

4A-1 Product is $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \end{bmatrix}$

4A-2 $AB = \begin{bmatrix} 4 & 1 \\ -2 & -4 \end{bmatrix}$ $BA = \begin{bmatrix} -3 & 1 \\ 5 & 3 \end{bmatrix}$

4A-3 $A^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 2 & -2 \\ -3 & 2 \end{bmatrix}$ by the formula

(since $|A| = -2$) $= \begin{bmatrix} -1 & 1 \\ 3/2 & -1 \end{bmatrix}$

check: $\begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3/2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4A-4 $\frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \frac{1}{|A|} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(similarly, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$)

4A-5 $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = A^2$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

4A-6 Using determinantal criterion for lin. dependence,

we want

$0 = \begin{vmatrix} 1 & 2 & c \\ -1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = 4 - 3c - 3$

$\therefore -3c + 1 = 0$
 $c = 1/3$

Adding: $(1 \ 2 \ c) \times 3$

$- (-1 \ 0 \ 1)$

$- (2 \ 3 \ 0) \times 2$

$(0 \ 0 \ 0)$

4B-1 a) $x'' + 5x' + tx^2 = 0 \rightarrow x' = y$
 $y' = -tx^2 - 5y$

b) $y'' - x^2 y' + (1-x^2)y = \sin x$

$\rightarrow y' = z$

$z' = (x^2-1)y + x^2 z + \sin x$

4B-2

$y'' + py'' + qy' + ry = 0$

let $y = y_1$

$y_1' = y_2$

$y_2' = y_3$

$y_3' = -py_3 - qy_2 - ry_1$

matrix form: $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

4B-3

$\begin{cases} x' = x + y \\ y' = 4x + y \end{cases}$

To eliminate y : $y = x' - x$ from 1st eqn.

$\therefore (x' - x)' = 4x + (x' - x)$ 2nd eqn.

or $x'' - x' = 4x + x' - x$

or $x'' - 2x' - 3x = 0$

converting to system:

let $x_1 = x$

system $\begin{cases} x_1' = x_2 \\ x_2' = 2x_2 + 3x_1 \end{cases}$

This system is not same as first, but is equivalent to it - just using different dep't variables.

The rel'n between the variables is:

$x_1 = x$

$x_2 = x + y$

or the other way: $\begin{cases} x = x_1 \\ y = x_2 - x_1 \end{cases}$

If you make this change of vars. the 1st system turns into the second.

4B-4

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ solve $\bar{x}' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \bar{x}$:

a) vectorially: $\frac{d}{dt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$
 $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t}$ these are equal. Other goes same way.

components: $x = e^{3t}$ solves $\begin{cases} x' = 4x - y \\ y' = 2x + y \end{cases}$ just plug in + check it.

b) linearly indep't: $\begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix} = e^{5t} \neq 0$

c) gen sol'n: $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ or $\begin{bmatrix} c_1 e^{3t} + c_2 e^{2t} \\ c_1 e^{3t} + 2c_2 e^{2t} \end{bmatrix}$
which is same as: $x = c_1 e^{3t} + c_2 e^{2t}$
 $y = c_1 e^{3t} + 2c_2 e^{2t}$

4B-5 $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ solve $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
 (do it same as $\frac{4.3}{1a}$ above). Linear indep: $\begin{vmatrix} e^{4t} & e^{-2t} \\ e^{4t} & -e^{-2t} \end{vmatrix} = -2e^{2t}$.
 IVP: $\vec{x}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, gives: (since $e^{4t}, e^{-2t} = 1$ when $t=0$)
 $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \therefore \begin{matrix} c_1 + c_2 = 5 \\ c_1 - c_2 = 1 \end{matrix} \quad \therefore \begin{matrix} c_1 = 3 \\ c_2 = 2 \end{matrix}$
 soln: $3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \vec{x}$.

