M.I.T. 18.03
Ordinary Differential Equations

Notes and Exercises

Arthur Mattuck, Haynes Miller, David Jerison,
Jennifer French, Jeremy Orloff

18.03 NOTES, EXERCISES, AND SOLUTIONS

NOTES

EXERCISES (exercises with * are not solved)

D. Definite Integral Solutions

You will find in your other subjects that solutions to ordinary differential equations (ODE's) are often written as definite integrals, rather than as indefinite integrals. This is particularly true when initial conditions are given, i.e., an initial-value problem (IVP) is being solved. It is important to understand the relation between the two forms for the solution.

As a simple example, consider the IVP

$$y' = 6x^2, \qquad y(1) = 5. \tag{1}$$

Using the usual indefinite integral to solve it, we get $y = 2x^3 + c$, and by substituting $x = 1,\ y = 5$, we find that $c = 3$. Thus the solution is

$$y = 2x^3 + 3 \tag{2}$$

However, we can also write down this answer in another form, as a definite integral

$$y = 5 + \int_1^x 6t^2 dt\ . \tag{3}$$

Indeed, if you evaluate the definite integral, you will see right away that the solution (3) is the same as the solution (2). But that is not the point. *Even before actually integrating,* you can see that (3) solves the IVP (1). For, according to the Second Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)\ .$$

If we use this and differentiate both sides of (3), we see that $y' = 6x^2$, so that the first part of the IVP (1) is satisfied. Also the initial condition is satisfied, since

$$y(1) = 5 + \int_1^1 6t^2\,dt = 5\ .$$

In the above example, the explicit form (2) seems preferable to the definite integral form (3). But if the indefinite integration that leads to (2) cannot be explicitly performed, we must use the integral. For instance, the IVP

$$y' = \sin(x^2), \quad y(0) = 1$$

is of this type: there is no explicit elementary antiderivative (indefinite integral) for $\sin(x^2)$; the solution to the IVP can however be written in the form (3):

$$y = 1 + \int_0^x \sin(t^2)\,dt\ .$$

The most important case in which the definite integral form (3) must be used is in scientific and engineering applications when the functions in the IVP aren't specified, but one still wants to write down the solution explicitly. Thus, the solution to the general IVP

$$y' = f(x), \qquad y(x_0) = y_0 \tag{4}$$

may be written

$$y \;=\; y_0 + \int_{x_0}^{x} f(t)\,dt\ . \tag{5}$$

If we tried to write down the solution to (4) using indefinite integration, we would have to say something like, “the solution is $y = \int f(x)\,dx + c$, where the constant c is determined by the condition $y(x_0) = y_0$” — an awkward and not explicit phrase.

In short, the definite integral (5) gives us an explicit solution to the IVP; the indefinite integral only gives us a procedure for finding a solution, and just for those cases when an explicit antiderivative can actually be found.

G. Graphical and Numerical Methods

In studying the first-order ODE

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

the main emphasis is on learning different ways of finding explicit solutions. But you should realize that most first-order equations cannot be solved explicitly. For such equations, one resorts to graphical and numerical methods. Carried out by hand, the graphical methods give rough qualitative information about how the graphs of solutions to (1) look geometrically. The numerical methods then give the actual graphs to as great an accuracy as desired; the computer does the numerical work, and plots the solutions.

1. Graphical methods.

The graphical methods are based on the construction of what is called a **direction field** for the equation (1). To get this, we imagine that through each point (x, y) of the plane is drawn a little line segment whose slope is $f(x, y)$. In practice, the segments are drawn in at a representative set of points in the plane; if the computer draws them, the points are evenly spaced in both directions, forming a lattice. If drawn by hand, however, they are not, because a different procedure is used, better adapted to people.

To construct a direction field by hand, draw in lightly, or in dashed lines, what are called the **isoclines** for the equation (1). These are the one-parameter family of curves given by the equations

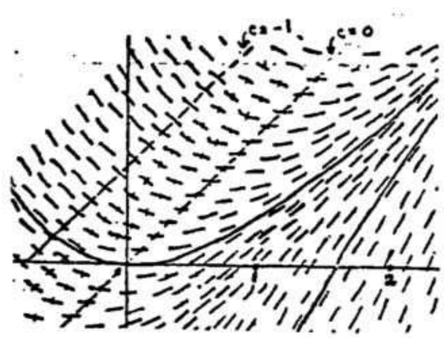
$$f(x,y) = c, \qquad c \text{ constant}. \tag{2}$$

Along the isocline given by the equation (2), the line segments all have the same slope c; this makes it easy to draw in those line segments, and you can put in as many as you want. (Note: "iso-cline" = "equal slope".)

The picture shows a direction field for the equation

$$y' = x - y\ .$$

The isoclines are the lines $x - y = c$, two of which are shown in dashed lines, corresponding to the values $c = 0, -1$. (Use dashed lines for isoclines).



Once you have sketched the direction field for the equation (1) by drawing some isoclines and drawing in little line segments along each of them, the next step is to draw in curves which are at each point tangent to the line segment at that point. Such curves are called **integral curves** or *solution curves* for the direction field. Their significance is this:

(3) *The integral curves are the graphs of the solutions to* $y' = f(x, y)$.

