

Section 6 Solutions

6A-1 All of these use the ratio test:
 if $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = L$, $\sum b_n$ converges if $L < 1$
 and $\sum b_n$ diverges if $L > 1$.

a) $n x \left| \frac{(n+1)x^{n+1}}{n x^n} \right| = \left(\frac{n+1}{n} \right) |x| \rightarrow |x|$
 as $n \rightarrow \infty$

\therefore converges if $|x| < 1$, so $R = 1$

b) $\left| \frac{x^{2(n+1)}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{x^{2n}} \right| = \frac{n}{(n+1)2} |x|^2$

$\rightarrow \frac{1}{2} |x|^2$, and $\frac{|x|^2}{2} < 1$
 as $n \rightarrow \infty$ if $|x| < \sqrt{2}$

\therefore converges if $|x| < \sqrt{2}$, so $R = \sqrt{2}$

c) $\left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = (n+1) |x| \rightarrow \infty$
 as $n \rightarrow \infty$ (if $x \neq 0$).

\therefore converges only when $x = 0$;
 $R = 0$.

d) $\left| \frac{[2(n+1)]!}{(n+1)!^2} \cdot x^{n+1} \cdot \frac{(n!)^2}{(2n)! x^n} \right|$

$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} |x| \rightarrow 4|x|$
 as $n \rightarrow \infty$

\therefore converges if $4|x| < 1$, i.e., $|x| < \frac{1}{4}$,
 so $R = \frac{1}{4}$

6A-2 a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\therefore \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$
 $= \sum_{n=0}^{\infty} (n+1) x^n$

(replacing n by $n+1$).

b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\therefore e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$x e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$

6A-2c $\frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

Integrating:

$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + c$

($c=0$: substitute $x=0$ on both sides)
 to see that $c=0$)

d) $\frac{d \ln(1+x)}{dx} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

Integrating:

$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + c = 0$

(see that $c=0$ by substituting $x=0$ on both sides)

[series could also be written $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$
 (putting n for $n+1$)

6A-2d $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$y' = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$y'' = \sum_{n=1}^{\infty} \frac{2n x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

the 0 term disappears $= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (changing $n \rightarrow n+1$)

This shows $y'' = y$, or $y'' - y = 0$.

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$

$\therefore \frac{e^x - e^{-x}}{2} = \frac{2x}{2} + \frac{2x^3}{2 \cdot 3!} + \frac{2x^5}{2 \cdot 5!} + \dots$

$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

4a) $\sum_{n=0}^{\infty} x^{3n+2} = x^2 \sum_{n=0}^{\infty} x^{3n}$

$= x^2 \cdot \frac{1}{1-x^3}$

(since $\sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty} (x^3)^n = \frac{1}{1-x^3}$)

6A-4(b) Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
 Integrate both sides:
 $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x) + C \stackrel{C=0}{\substack{\text{(substitute)} \\ x=0}}$
 $\therefore \sum_{n=0}^{\infty} \frac{x^n}{n+1} = -\frac{\ln(1-x)}{x}$

4c) Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
 Differentiating, $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$
 $\therefore \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$
 note $\rightarrow 1$ or 0 (makes no difference)

6B-1
 a) Since $y(0) = 1$,
 $y = 1 + a_1x + a_2x^2 + a_3x^3$
 $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$
 $y^2 = (1 + a_1x + a_2x^2 + \dots)(1 + a_1x + a_2x^2 + \dots)$
 $= 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_2a_1)x^3 + \dots$ (this is far enough to get a_3)
 $y' = x + y^2$ says that
 $a_1 + 2a_2x + 3a_3x^2 + \dots = 1 + (2a_1 + 1)x + (2a_2 + a_1^2)x^2 + \dots$

\therefore equating coefficients of like powers of x gives us:
 $a_1 = 1, 2a_2 = 2a_1 + 1 = 3, \therefore a_2 = 3/2$
 $3a_3 = 2a_2 + a_1^2 = 4, \therefore a_3 = 4/3$

So: $y = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots$

b) Using Taylor's formula: $y(0) = 1$
 $y' = x + y^2 \quad \therefore y'(0) = 0 + 1^2 = 1$
 $\therefore y'' = 1 + y' \cdot 2y \quad y''(0) = 1 + 1 \cdot (2 \cdot 1) = 3$
 $y''' = y'' \cdot 2y + y' \cdot 2y' \quad y'''(0) = 3 \cdot 2 + 1 \cdot 2 = 8$

6B-2

a) $y = \sum_{n=0}^{\infty} a_n x^n$
 $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$
 $y' - y = x$ says that
 $(n+1)a_{n+1} - a_n = 0$ if $n \neq 1$
 $= 1$ if $n = 1$,

that is, (since $y(0) = 0$):
 $a_0 = 0, a_{n+1} = \frac{a_n}{n+1}$ if $n \neq 1$
 and $2a_2 - a_1 = 1$.