4B-6 $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. or $\begin{matrix} x' = x + y \\ y' = y \end{matrix}$
 a) From second eqn, $y = c_1 e^t$
 $\therefore x' - x = c_1 e^t$ soln: $x = c_2 e^t + c_1 t e^t$
 $y = c_1 e^t$
 b) Here we eliminate y instead:
 $y = x' - x$ $\therefore (x' - x)' = x' - x$
 $\stackrel{1^{st} \text{ eqn.}}{=} x'' - 2x' + x = 0 \quad \therefore x = c_1 e^t + c_2 t e^t$
 $(m-1)^2 = 0 \quad \therefore y = c_2 e^t$
 same as before (just switch c_1, c_2). $\therefore y = x' - x$ since $y = x' - x$

4B-7 $\begin{matrix} x' = -ax & (\text{straight decay}) \\ y' = -by + ax \end{matrix}$ matrix: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
 $\underbrace{\quad}_{\text{decay rate}} \quad \underbrace{\quad}_{\text{rate at which decay of } x \text{ produces } y}$
 Soln by elimination: eliminate x : $x = \frac{1}{a} y' + \frac{b}{a} y$
 subst. into 1st eqn, get
 $\frac{1}{a} y'' + \frac{b}{a} y' = -y' - by$
 $y'' + (b+a)y' + by = 0 \quad m^2 + (a+b)m + ab = 0$
 $\therefore y = c_1 e^{-at} + c_2 e^{-bt} \quad m = -a, m = -b$
 $\begin{cases} y = c_1 e^{-at} + c_2 e^{-bt} \\ x = c_1 \left(1 + \frac{b}{a}\right) e^{-at} \end{cases} \leftarrow \begin{bmatrix} x = \frac{1}{a} (y' + by) \\ = \frac{1}{a} \begin{pmatrix} -ac_1 e^{-at} \\ -bc_2 e^{-bt} \\ +bc_1 e^{-at} \\ +bc_2 e^{-bt} \end{pmatrix} \end{bmatrix}$
 [NOTE: having found y , you can't just say $x' = -ax$, $\therefore x = c_3 e^{-at}$ since c_3 is not arbitrary - x must also satisfy the 2nd eqn !!

4C-1 a)

a) $\vec{x}' = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \vec{x}$

Eigenvalues:

eigenvalues: $\begin{vmatrix} -3-m & 4 \\ -2 & 3-m \end{vmatrix} = 0$

$\therefore -(3+m)(3-m) + 8 = 0$
 $m^2 - 1 = 0 \quad m = \pm 1$

if $m = 1$,

$\begin{cases} -4\alpha_1 + 4\alpha_2 = 0 \\ -2\alpha_1 + 2\alpha_2 = 0 \end{cases}$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its mult. are soln; eigenvalue.

if $m = -1$:

$\begin{cases} -2\alpha_1 + 4\alpha_2 = 0 \\ -2\alpha_1 + 4\alpha_2 = 0 \end{cases}$

soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

eigenvalue.

NOTE: can also write down char. poly. (vs):

$m^2 - (a_{11} + b_{22})m + \det A = 0$

b) $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$

$\begin{vmatrix} 4-m & -3 \\ 8 & -6-m \end{vmatrix} = 0$ gives

$m^2 + 2m = 0 \quad m = -2, m = 0$

$m = 0$:

$\begin{cases} 4\alpha_1 - 3\alpha_2 = 0 \\ 8\alpha_1 - 6\alpha_2 = 0 \end{cases} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

eigenvalue.

$m = -2$

$\begin{cases} 6\alpha_1 - 3\alpha_2 = 0 \\ 8\alpha_1 - 4\alpha_2 = 0 \end{cases} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

eigenvalue.

$\therefore \vec{x} = C_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$

4C-1

c)

eigenvalues: $\begin{vmatrix} 1-m & -1 & 0 \\ 1 & 2-m & 1 \\ -2 & 1 & -1-m \end{vmatrix} = -1(1-m)(1-m) + 2 + (m-1) - 1 - m = 0$

$-(1-m)(2-m)(1+m) = 0$

eigenvalues: \therefore are $m = 1, m = 2, m = -1$

$m = 1$

$\begin{cases} 0\alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \end{cases}$

soln: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

eigenvalue.