Proof. Suppose the integral curve C is represented near the point (x, y) by the graph of the function $y = y(x)$. To say that C is an integral curve is the same as saying

$$\text{slope of } C \text{ at } (x,y) \quad = \quad \text{slope of the direction field at } (x,y);$$

from the way the direction field is defined, this is the same as saying

$$y'(x) \quad = \quad f(x,y)\ .$$

But this last equation exactly says that $y(x)$ is a solution to (1). □

We may summarize things by saying, the direction field gives a picture of the first-order equation (1), and its integral curves give a picture of the solutions to (1).

Two integral curves (in solid lines) have been drawn for the equation $y' = x - y$. In general, by sketching in a few integral curves, one can often get some feeling for the behavior of the solutions. The problems will illustrate. Even when the equation can be solved exactly, sometimes you learn more about the solutions by sketching a direction field and some integral curves, than by putting numerical values into exact solutions and plotting them.

There is a theorem about the integral curves which often helps in sketching them.

Integral Curve Theorem.

(i) If $f(x,y)$ is defined in a region of the xy-plane, then integral curves of $y' = f(x,y)$ cannot cross at a positive angle anywhere in that region.

(ii) If $f_y(x,y)$ is continuous in the region, then integral curves cannot even be tangent in that region.

A convenient summary of both statements is (here "smooth"= continuously differentiable):

Intersection Principle

(4) *Integral curves of $y' = f(x,y)$ cannot intersect wherever $f(x,y)$ is smooth.*

Proof of the Theorem. The first statement (i) is easy, for at any point (x_0, y_0) where they crossed, the two integral curves would have to have the same slope, namely $f(x_0, y_0)$. So they cannot cross at a positive angle.

The second statement (ii) is a consequence of the uniqueness theorem for first-order ODE's; it will be taken up then when we study that theorem. Essentially, the hypothesis guarantees that through each point (x_0, y_0) of the region, there is a unique solution to the ODE, which means there is a unique integral curve through that point. So two integral curves cannot intersect — in particular, they cannot be tangent — at any point where $f(x,y)$ has continuous derivatives.

Exercises: Section 1C

2. The ODE of a family. Orthogonal trajectories.

The solution to the ODE (1) is given analytically by an xy-equation containing an arbitrary constant c; either in the explicit form (5a), or the implicit form (5b):

$$(5) \qquad (a) \quad y = g(x,c) \qquad\qquad (b) \quad h(x,y,c) = 0\ .$$

In either form, as the parameter c takes on different numerical values, the corresponding graphs of the equations form a one-parameter family of curves in the xy-plane.

We now want to consider the inverse problem. Starting with an ODE, we got a one-parameter family of curves as its integral curves. Suppose instead we start with a one-parameter family of curves defined by an equation of the form (5a) or (5b), can we find a first-order ODE having these as its integral curves, i.e. the equations (5) as its solutions?

The answer is yes; the ODE is found by differentiating the equation of the family (5) (using implicit differentiation if it has the form (5b)), and then using (5) to eliminate the arbitrary constant c from the differentiated equation.

Example 1. Find a first-order ODE whose general solution is the family

$$(6) \qquad y = \frac{c}{x-c} \qquad (c \text{ is an arbitrary constant}).$$

Solution. We differentiate both sides of (6) with respect to x, getting $y' = -\dfrac{c}{(x-c)^2}$. We eliminate c from this equation, in steps. By (6), $x - c = c/y$, so that

$$(7) \qquad y' = -\frac{c}{(x-c)^2} = -\frac{c}{(c/y)^2} = -\frac{y^2}{c}\ ;$$

To get rid of c, we solve (6) algebraically for c, getting $c = \dfrac{yx}{y+1}$; substitute this for the c on the right side of (7), then cancel a y from the top and bottom; you get as the ODE having the solution (6)

$$(8) \qquad y' = -\frac{y(y+1)}{x}\ .$$

Remark. The c must not appear in the ODE, since then we would not have a single ODE, but rather a one-parameter family of ODE's — one for each possible value of c. Instead, we want just one ODE which has each of the curves (5) as an integral curve, regardless of the value of c for that curve; thus the ODE cannot itself contain c.

Orthogonal trajectories.

Given a one-parameter family of plane curves, its **orthogonal trajectories** are another one-parameter family of curves, each one of which is perpendicular to all the curves in the original family. For instance, if the original family consisted of all circles having center at the origin, its orthogonal trajectories would be all rays (half-lines) starting at the origin.

Orthogonal trajectories arise in different contexts in applications. For example, if the original family represents the **lines of force** in a gravitational or electrostatic field, its orthogonal trajectories represent the **equipotentials**, the curves along which the gravitational or electrostatic potential is constant.

In a temperature map of the U.S., the original family would be the **isotherms**, the curves along which the temperature is constant; their orthogonal trajectories would be the **temperature gradients**, the curves along which the temperature is changing most rapidly.

More generally, if the original family is of the form $h(x,y) = c$, it represents the **level curves** of the function $h(x,y)$; its orthogonal trajectories will then be the **gradient curves** for this function — curves which everywhere have the direction of the gradient vector ∇h. This follows from the 18.02 theorem which says the gradient ∇h at any point (x,y) is perpendicular to the level curve of $h(x,y)$ passing through that point.

To find the orthogonal trajectories for a one-parameter family (5):

1. Find the ODE $y' = f(x,y)$ satisfied by the family.

2. The new ODE $y' = -\dfrac{1}{f((x,y)}$ will have as its integral curves the orthogonal trajectories to the family (5); solve it to find the equation of these curves.