This gives:
 $a_0 = 0, a_1 = 0, a_2 = 1/2, a_3 = 1/3 \cdot 1/2,$
 $a_4 = 1/4 \cdot 1/3 \cdot 1/2, \text{ etc. } \dots$

so $y = \sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$

b) $y = \sum_{n=0}^{\infty} a_n x^n \xrightarrow{-xy} -xy = \sum_{n=0}^{\infty} a_n x^{n+1}$
 $y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \xrightarrow{n \rightarrow n+2} \sum_{n=2}^{\infty} (n+2) a_{n+2} x^{n+1}$ (note)
 $y' = -xy \Rightarrow$
 $(n+2)a_{n+2} = -a_n \quad n=0, 1, 2, \dots$
 $a_1 = 0$ (unresponding to $n=-1$)
 $a_0 = 1$ (since $y(0) = 1$)
 $\therefore a_{n+2} = \frac{-a_n}{n+2} \quad n=0, 1, 2, \dots$

so $a_0 = 1, a_2 = -1/2, a_4 = 1/4 \cdot 1/2, a_6 = -1/6 \cdot 4 \cdot 2$
 $a_1 = a_3 = a_5 = \dots = 0$.

so $y = \sum_{n=0}^{\infty} \frac{x^{2n} (-1)^n}{2^n \cdot n!} = e^{-x^2/2}$

By Taylor's formula,
 $y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$
 $\therefore y = 1 + x + \frac{3}{2}x^2 + \frac{8}{6}x^3 + \dots$
 just as in part (a).

6B-2

$$c) \quad y = \sum_0^{\infty} a_n x^n$$

$$y' = \sum_0^{\infty} n a_n x^{n-1} \rightsquigarrow \sum_0^{\infty} (n+1) a_{n+1} x^n$$

$$x y' = \sum_0^{\infty} n a_n x^n$$

$$\therefore (1-x)y' - y = 0 \Rightarrow \text{(equating the coeff of } x^n \text{ to 0)}$$

$$(n+1)a_{n+1} - n a_n - a_n = 0$$

$$\text{or } a_{n+1} = \frac{(n+1)}{n+1} a_n = a_n$$

$$y(0) = 1 \Rightarrow a_0 = 1$$

$$\therefore y = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

6C-1

$$a) \quad \sum_1^{\infty} a_n x^{n+3} = \sum_{n \rightarrow n-3}^{\infty} a_{n-3} x^n$$

↑
this starts with x^4 , so this must also

$$b) \quad \sum_0^{\infty} n(n-1)a_n x^{n-2} = \sum_{n \rightarrow n+2}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

↑
starts with x^0 , so this must also ↑

$$c) \quad \sum_1^{\infty} (n+1)a_n x^{n-1} = \sum_{n \rightarrow n+1}^{\infty} (n+2)a_{n+1} x^n$$

↑
starts with x^0 , so this must also ↑

6C-2

$$y = \sum_0^{\infty} a_n x^n \rightsquigarrow 4y = \sum_0^{\infty} 4a_n x^n$$

$$y'' = \sum_0^{\infty} a_n \cdot n(n-1) x^{n-2} \rightsquigarrow \sum_{n \rightarrow n+2}^{\infty} a_{n+2} (n+2)(n+1) x^n$$

$$y'' - 4y = 0 \Rightarrow a_{n+2} (n+2)(n+1) - 4a_n = 0$$

$$\text{or } \boxed{a_{n+2} = \frac{4 a_n}{(n+2)(n+1)}} \quad \text{Recursion formula}$$

$$\therefore a_2 = \frac{4 a_0}{2 \cdot 1}, \quad a_4 = \frac{4}{4 \cdot 3} \cdot \frac{4}{2 \cdot 1} a_0 = \frac{4^2}{4!} a_0$$

$$a_3 = \frac{4 a_1}{3 \cdot 2}, \quad a_5 = \frac{4}{5 \cdot 4} \cdot \frac{4}{3 \cdot 2} a_1 = \frac{4^2}{5!} a_1$$

continued above ↑

6C-2

(continued)

Get one series by taking $a_0=1, a_1=0$:

$$y_0 = 1 + \frac{4}{2!} x^2 + \frac{4^2}{4!} x^4 + \frac{4^3}{6!} x^6 + \dots$$

Other series: take $a_0=0, a_1=1$

$$y_1 = x + \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} + \dots$$

In summation notation:

$$y_0 = \sum_0^{\infty} \frac{4^n x^{2n}}{n!}, \quad y_1 = \sum_0^{\infty} \frac{4^n x^{2n+1}}{(2n+1)!}$$

Can also write numerator as $(2x)^{2n}$

6C-3

Not solved.

6C-4

$$y'' - 2xy' + ky = 0, \quad \boxed{k=2m}$$

$$y = \sum_0^{\infty} a_n x^n \rightsquigarrow \sum_0^{\infty} 2m a_n x^n$$

$$y' = \sum_0^{\infty} n a_n x^{n-1} \rightsquigarrow \sum_0^{\infty} -2n a_n x^n$$

$$y'' = \sum_0^{\infty} n(n-1) a_n x^{n-2} \rightsquigarrow \sum_{n \rightarrow n+2}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Since $y'' - 2xy' + ky = 0$, this gives

$$(n+2)(n+1)a_{n+2} - 2n a_n + 2m a_n = 0$$

$$\text{or } \boxed{a_{n+2} = \frac{2(n-m)}{(n+2)(n+1)} a_n}$$

If $n=m$, then $a_{m+2} = 0$, etc.