$m = -1$

$\begin{cases} 2\alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 = 0 \end{cases}$

soln: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

eigenvalue.

$m = 2$

$\begin{cases} -\alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 = 0 \end{cases}$

$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

eigenvalue.

soln:

$\vec{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{-t}$

4C-2

Proof #1:

$\therefore 0$ is an eigenvalue if and only if $A\vec{x} = 0\vec{x}$ has a nontriv. soln for \vec{x}
 $\Leftrightarrow A\vec{x} = \vec{0}$ " " " " "
 $\Leftrightarrow \det A = 0$ (see notes p.2 (5)).

Proof #2: The characteristic equation is

$\det(A - mI) = 0$.

if $m = 0$ is a root, this says (substituting $m = 0$) $\det(A) = 0$

4C-3

$\begin{vmatrix} a-m & * & * \\ 0 & b-m & * \\ 0 & 0 & c-m \end{vmatrix} = (a-m)(b-m)(c-m) = 0$
 $\therefore m = a, b, c$ are eigenvalues

This always holds: using a Laplace expansion by the minors of first column:

$\begin{vmatrix} a_1-m & * & \dots & * \\ 0 & a_2-m & & * \\ \vdots & & \ddots & * \\ 0 & \dots & & a_k-m \end{vmatrix} = (a_1-m) \begin{vmatrix} a_2-m & * & \dots & * \\ \vdots & & \ddots & * \\ 0 & \dots & & a_k-m \end{vmatrix}$

$= (a_1-m)(a_2-m) \dots (a_k-m)$

by mathematical induction on the size of matrix (i.e., k)

\therefore eigenvalues are the roots:

$m = a_1, a_2, \dots, a_k = \text{diagonal elements.}$

4C-4

By hypothesis, $A\vec{x} = m\vec{x}$.

Multiply both sides by A :

$AA\vec{x} = mA\vec{x} = m(m\vec{x})$

$\therefore A^2\vec{x} = m^2\vec{x}$

so \vec{x} is eigenv. of A^2 , assoc. to eigenvalue m^2 .

[Continuing, one sees that

$A^k\vec{x} = m^k\vec{x}$

the eigenvalues of A^k are the k th powers of the eigenvalues of A].

4C-5

$\vec{x}' = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix} \vec{x}$

Eigenvalues: $-a, -b$ (by previous problem, or directly)

$m = -a$

$a\alpha_1 + (b+a)\alpha_2 = 0$

soln: $\begin{bmatrix} a-b \\ a \\ 1 \end{bmatrix}$

eigenvalue

$m = -b$

$(-a-b)\alpha_1 = 0$

$a\alpha_1 + 0\alpha_2 = 0$

$\therefore \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ soln. eigenvalue.

$\therefore \vec{x} = C_1 \begin{bmatrix} a-b \\ a \\ 1 \end{bmatrix} e^{-at} + C_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-bt}$

When written out with

components, this is identical to our earlier solution.

4C-6

$S' = S - aS + bJ$
 $J' = J - bJ + aS$
 $S' = (1-a)S + bJ$
 $J' = aS + (1-b)J$
 if $a = b = \frac{1}{2}$,
 $\begin{bmatrix} S' \\ J' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S \\ J \end{bmatrix}$
 Eigenvalues: $\begin{vmatrix} \frac{1}{2}-m & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-m \end{vmatrix} = m^2 - m = 0$
 $m=0, m=1$ eigenvalues:
 $\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
 $\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ IV: $c_1 + c_2 = 20$ $c_2 = 15$
 $c_1 = 5$
 since $t=0$
 Nonoscillation: show char. polynomial has complex roots:
 $m^2 + (a+b-2)m + (1-a-b) = 0$