The method works because at any point (x,y), the orthogonal trajectory passing through (x,y) is perpendicular to the curve of the family (5a) passing through (x,y). Therefore the slopes of the two curves are negative reciprocals of each other. Since the slope of the original curve at (x,y) is $f(x,y)$, the slope at (x,y) of the orthogonal trajectory has to be $-1/f(x,y)$. The ODE for the orthogonal trajectories then gives their slope at (x,y), thus it is

$$(9) \qquad y' = -\frac{1}{f(x,y)} \qquad \textit{ODE for orthogonal trajectories to (5a)} .$$

More generally, if the equation of the original family is given implicitly by (5b), and its ODE is also in implicit form, the procedure and its justification are essentially the same:
1. Find the ODE in implicit form $F(x,y,y') = 0$ satisfied by the family (5).
2. Replace y' by $-1/y'$; solve the new ODE $F(x,y,-1/y') = 0$ to find the orthogonal trajectores of the original family.

Example 2. Find the orthogonal trajectories to the family of curves $y = c\,x^n$, where n is a fixed positive integer and c an arbitrary constant.

Solution. If $n = 1$, the curves are the family of rays from the origin, so the orthogonal trajectories should be the circles centered at the origin – this will help check our work.

We first find the ODE of the family. Differentiating the equation of the family gives $y' = nc\,x^{n-1}$; we eliminate c by using the equation of the family to get $c = y/x^n$ and substituting this into the differentiated equation, giving

$$(10) \qquad y' = \frac{n\,y}{x} \quad (\textit{ODE of family}); \qquad\qquad y' = -\frac{x}{n\,y} \quad (\textit{ODE of orthog. trajs.}) .$$

Solving the latter equation by separation of variables leads first to $n\,ydy = -xdx$, then after integrating both sides, transposing, and multiplying through by 2, to the solution

$$(11) \qquad x^2 + n\,y^2 = k, \qquad (k \geq 0 \textit{ is an arbitrary non-negative constant; } n \textit{ is fixed.})$$

For different k-values, the equations (11) represent the family of ellipses centered at the origin, and having x-intercepts at $\pm\sqrt{k}$ and y-intercepts at $\pm\sqrt{k/n}$.

If $n = 1$, these intercepts are equal, and the ellipses are circles centered at the origin, as predicted.

Exercises: Section 1D-2

3. Euler's numerical method. The graphical method gives you a quick feel for how the integral curves behave. But when they must be known accurately and the equation cannot be solved exactly, numerical methods are used. The simplest method is called **Euler's method**. Here is its geometric description.

We want to calculate the solution (integral curve) to $y' = f(x,y)$ passing through (x_0, y_0). It is shown as a curve in the picture.

We choose a step size h. Starting at (x_0, y_0), over the interval $[x_0, x_0 + h]$, we approximate the integral curve by the tangent line: the line having slope $f(x_0, y_0)$. (This is the slope of the integral curve, since $y' = f(x,y)$.)

This takes us as far as the point (x_1, y_1), which is calculated by the equations (see the picture)

$$x_1 = x_0 + h, \qquad y_1 = y_0 + h\, f(x_0, y_0)\ .$$

Now we are at (x_1, y_1). We repeat the process, using as the new approximation to the integral curve the line segment having slope $f(x_1, y_1)$. This takes us as far as the next point (x_2, y_2), where

$$x_2 = x_1 + h, \qquad y_2 = y_1 + h\, f(x_1, y_1)\ .$$

We continue in the same way. The general formulas telling us how to get from the $(n-1)$-st point to the n-th point are

$$x_n = x_{n-1} + h, \qquad y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})\ . \tag{12}$$

In this way, we get an approximation to the integral curve consisting of line segments joining the points $(x_0, y_0),\ (x_1, y_1), \ldots$.

In doing a few steps of Euler's method by hand, as you are asked to do in some of the exercises to get a feel for the method, it's best to arrange the work systematically in a table.

Example 3. For the IVP: $y' = x^2 - y^2, \quad y(1) = 0$, use Euler's method with step size .1 to find y(1.2).

Solution. We use $f(x,y) = x^2 - y^2, \quad h = .1$, and (12) above to find x_n and y_n:

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$
0	1	0	1	.1
1	1.1	.1	1.20	.12
2	1.2	.22		

Remarks. Euler's method becomes more accurate the smaller the step-size h is taken. But if h is too small, round-off errors can appear, particularly on a pocket calculator.

As the picture suggests, the errors in Euler's method will accumulate if the integral curve is convex (concave up) or concave (concave down). Refinements of Euler's method are aimed at using as the slope for the line segment at (x_n, y_n) a value which will correct for the convexity or concavity, and thus make the next point (x_{n+1}, y_{n+1}) closer to the true integral curve. We will study some of these. The book in Chapter 6 has numerical examples illustrating Euler' s method and its refinements.

Exercises: Section 1C

C. Complex Numbers

1. Complex arithmetic.

Most people think that complex numbers arose from attempts to solve quadratic equations, but actually it was in connection with cubic equations they first appeared. Everyone knew that certain quadratic equations, like

$$x^2 + 1 = 0, \qquad \text{or} \qquad x^2 + 2x + 5 = 0,$$

had no solutions. The problem was with certain cubic equations, for example

$$x^3 - 6x + 2 = 0.$$

This equation was known to have three real roots, given by simple combinations of the expressions

(1) $$A = \sqrt[3]{-1 + \sqrt{-7}}, \qquad B = \sqrt[3]{-1 - \sqrt{-7}};$$

one of the roots for instance is $A + B$: it may not look like a real number, but it turns out to be one.