So: if m is odd,

take $a_0=0, a_1=1$; then

all $a_0 = a_2 = a_4 = \dots = 0$

and all $a_{m+2} = a_{m+4} = 0 = \dots$

so $y_1 = a_1 x + a_3 x^3 + \dots + a_m x^m$

If m is even, take $a_1=0$.

then similarly, (so $a_3=0, a_5=0, \dots$)

$$y_0 = a_0 + a_2 x^2 + \dots + a_m x^m$$

6C-5

$y'' = xy$

$y = \sum_0^{\infty} a_n x^n \rightsquigarrow xy = \sum_0^{\infty} a_n x^{n+1} = \sum_1^{\infty} a_{n-1} x^n$
 $y'' = \sum_0^{\infty} n(n-1) a_n x^{n-2} \rightsquigarrow \sum_{n \rightarrow n+2}^{\infty} (n+2)(n+1) a_{n+2} x^n$

Equating coeff's of like powers of x (since $y'' = xy$)

gives $(n+2)(n+1) a_{n+2} = a_{n-1} \quad (n \geq 1) \rightsquigarrow \therefore$
 $= 0 \quad (n=0)$

Recursion formula

$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.$

$\therefore a_0, a_1$ are arbitrary, $2a_2 = 0$ (so $a_2 = 0$),

$a_2 = 0$

and other terms are: $a_3 = \frac{a_0}{3 \cdot 2}, \quad a_5 = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2} \dots$

$(\therefore a_7 = a_8 = a_{11} = \dots = 0$
by the recursion formula)

$a_4 = \frac{a_1}{4 \cdot 3}, \quad a_7 = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3} \dots$

Taking $a_0 = 1, a_1 = 0$

gives $y_0 = 1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \dots + \frac{x^{3n}}{3n \cdot (3n-1) \cdot (3n-3) \cdot \dots \cdot 3 \cdot 2} + \dots$

taking $a_0 = 0, a_1 = 1$

gives $y_1 = x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \dots + \frac{x^{3n+1}}{(3n+1) \cdot 3n \cdot (3n-2) \cdot \dots \cdot 4 \cdot 3} + \dots$

6C-6

$y = \sum_0^{\infty} a_n x^n \rightsquigarrow 6y = \sum_0^{\infty} 6a_n x^n$
 $y' = \sum_0^{\infty} n a_n x^{n-1} \rightsquigarrow -2xy' = \sum_0^{\infty} -2n a_n x^n$
 $y'' = \sum_0^{\infty} n(n-1) a_n x^{n-2} \rightsquigarrow y'' = \sum_0^{\infty} (n+2)(n+1) a_{n+2} x^n$
 $\rightsquigarrow -x^2 y'' = -\sum_0^{\infty} n(n-1) a_n x^n$

$y'' - x^2 y'' - 2xy' + 6y = 0$

Equating coeff's of x^n to 0

gives:

$(n+2)(n+1) a_{n+2} - n(n-1) a_n - 2n a_n + 6a_n = 0$

or $a_{n+2} = a_n \frac{[n(n-1) + 2n - 6]}{(n+2)(n+1)}$

or $a_{n+2} = \frac{(n+3)(n-2)}{(n+2)(n+1)} a_n$

RECURSION FORMULA.

This gives solutions

$y_0 = 1 - 3x^2 \quad (a_0 = 1, a_1 = 0 = a_3 = a_5 = \dots)$

$y_1 = x - \frac{3}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 - \dots$

Radius of convergence for y_1 is determined by

ratio test: $\left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \frac{(n+3)(n-2)}{(n+2)(n+1)} |x|^2 \rightarrow x^2$ as $n \rightarrow \infty$, if $|x| < 1$

$\therefore R = 1$. This is expected, since in standard form, ODE is $y'' - \frac{2x}{1-x^2} y' + \frac{6}{1-x^2} y = 0$, and coefficients become infinite at $|x| = 1$.

6C-7

$y = \sum_0^{\infty} a_n x^n, \rightsquigarrow xy = \sum_1^{\infty} a_{n-1} x^n$
 $y' = \sum_0^{\infty} n a_n x^{n-1} \rightsquigarrow 2y' = 2 \sum_0^{\infty} (n+1) a_{n+1} x^n$
 $y'' = \sum_0^{\infty} (n+2)(n+1) a_{n+2} x^n$

$\therefore y'' + 2y' + (x-1)y = 0$ leads to the recursion:

$(n+2)(n+1) a_{n+2} + 2(n+1) a_{n+1} + a_{n-1} - a_n = 0$

leading to: $y_0 = 1 + \frac{x^2}{2} - \frac{x^3}{2} + \dots \quad (a_0 = 1, a_1 = 0)$
two sides $y_1 = x - x^2 + \frac{5}{6}x^3 + \dots \quad (a_0 = 0, a_1 = 1)$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises

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