4C-7

From the "picture":
 $\frac{1}{4}(x_1' - x_1) = x_2$
 $x_2' - x_2 = x_1$
 $\begin{cases} x_1' = x_1 + 4x_2 \\ x_2' = x_1 + x_2 \end{cases}$
 solving:
 eigenvalues: $\begin{vmatrix} 1-m & 4 \\ 1 & 1-m \end{vmatrix} = (1-m)^2 - 4 = 0 \Rightarrow 1-m = \pm 2$
 $m=3, m=-1$
 $m=3: -2\alpha_1 + 4\alpha_2 = 0 \Rightarrow \alpha_1 = 2\alpha_2$
 $m=-1: 2\alpha_1 + 4\alpha_2 = 0 \Rightarrow \alpha_1 = -2\alpha_2$
 soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
 $\therefore \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$
 Initial condition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
 $2c_1 - 2c_2 = 1$
 $c_1 + c_2 = 0$
 Soln: $\vec{x} = \frac{1}{4} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{4} \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$

4D-1

2b. Characteristic equation: $m^2 + 4$; $m = 2i$ eigenvalue
 Corresponding eigenvector:
 $\begin{cases} (1-2i)\alpha_1 - 5\alpha_2 = 0 \\ \alpha_1 + (-1-2i)\alpha_2 = 0 \end{cases}$ these are multiples of each other
 Possible choices for eigenvector: $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$ or $\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$
 The second choice gives as the soln $(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} i) (\cos 2t + i \sin 2t)$
 with real part $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t$, imag. part $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t$
 $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = c_1 (\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t) + c_2 (\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin 2t)$
 $\therefore x = (c_1 + 2c_2) \cos 2t + (c_2 - c_1) \sin 2t$
 $y = c_1 \cos 2t + c_2 \sin 2t$
 The other choice leads to $x = 5a_1 \cos 2t + 5a_2 \sin 2t$
 $y = (a_1 - 2a_2) \cos 2t + (2a_1 + a_2) \sin 2t$
 (an equivalent solution).

4D-2

Characteristic equation is $m^2 - 6m + 25 = 0$
 $\therefore m = 3 \pm 4i$, by quadratic formula
 Using $3+4i$ as complex eigenvalue, corresponding eigenvector comes from equation $(3-m)\alpha_1 + 4\alpha_2 = 0 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$
 Corresponding solution is formed from real + imag. parts of $(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i) e^{3t} (\cos 4t + i \sin 4t)$, giving
 $x = e^{3t} (c_1 \cos 4t + c_2 \sin 4t)$
 $y = e^{3t} (c_1 \sin 4t - c_2 \cos 4t)$

4D-3

Char. equation is $(m-2)^2(m+1) = 0$
 Eigenvalue -1 gives eqns $3\alpha_1 + 3\alpha_2 + 3\alpha_3 = 0$
 $-3\alpha_2 = 0 \Rightarrow \alpha_2 = 0$
 $3\alpha_3 = 0 \Rightarrow \alpha_3 = 0$
 $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
 w/td eigenvalue 2 gives eqns $\begin{cases} 3\alpha_2 + 3\alpha_3 = 0 \\ -3\alpha_2 - 3\alpha_3 = 0 \\ 0 = 0 \end{cases}$
 which have 2 lin. indep. solns. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ — thus 2 is a complete eigenvalue
 $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$
 $x = c_1 e^{-t} + c_2 e^{2t}$
 $y = -c_1 e^{-t} + c_3 e^{2t}$
 $z = -c_3 e^{2t}$