What was to be made of the expressions A and B? They were viewed as some sort of "imaginary numbers" which had no meaning in themselves, but which were useful as intermediate steps in calculations that would ultimately lead to the real numbers you were looking for (such as $A + B$).

This point of view persisted for several hundred years. But as more and more applications for these "imaginary numbers" were found, they gradually began to be accepted as valid "numbers" in their own right, even though they did not measure the length of any line segment. Nowadays we are fairly generous in the use of the word "number": numbers of one sort or another don't have to measure anything, but to merit the name they must belong to a system in which some type of addition, subtraction, multiplication, and division is possible, and where these operations obey those laws of arithmetic one learns in elementary school and has usually forgotten by high school — the commutative, associative, and distributive laws.

To describe the complex numbers, we use a formal symbol i representing $\sqrt{-1}$; then a **complex number** is an expression of the form

(2) $$a + bi, \qquad a, b \quad \text{real numbers.}$$

If $a = 0$ or $b = 0$, they are omitted (unless both are 0); thus we write

$$a + 0i = a, \qquad 0 + bi = bi, \qquad 0 + 0i = 0\ .$$

The definition of *equality* between two complex numbers is

(3) $$a + bi = c + di \quad \Leftrightarrow \quad a = c,\ b = d\ .$$

This shows that the numbers a and b are uniquely determined once the complex number $a + bi$ is given; we call them respectively the **real** and **imaginary** parts of $a + bi$. (It would be more logical to call bi the imaginary part, but this would be less convenient.) In symbols,

(4) $$a = \mathrm{Re}\,(a + bi), \qquad b = \mathrm{Im}\,(a + bi)$$

Addition and multiplication of complex numbers are defined in the familiar way, making use of the fact that $i^2 = -1$:

$$\text{(5a)} \qquad \textbf{Addition} \qquad (a+bi)+(c+di) \;=\; (a+c)+(b+d)i$$

$$\text{(5b)} \qquad \textbf{Multiplication} \qquad (a+bi)(c+di) \;=\; (ac-bd)+(ad+bc)i$$

Division is a little more complicated; what is important is not so much the final formula but rather the procedure which produces it; assuming $c + di \neq 0$, it is:

$$\text{(5c)} \qquad \textbf{Division} \qquad \frac{a+bi}{c+di} \;=\; \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} \;=\; \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\,i$$

This division procedure made use of *complex conjugation*: if $z = a + bi$, we define the **complex conjugate** of z to be the complex number

$$\text{(6)} \qquad \bar{z} = a - bi \qquad \text{(note that } z\bar{z} = a^2 + b^2 \text{).}$$

The size of a complex number is measured by its **absolute value**, or *modulus*, defined by

$$\text{(7)} \qquad |z| = |a+bi| = \sqrt{a^2+b^2}; \qquad (\text{thus}: \quad z\bar{z} = |z|^2 \text{).}$$

Remarks. For the sake of computers, which do not understand what a "formal expression" is, one can define a complex number to be just an ordered pair (a, b) of real numbers, and define the arithmetic operations accordingly; using (5b), multiplication is defined by

$$(a,b)(c,d) \;=\; (ac-bd,\; ad+bc)\ .$$

Then if we let i represent the ordered pair $(0, 1)$, and a the ordered pair $(a, 0)$, it is easy to verify using the above definition of multiplication that

$$i^2 = (0,1)(0,1) = (-1,0) = -1 \qquad \text{and} \qquad (a,b) = (a,0)+(b,0)(0,1) = a+bi\ ,$$

and we recover the human way of writing complex numbers.

Since it is easily verified from the definition that multiplication of complex numbers is commutative: $z_1 z_2 = z_2 z_1$, it does not matter whether the i comes before or after, i.e., whether we write $z = x + yi$ or $z = x + iy$. The former is used when x and y are simple numbers because it looks better; the latter is more usual when x and y represent functions (or values of functions), to make the i stand out clearly or to avoid having to use parentheses:

$$2+3i, \quad 5-2\pi i; \qquad \cos\tfrac{\pi}{2}+i\,\sin\tfrac{\pi}{2}, \quad x(t)+i\,y(t)\ .$$

2. Polar representation.

Complex numbers are represented geometrically by points in the plane: the number $a+ib$ is represented by the point (a, b) in Cartesian coordinates. When the points of the plane represent complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we get the **polar form** for a non-zero complex number: assuming $x + iy \neq 0$,

$$\text{(8)} \qquad x+iy \;=\; r(\cos\theta + i\sin\theta)\ .$$

When the complex number is written in polar form, we see from (7) that

$$r = |x+iy|. \qquad \textit{(absolute value, modulus)}$$

We call θ the *polar angle* or the *argument* of $x + iy$. In symbols, one sometimes sees

$$\theta = \arg\,(x+iy) \qquad \textit{(polar angle, argument)}\ .$$

The absolute value is uniquely determined by $x + iy$, but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \le \theta < 2\pi \qquad \textit{principal value} \text{ of the polar angle.}$$

This so-called principal value of the angle is sometimes indicated by writing $\operatorname{Arg}\,(x + iy)$. For example,

$$\operatorname{Arg}\,(-1) = \pi, \qquad\qquad \arg\,(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots\ .$$

Changing between Cartesian and polar representation of a complex number is essentially the same as changing between Cartesian and polar coordinates: the same equations are used.

