of the undamped spring is $\omega_0 = \sqrt{18/2} = 3$

This frequency occurs in the Fourier series for $F(t)$ (see problem 3). Thus the $n=3$ term should dominate. (The actual series is

$$x_{sp}(t) \approx .25\sin(t-.0063) - .20\sin(2t-.02) + 4.44\sin(3t-1.5708) - .07\sin(4t-3.1130)\dots$$

(steady periodic soln – no transients)

## 7C-3b

The natural frequency of the undamped spring is $\sqrt{30/3} = \sqrt{10}$

Expanding the force in a Fourier series, since $L = 1$ (half-period), + $F(t)$ is odd, it will be $F(t) = \sum b_n \sin n\pi t$

It's virtually certain all terms will occur (since $F(t)$ looks so messy) – (check soln to 8.4/5 in back of book)

$\therefore$ since $\sqrt{10} \approx \pi$, $b_1 \sin \pi t$ should be the dominant term in the $x_{sp}(t)$ series (this checks with answer given in back of book)

[NOTE: Edwards + Penney 4th edn: 8.4 (16), p. 590 has a sign error in denominators – cf. (13), which is correct.]

# M.I.T. 18.03 Ordinary Differential Equations
# 18.03 Notes and Exercises