4D-4

a) $A_1' = (A_2 - A_1) + (A_3 - A_1)$ $A_2 - A_1 = x_2 - x_1$
 $A_3 - A_1 = x_3 - x_1$
 $\therefore x_1' = x_2 - x_1 + x_3 - x_1 = -2x_1 + x_2 + x_3$
 Similarly, $x_2' = x_1 - 2x_2 + x_3$
 $x_3' = x_1 + x_2 - 2x_3$

b) Characteristic eqn is $m^3 + 6m^2 + 9m = 0$
 $= m(m+3)^2$
 Eigenvalue 0 gives eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, normal mode is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
 $(e^{0t} = 1, \text{ notice})$
 Eigenvalue -3 gives for eigenvector equations just $\alpha_1 + \alpha_2 + \alpha_3 = 0$ (all 3 eqns are same)
 This is a complete eigenvalue: it has multiplicity 2 and 2 lin indep solns: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.
 normal modes: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-3t}$
 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: all 3 cells have same amt of self-steps
 $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-3t}$ — one cell is at steady concentration A_0 + steps that stay; other two cells are equally above + below A_0 at start; self form from one + other until "at ∞ " they all have A_0 self in form.

4E-1 $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ solving to get eigenvectors:

$\lambda^2 - 3\lambda - 10 = 0$ $\lambda = 5$ gives $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
 $(\lambda - 5)(\lambda + 2) = 0$ $\lambda = -2$ gives $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

[eqns are: $-a_1 + 2a_2 = 0$ and $6a_1 + 2a_2 = 0$, respectively]

\therefore coord. change is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

[can multiply each column by a constant and it's still OK]

Check it decouples: $x = 2u + v$
 $y = u - 3v$

\therefore substituting into system:

$$2u' + v' = 4(2u + v) + 2(u - 3v) = 10 - 2v$$

$$u' - 3v' = 5u + 6v, \text{ similarly}$$

Multiply top eqn by 3 and add
 bot. eqn by 2 and subtract

and you get $u' = 5u$ decoupled!
 $v' = -2v$

4E-2 $\vec{x}' = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ use the eigenvectors given in 4D-4:

variable change matrix is:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}; \vec{x} = E\vec{u} \text{ is the change of vars. (cols are eigenvectors)}$$

To check, use matrices: $\vec{u}' = E^{-1}A\vec{u}$ + subst. into system

$$\vec{u}' = E^{-1}A\vec{u}$$

is the new system. Calculating:

$$\vec{u}' = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 3 \end{bmatrix} \vec{u}$$

$$\vec{u}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \vec{u}$$

So system is decoupled: $u_1' = 0$
 $u_2' = -3u_2$
 $u_3' = -3u_3$

4F-1 $x'' + px' + qx = 0$

a) $x' = y$ $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$

b) $y' = -qx - py$
 \therefore Wronskian of two solutions \vec{x}_1 and \vec{x}_2 is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, or $\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$, since $y_i = x_i'$, which is the usual Wronskian of x_1 and x_2 .

4F-2

a) Neither is a constant multiple of the other.

b) $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

c) Since $W = 0$ when $t = 0$, \vec{x}_1 and \vec{x}_2 cannot be solutions of $\vec{x}' = A(t)\vec{x}$, where the entries of $A(t)$ are continuous.

d)

To find $A(t)$ explicitly, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\vec{x}' = A\vec{x}$

Then since $\begin{bmatrix} t \\ 1 \end{bmatrix}$ is soln, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$ $\therefore \begin{cases} 1 = at + b \\ 0 = ct + d \end{cases}$

Since $\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ is soln, $\begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ $\therefore \begin{cases} 2t = at^2 + b2t \\ 2 = ct^2 + d2t \end{cases}$

There are 4 equations for a, b, c, d . Solving: $a = 0, b = 1, c = -2/t^2, d = 2/t$ So not contin. at $t = 0$

4F-3

a) $\begin{vmatrix} \alpha_1 e^{m_1 t} & \alpha_2 e^{m_2 t} \\ \beta_1 e^{m_1 t} & \beta_2 e^{m_2 t} \end{vmatrix} = e^{(m_1+m_2)t} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$

$\vec{\alpha}_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ $\vec{\alpha}_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ $= 0 \Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0$