Example 1. Give the polar form for: $-i$, $1 + i$, $1 - i$, $-1 + i\sqrt{3}$.

Solution.

$$\begin{aligned} -i &= i\,\sin\tfrac{3\pi}{2} & \qquad 1+i &= \sqrt{2}\,(\cos\tfrac{\pi}{4} + i\,\sin\tfrac{\pi}{4}) \\ -1+i\sqrt{3} &= 2\,(\cos\tfrac{2\pi}{3} + i\,\sin\tfrac{2\pi}{3}) & \qquad 1-i &= \sqrt{2}\,(\cos\tfrac{-\pi}{4} + i\,\sin\tfrac{-\pi}{4}) \end{aligned}$$

The abbreviation $\operatorname{cis}\theta$ is sometimes used for $\cos\theta + i\sin\theta$; for students of science and engineering, however, it is important to get used to the exponential form for this expression:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{Euler's formula}. \tag{9}$$

Equation (9) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it however is found in the infinite series

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ .$$

If we substitute $i\theta$ for t in the series, and collect the real and imaginary parts of the sum (remembering that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ldots\ ,$$

and so on, we get

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\ &= \qquad \cos\theta \quad + \quad i\sin\theta\ , \end{aligned}$$

in view of the infinite series representations for $\cos\theta$ and $\sin\theta$.

> Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (9). What it shows is that Euler's formula (9) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Using the complex exponential, the polar representation (8) is written

$$x + iy = r\,e^{i\theta} \tag{10}$$

The most important reason for polar representation is that multiplication and division of complex numbers is particularly simple when they are written in polar form. Indeed, by using Euler's formula (9) and the trigonometric addition formulas, it is not hard to show

$$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')} . \tag{11}$$

This gives another justification for the definition (9) — it makes the complex exponential follow the same exponential addition law as the real exponential. The law (11) leads to the simple rules for multiplying and dividing complex numbers written in polar form:

(12a) **multiplication rule** $$r\,e^{i\theta}\cdot r'e^{i\theta'} = r\,r'\,e^{i(\theta+\theta')} ;$$

to multiply two complex numbers, you multiply the absolute values and add the angles.

(12b) **reciprocal rule** $$\frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta};$$

(12c) **division rule** $$\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}\,e^{i\,(\theta-\theta')};$$

to divide by a complex number, divide by its absolute value and subtract its angle.

The reciprocal rule (12b) follows from (12a), which shows that $\dfrac{1}{r}e^{-i\theta}\cdot re^{i\theta} = 1$.

The division rule follows by writing $\dfrac{r\,e^{i\theta}}{r'e^{i\theta'}} = \dfrac{1}{r'e^{i\theta'}}\cdot r\,e^{i\theta}$ and using (12b) and then (12a).

Using (12a), we can raise $x+iy$ to a positive integer power by first using $x+iy = r\,e^{i\theta}$; the special case when $r = 1$ is called *DeMoivre's formula*:

(13) $(x+iy)^n = r^n e^{in\theta}$; **DeMoivre's formula:** $(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$.

Example 2. Express a) $(1+i)^6$ in Cartesian form; b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

Solution. a) Change to polar form, use (13), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}\,e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i\,6\pi/4} = 8\,e^{i\,3\pi/2} = -8i .$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (12c).

You can check the answer to (a) by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts; to (b) by doing the division in Cartesian form (5c), then converting the answer to polar form.

3. Complex exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them.

Euler's formula (9) defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b). \tag{14}$$

We stress that the equation (14) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (14) we see that

$$\mathrm{Re}\,(e^{a+ib}) = e^a\cos b, \qquad \mathrm{Im}\,(e^{a+ib}) = e^a\sin b . \tag{15}$$

The complex exponential obeys the usual law of exponents:

$$e^{z+z'} = e^z e^{z'} , \tag{16}$$

as is easily seen by combining (14) and (11).

The complex exponential is expressed in terms of the sine and cosine by Euler's formula (9). Conversely, the sin and cos functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

$$\cos x = \mathrm{Re}\,(e^{ix}), \qquad \sin x = \mathrm{Im}\,(e^{ix}) \; ; \tag{17}$$

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}) \; . \tag{18}$$

The equations in (18) follow easily from Euler's formula (9); their derivation is left for the exercises. Here are some examples of their use.

Example 3. Express $\cos^3 x$ in terms of the functions $\cos nx$, for suitable n.