$\Leftrightarrow \vec{\alpha}_1, \vec{\alpha}_2$ are lin. dep't

b) Suppose $c_1 \vec{\alpha}_1 + c_2 \vec{\alpha}_2 = \vec{0}$ Multiply by A :

$$c_1 A \vec{\alpha}_1 + c_2 A \vec{\alpha}_2 = A \vec{0}$$

$$\therefore c_1 m_1 \vec{\alpha}_1 + c_2 m_2 \vec{\alpha}_2 = \vec{0}$$

Multiply top eqn by m_1 , subtract from 3rd eqn, get

$$c_2 (m_2 - m_1) \vec{\alpha}_2 = \vec{0}$$

But $m_1 \neq m_2, \vec{\alpha}_2 \neq \vec{0}$ (since it's an eigenvector)

$$\therefore c_2 = 0$$

\therefore also $c_1 = 0$ (since $c_1 \vec{\alpha}_1 = \vec{0} \neq \vec{\alpha}_1 \neq \vec{0}$)

4F-4

If $\vec{x}'(0) = \vec{0}$, then since $\vec{x}' = A\vec{x}$,
it follows that $A\vec{x}(0) = \vec{0}$, also.

Since A is nonsingular, we can multiply by A^{-1} , + get
 $\vec{x}(0) = \vec{0}$.

∴ by the uniqueness theorem, $\vec{x}(t) = \vec{0}$ for all t .

Hypotheses needed: A can be a function of t
(with continuous entries); require only that at time $t=0$,
 $A(0)$ is nonsingular — then above reasoning still applies.

4G-1

a) gen soln \vec{x} : $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

$\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

∴ $c_1 + c_2 = 0$
 $c_1 + 2c_2 = 1$ ∴ $c_2 = 1, c_1 = -1$

∴ $\vec{x}_2 = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ solves $\vec{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) $\vec{x}_1 = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ solves $\vec{x}_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

∴ soln + $\vec{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$ is: $a\vec{x}_1 + b\vec{x}_2$

(since $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

$a\vec{x}_1 + b\vec{x}_2 = (2a-b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + (b-a)\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

4G-2

a) $\vec{x}'' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$ Eigenvalues: $\begin{vmatrix} 5-m & -1 \\ 3 & 1-m \end{vmatrix} = m^2 - 6m + 8 = 0$
 $m = 4, 2$

$m=4$: $\alpha_1 - \alpha_2 = 0$ $m=2$: $3\alpha_1 - \alpha_2 = 0$
 $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ eigenvec sol'n $\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}$ eigenvec solution.

Fund. matrix: $\begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix} = F(t)$ $F(0) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ $F(0)^{-1} =$

Soln + IVP: $F(0)^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
 $= \begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}$

b) Normalized fund. mx: $\begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{2t} + \frac{1}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t} + \frac{1}{2}e^{4t} & \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \end{bmatrix}$

Multiply this on right by $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ to get same answer.

4H-1

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$, ... $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ by rules for mx. mult.

∴ $e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$
 $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix} + \begin{bmatrix} \frac{a^2 t^2}{2!} & 0 \\ 0 & \frac{b^2 t^2}{2!} \end{bmatrix} + \dots$
 $= \begin{bmatrix} 1+at+\frac{a^2 t^2}{2!}+\dots & 0 \\ 0 & 1+bt+\frac{b^2 t^2}{2!}+\dots \end{bmatrix} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$

$\vec{x} = e^{At} \vec{x}_0 = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 e^{at} \\ k_2 e^{bt} \end{bmatrix}$

Verify: $x = k_1 e^{at}$
 $y = k_2 e^{bt}$ is soln of: $\begin{cases} x' = ax \\ y' = by \end{cases}$ obvious!
with $x(0) = k_1$
 $y(0) = k_2$

4H-2

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

after this it repeats (since $A^4 = I$)
ie, $A^5 = A$, $A^6 = A^2$, etc.

$e^{At} = \begin{bmatrix} 1 - t^2/2! + t^4/4! - \dots & t - t^3/3! + t^5/5! - \dots \\ -t + t^3/3! - \dots & 1 - t^2/2! + t^4/4! - \dots \end{bmatrix}$
 $= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$\vec{x} = e^{At} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \cos t + k_2 \sin t \\ -k_1 \sin t + k_2 \cos t \end{bmatrix}$

This obviously satisfies the system: $x' = y$, $x(0) = k_1$
 $y' = -x$, $y(0) = k_2$ (I.V.P.)

4H-4

$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$ (*)

In general, for matrices B, C , (square),

$\frac{d}{dt} B(t)C(t) = \frac{dB}{dt} C + B \frac{dC}{dt}$

∴ $\frac{d}{dt} A(t)A(t) = \frac{dA}{dt} A + A \frac{dA}{dt}$

$\neq 2A \frac{dA}{dt}$ since above two matrices are not =!!

∴ in general,

$\frac{d}{dt} A^n(t) \neq nA^{n-1} \frac{dA}{dt}$

and so you can't differentiate (*) term-by-term to get Ae^{At} .

4I-7

a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \dots$

similarly, $A^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$

$\therefore e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 2t & t \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2!} & 0 \\ 4\frac{t^2}{2!} & \frac{t^2}{2!} \end{bmatrix} + \dots$
 $= \begin{bmatrix} 1+t+\frac{t^2}{2!}+\dots & 0 \\ 2t+4\frac{t^2}{2!}+6\frac{t^3}{3!}+\dots & 1+t+\frac{t^2}{2!}+\dots \end{bmatrix}$

But lower-left corner

$= 2t(1 + \frac{2t}{2!} + \frac{3t^2}{3!} + \frac{4t^3}{4!} + \dots) = 2te^t$

$\therefore e^{At} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix} \otimes$

b) $e^{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}t} = e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t} \cdot e^{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}t}$

$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2t & 0 \end{bmatrix} \right)$

(see book ex. 1 p. 951) (higher power of nx are 0)

$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} = \otimes$

c) Find F by solving the system:

$x' = x \Rightarrow x = c_1 e^t$
 $y' = 2x + y \Rightarrow y' - y = 2c_1 e^t$

solving 2nd equation e.g. a linear eqn:

$(y e^{-t})' = 2c_1$

$y e^{-t} = 2c_1 t + c_2$

$y = c_1 \cdot 2te^t + c_2 e^t$

$\therefore F = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix} \quad F(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

so $e^{At} = F \cdot F(0)^{-1} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix}$

4I-1

$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} -5 \\ -8 \end{bmatrix} t + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

① Solve the reduced equation $\vec{x}' = A\vec{x}$

char. eqn is $m^2 + m - 6 = 0$ roots: $m = -3, m = 2$
 $(m+3)(m-2) = 0$

$\frac{m=-3}{+4\alpha_1 + \alpha_2 = 0}$ soln: $\begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-3t}$ $\frac{m=2}{-\alpha_1 + \alpha_2 = 0}$ soln: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

fund. mx: $\begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = F$ $F^{-1} = \begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{3t} & e^{3t} \end{bmatrix} \frac{1}{5e^{3t}}$

$v' = F^{-1} \begin{bmatrix} -5t+2 \\ -8t-8 \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}}{5}(-5t+2) - \frac{e^{3t}}{5}(-8t-8) \\ \frac{4}{5}e^{3t}(-5t+2) + \frac{e^{3t}}{5}(-8t-8) \end{bmatrix} = \begin{bmatrix} \frac{3e^{3t}}{5}t + 2e^{3t} \\ -\frac{2}{5}e^{3t}t \end{bmatrix}$

$\therefore v = \begin{bmatrix} \frac{t^2}{5} + \frac{2}{5}e^{3t} \\ -\frac{2}{5}e^{3t}t + \frac{2}{5}e^{3t} \end{bmatrix}$

$\vec{x}_p = Fv = \begin{bmatrix} \frac{t}{5} + \frac{2}{5} + \frac{1}{5}t + \frac{2}{5} \\ -\frac{4t}{5} - \frac{12}{5} + \frac{1}{5}t + \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 3t+2 \\ 2t-1 \end{bmatrix}$ Ans.