Solution. We use (18) and the binomial theorem, then (18) again:

$$\begin{aligned} \cos^3 x &= \tfrac{1}{8}(e^{ix} + e^{-ix})^3 \\ &= \tfrac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x \; . \qquad \square \end{aligned}$$

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos x + i \sin x$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + i\,v(x), \qquad u,\ v \text{ real-valued}$$

and its derivative and integral with respect to x are defined to be

$$\text{(19a,b)} \qquad a)\ D(u+iv) = Du + iDv, \qquad b)\ \int (u+iv)\,dx = \int u\,dx + i\int v\,dx \; .$$

From this it follows by a calculation that

$$\text{(20)} \quad D(e^{(a+ib)x} = (a+ib)e^{(a+ib)x}, \qquad \text{and therefore} \qquad \int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} \; .$$

Example 4. Calculate $\int e^x \cos 2x\,dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$e^x \cos 2x = \mathrm{Re}\left(e^{(1+2i)x}\right), \qquad \text{by (14) or (15); therefore}$$

$$\int e^x \cos 2x\,dx = \mathrm{Re}\left(\int e^{(1+2i)x}\,dx\right), \qquad \text{by (19b).}$$

Calculating the integral,

$$\begin{aligned} \int e^{(1+2i)x}\,dx &= \frac{1}{1+2i}\,e^{(1+2i)x} \qquad \text{by (20);} \\ &= \left(\frac{1}{5} - \frac{2}{5}i\right)\left(e^x \cos 2x + i\,e^x \sin 2x\right), \end{aligned}$$

using (14) and complex division (5c). According to the second line above, we want the real part of this last expression. Multiply using (5b) and take the real part; you get

$$\tfrac{1}{5}e^x \cos 2x + \tfrac{2}{5}e^x \sin 2x. \qquad \square$$

In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and you need to get used to it.

4. Finding n-th roots.

To solve linear differential equations with constant coefficients, you need to be able find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where α is a complex number, i.e., finding the n-th roots of α. Polar representation will be a big help in this.

Let's begin with a special case: the **n-th roots of unity**: the solutions to

$$z^n = 1\ .$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

$$r^n e^{in\theta} \ = \ 1 \cdot e^{(2k\pi i)}, \qquad k = 0, \pm 1, \pm 2, \dots\ .$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \qquad n\theta = 2k\pi\ , \qquad k = 0, \pm 1, \pm 2, \dots\ ,$$

from which we conclude that

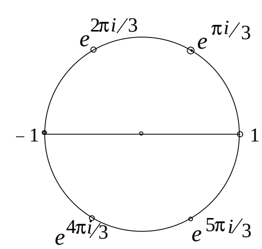
$$(*) \qquad r = 1, \qquad \theta \ = \ \frac{2k\pi}{n}\ , \qquad k = 0, 1, \dots, n-1\ .$$

In the above, we get only the value $r = 1$, since r must be real and non-negative. We don't need any integer values of k other than $0, \dots, n-1$ since they would not produce a complex number different from the above n numbers. That is, if we add an, an integer multiple of n, to k, we get the same complex number:

$$\theta' \ = \ \frac{2(k+an)\pi}{n} \ = \ \theta + 2a\pi; \qquad \text{and} \qquad e^{i\theta'} = e^{i\theta}, \quad \text{since} \ \ e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from $(*)$ therefore that

(21) *the n-th roots of 1 are the numbers* $e^{2k\pi i/n}, \quad k = 0, \dots, n-1.$



This shows there are n complex n-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with 1; the angle between two consecutive ones is $2\pi/n$. These facts are illustrated on the right for the case $n = 6$.

From (21), we get another notation for the roots of unity (ζ is the Greek letter "zeta"):

(22) *the n-th roots of 1 are* $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$.

We now generalize the above to find the n-th roots of an arbitrary complex number w. We begin by writing w in polar form:

$$w \;=\; r\, e^{i\theta}\;; \qquad \theta = \text{Arg}\; w,\; 0 \le \theta < 2\pi,$$

i.e., θ is the principal value of the polar angle of w. Then the same reasoning as we used above shows that if z is an n-th root of w, then

(23) $$z^n = w = r\, e^{i\theta}, \qquad \text{so} \qquad z = \sqrt[n]{r}\, e^{i(\theta+2k\pi)/n}, \qquad k = 0, 1, \ldots, n-1.$$

Comparing this with (22), we see that these n roots can be written in the suggestive form

(24) $$\sqrt[n]{w} \;=\; z_0,\; z_0\zeta,\; z_0\zeta^2,\; \ldots,\; z_0\zeta^{n-1}, \qquad \text{where } \; z_0 = \sqrt[n]{r}\, e^{i\theta/n}\;.$$

As a check, we see that all of the n complex numbers in (24) satisfy $z^n = w$:

$$\begin{aligned} (z_0\zeta^i)^n \;&=\; z_0^n\zeta^{ni} \;=\; z_0^n \cdot 1^i, && \text{since } \zeta^n = 1, \text{ by (22);} \\ &=\; w, && \text{by the definition (24) of } z_0 \text{ and (23).} \end{aligned}$$

Example 5. Find in Cartesian form all values of a) $\sqrt[3]{1}$ b) $\sqrt[4]{i}$.

Solution. a) According to (22), the cube roots of 1 are $1, \omega$, and ω^2, where

$$\begin{aligned} \omega \;&=\; e^{2\pi i/3} &&=\; \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} &&=\; -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ \omega^2 \;&=\; e^{-2\pi i/3} &&=\; \cos\frac{-2\pi}{3} + i \sin\frac{-2\pi}{3} &&=\; -\frac{1}{2} - i\frac{\sqrt{3}}{2}\;. \end{aligned}$$

The greek letter ω ("omega") is traditionally used for this cube root. Note that for the polar angle of ω^2 we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x\;.$$

Note that $\omega^2 = \bar{\omega}$. Another way to do this problem would be to draw the position of ω^2 and ω on the unit circle, and use geometry to figure out their coordinates.

b) To find $\sqrt[4]{i}$, we can use (24). We know that $\sqrt[4]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture, or by using (22)). Therefore by (24), we get

$$\begin{aligned} \sqrt[4]{i} \;&= z_0,\; z_0 i,\; -z_0,\; -z_0 i, && \text{where } \; z_0 = e^{\pi i/8} \;=\; \cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\;; \\ &= a + ib,\; -b + ia,\; -a - ib,\; b - ia, && \text{where } \; z_0 = a + ib \;=\; \cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\;. \end{aligned}$$

Example 6. Solve the equation $x^6 - 2x^3 + 2 = 0$.

Solution. Treating this as a quadratic equation in x^3, we solve the quadratic by using the quadratic formula, the two roots are $1+i$ and $1-i$ (check this!), so the roots of the original equation satisfy either

$$x^3 = 1+i, \qquad \text{or} \qquad x^3 = 1-i\ .$$

This reduces the problem to finding the cube roots of the two complex numbers $1 \pm i$. We begin by writing them in polar form:

$$1+i = \sqrt{2}\,e^{\pi i/4}, \qquad 1-i = \sqrt{2}\,e^{-\pi i/4}\ .$$

(Once again, note the use of the negative polar angle for $1-i$, which is more convenient for calculations.) The three cube roots of the first of these are (by (23)),

$$\begin{aligned}
\sqrt[6]{2}\;e^{\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\
\sqrt[6]{2}\;e^{3\pi i/4} &= \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \qquad \text{since } \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4}; \\
\sqrt[6]{2}\;e^{-7\pi i/12} &= \sqrt[6]{2}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right), \qquad \text{since } \frac{\pi}{12} - \frac{2\pi}{3} = -\frac{7\pi}{12}.
\end{aligned}$$

The second cube root can also be written as $\sqrt[6]{2}\left(\dfrac{-1+i}{\sqrt{2}}\right) = \dfrac{-1+i}{\sqrt[3]{2}}$.

This gives three of the cube roots. The other three are the cube roots of $1-i$, which may be found by replacing i by $-i$ everywhere above (i.e., taking the complex conjugate).

The cube roots can also according to (24) be described as

$$z_1,\ z_1\omega,\ z_1\omega^2 \quad \text{and} \quad z_2,\ z_2\omega,\ z_2\omega^2, \qquad \text{where } z_1 = \sqrt[6]{2}\;e^{\pi i/12},\ \ z_2 = \sqrt[6]{2}\;e^{-\pi i/12}.$$

Exercises: Section 2E

IR. Input-Response Models

1. First-order linear ODE's with positive constant coefficient. This is probably the most important first order equation; we use t and y as the variables, and think of the independent variable t as representing time. The IVP in standard form then is

$$y' + ky = q(t), \quad k > 0; \qquad y(0) = y_0 \; . \tag{1}$$

The integrating factor for the ODE in (1) is e^{kt}; using it, the general solution is

$$y = e^{-kt}\left(\int q(t)e^{kt}dt + c\right). \tag{2}$$

To get from this an explicit solution to (1), we change (cf. Notes D) the indefinite integral in (2) to a definite integral from 0 to t, which requires us to change the t in the integrand to a different dummy variable, u say; then the explicit solution to (1) is

$$y = e^{-kt}\int_0^t q(u)e^{ku}du + y_0e^{-kt}. \tag{3}$$

In this form, note that the first term on the right is the solution to the IVP (1) corresponding to the initial condition $y_0 = 0$.

What we have done so far does not depend on whether k is positive or negative. However, the terminology we will now introduce makes sense only when $k > 0$, which we shall assume from now on.

Looking at (3) and assuming $k > 0$, we observe that as $t \to \infty$, the second term of the solution $y_0e^{-kt} \to 0$, regardless of the initial value y_0. It is therefore called the **transient** since its effect on the solution dies away as time increases. As it dies away, what is left of the solution is the integral term on the right, which does not involve the initial value y_0; it is called the **steady-state** or **long-term** solution to (1).

$$y \;= \underbrace{e^{-kt}\int_0^t q(u)e^{ku}du}_{\textit{steady-state}} + \underbrace{y_0e^{-kt}}_{\textit{transient}}, \qquad k > 0. \tag{4}$$

Despite the use of the definite article, the steady-state solution is not unique: since all the solutions approach the steady-state solution as $t \to \infty$, they all approach each other, and thus any of them can be called the steady-state solution. In practice, it is usually the simplest-looking solution which is given this honor.

2. Input-response; superposition of inputs. When the ODE (1) is used to model a physical situation, the left-hand side usually is concerned with the physical set-up — the "system" — while the right-hand side represents something external which is driving or otherwise affecting the system from the outside. For this reason, the function $q(t)$ is often called the **input**, or in some engineering subjects, the **signal**; the corresponding general solution (2) is called the **response** of the system to this input.

We will indicate the relation of input to response symbolically by

$$q(t) \rightsquigarrow y(t) \qquad (\textit{input} \rightsquigarrow \textit{response}).$$

Superposition principle for inputs.

For the ODE $y' + ky = q(t)$, *let* $q_1(t)$ *and* $q_2(t)$ *be inputs, and* $c_1,\ c_2$ *constants. Then*

$$q_1 \rightsquigarrow y_1, \quad q_2 \rightsquigarrow y_2 \quad \Longrightarrow \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2. \tag{5}$$

Proof. This follows in one line from the sum property of indefinite integrals; note that the proof does not require k to be positive.

$$y = e^{-kt}\int (q_1+q_2)e^{kt}dt = e^{-kt}\int q_1e^{kt}dt + e^{-kt}\int q_2e^{kt}dt = y_1 + y_2\ .$$

The superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces. Here is an easy example.