4I-2

a) Using the work from above:

$v' = \frac{1}{5} \begin{bmatrix} e^{2t} & -e^{2t} \\ 4e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^t + 2e^{4t} \\ 4e^{4t} - 2e^t \end{bmatrix}$

$v = \frac{1}{5} \begin{bmatrix} e^t + \frac{e^{4t}}{2} \\ 4e^{4t} + 2e^t \end{bmatrix} \quad \vec{x} = Fv = \frac{1}{5} \begin{bmatrix} e^{-2t} + \frac{e^{4t}}{2} - e^{2t} + 2e^t \\ -4e^{-2t} - 2e^t - e^{2t} + 2e^t \end{bmatrix}$

$\therefore \vec{x}_p = \frac{1}{5} \begin{bmatrix} \frac{5}{2}e^t \\ -5e^{2t} \end{bmatrix} = \begin{bmatrix} \frac{e^t}{2} \\ -e^{2t} \end{bmatrix}$

All to \vec{x}_p the $\vec{x}_h = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$\vec{x}_p = c \vec{e}^{-2t} + d \vec{e}^t$ Substitute in the equation:

$-2c \vec{e}^{-2t} + d \vec{e}^t = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} c \vec{e}^{-2t} + \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} d \vec{e}^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$

$\therefore -2c = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} c + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $d = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} d + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Writing the left side of the 1st system as $-2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} c$, it becomes (I'm just being cute - you could just write it all out + hack away) on subtracting $\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} c$ from both sides

$\begin{bmatrix} -3 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or: $-3c_1 - c_2 = 1$
 $-4c_1 = 0 \Rightarrow c_1 = 0, c_2 = -1$

Similarly for the other system:

$\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ $-d_2 = 0 \Rightarrow d_2 = 0$
 $-4d_1 + 3d_2 = -2 \Rightarrow d_1 = \frac{1}{2}$

Thus $\vec{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t = \begin{bmatrix} e^{t/2} \\ -e^{-2t} \end{bmatrix}$ same as before, to do.

4I-4

Solve reduced equation first: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$

char eqn: $m^2 - 1 = 0$

$m=1$: $\alpha_1 - \alpha_2 = 0$

$m=-1$: $3\alpha_1 - \alpha_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ soln.

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ soln.

To find particular soln, since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ is a soln of reduced equation, we have to use as the trial soln not just $\vec{c}e^t$ but

$$\vec{x}_p = \vec{c}e^t + \vec{d}te^t$$

Substituting into the ODE's:

$$\vec{c}e^t + \vec{d}e^t + \vec{d}te^t = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} (\vec{c}e^t + \vec{d}te^t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

$$\therefore \vec{c} + \vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{c} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{d}$$

Solving second system:

(as done in prob. 2b)

$$\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k$$

Solving first system:

$$\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -1-k \end{bmatrix}$$

Subtract 3x first row from second:

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -4+2k \end{bmatrix} \quad \therefore k=2$$

$$\text{get: } -c_1 + c_2 = -1 \quad \text{so take } \vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (other } \vec{c} \text{ are possible)}$$

$$\text{soln: } \vec{x}_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

↑ to this could be added ↑

4I-5

$\vec{x}' = A\vec{x} + \vec{x}_0$. Try $\vec{x}_p = \vec{c}$. Substituting:

$$A\vec{c} + \vec{x}_0 = 0. \quad \therefore \vec{x}_p = -A^{-1}\vec{x}_0 \quad \text{if } A \text{ is nonsingular!}$$

[If A is singular, you only get soln $\vec{x}_p = \vec{c}$ if $A\vec{c} = -\vec{x}_0$ is consistent. In general, if rank $A = n-r$, you use $\vec{x}_p = \vec{c}_0 + \vec{c}_1 t + \dots + \vec{c}_{r-1} t^{r-1}$

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises

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