Example 1. Find the response of $y' + 2y = q(t)$ to $q = 1 + e^{-2t}$.

Solution. The input $q = 1$ generates the response $y = 1/2$, by inspection; the input e^{-2t} generates the response te^{-2t}, by solving; therefore the response to $1+e^{-2t}$ is $1/2+te^{-2t}$. □

3. Physical inputs; system responses to linear inputs.

We continue to assume $k > 0$. We want to get some feeling for how the system responds to a variety of inputs. The temperature model for (1) will be a good guide: in two notations – suggestive and neutral, respectively – the ODE is

$$T' + kT = kT_e(t), \qquad\qquad y' + ky = kq_e(t) = q(t). \tag{6}$$

Note that the neutral notation writes the input in two different forms: the $q(t)$ we have been using, and also in the form $kq_e(t)$ with the k factored out. This last corresponds to the way the input appears in certain physical problems (temperature and diffusion problems, for instance) and leads to more natural formulas: for example, q_e and y have the same units, whereas q and y do not.

For this class of problems, the relation of response to input will be clearer if we relate y with q_e, rather than with q. We will use for q_e the generic name **physical input**, or if we have a specific model in mind, the *temperature* input, *concentration* input, and so on.

The expected behavior of the temperature model suggests general questions such as:

Is the response the same type of function as the physical input? What controls its magnitude?

Does the graph of the response lag behind that of the physical input?

What controls the size of the lag?

Our plan will be to get some feeling for the situation by answering these questions for several simple physical inputs. We begin with linear inputs. Throughout, keep the temperature model in mind to guide your intuition.

Example 2. Find the response of the system (6) to the physical inputs 1 and t.

Solution. The ODE is $y' + ky = kq_e$.

If $q_e = 1$, a solution by inspection is $y = 1$, so the response is 1.

If $q_e = t$, the ODE is $y' + ky = kt$; using the integrating factor e^{kt} and subsequent integration by parts leads (cf. (2)) to the simplest steady-state solution

$$\begin{aligned} y &= e^{-kt} \int kte^{kt} dt \\ &= k\, e^{-kt} \left(\frac{te^{kt}}{k} - \frac{e^{kt}}{k^2} \right) \\ &= t - \frac{1}{k}. \end{aligned}$$

Thus the response of (6) is identical to the physical input t, but with a time lag $1/k$. This is reasonable when one thinks of the temperature model: the internal temperature increases linearly at the same rate as the temperature of the external water bath, but with a time lag dependent on the conductivity: the higher the conductivity, the shorter the time lag.

Using the superposition principle for inputs, it follows from Example 2 that for the ODE $y' + ky = kq_e$, its response to a general linear physical input is given by:

(7) **linear input** physical input: $q_e = a + bt$ response: $a + b\left(t - \dfrac{1}{k}\right)$.

In the previous example, we paid no attention to initial values. If they are important, one cannot just give the steady-state solution as the response, one has to take account of them, either by using a definite integral as in (3), or by giving the value of the arbitrary constant in (2). Examples in the next section will illustrate.

4. Response to discontinuous inputs, $k > 0$.

The most basic discontinuous function is the unit-step function at a point, defined by

(8) $$u_a(t) = \begin{cases} 0, & t < a; \\ 1, & t > a. \end{cases}$$ **unit-step function at a**

(We leave its value at a undefined, though some books give it the value 0, others the value 1 there.)

Example 3. Find the response of the IVP $y' + ky = kq_e$, $y(0) = 0$, for $t \geq 0$, to the unit-step physical input $u_a(t)$, where $a \geq 0$.

Solution. For $t < a$ the input is 0, so the response is 0. For $t \geq a$, the steady-state solution for the physical input $u_a(t)$ is the constant function 1, according to Example 2 or (7).

We still need to fit the value $y(a) = 0$ to the response for $t \geq a$. Using (2) to do this, we get $1 + ce^{-ka} = 0$, so that $c = -e^{ka}$. We now assemble the results for $t < a$ and $t \geq a$ into one expression; for the latter we also put the exponent in a more suggestive form. We get finally

unit-step input

(9) physical input: $u_a(t),\ a \geq 0$ response: $$y(t) = \begin{cases} 0, & 0 \leq t < a; \\ 1 - e^{-k(t-a)}, & t \geq a. \end{cases}$$

Note that the response is just the translation a units to the right of the response to the unit-step input at 0.

Another way of getting the same answer would be to use the definite integral in (3); we leave this as an exercise.

As another example of discontinuous input, we focus on the temperature model, and obtain the response to the temperature input corresponding to the external bath initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by the ice-water bath.

Example 4. Find the response of $y' + ky = kq_e$ to the physical input

$$(10) \qquad u_{ab} = \begin{cases} 1, & a \le t \le b; \\ 0, & \text{otherwise,} \end{cases} \qquad 0 \le a < b; \qquad \textbf{unit-box function} \text{ on } [a, b].$$

Solution. There are at least three ways to do this:

a) Express u_{ab} as a sum of unit step functions and use (9) together with superposition of inputs;

b) Use the function u_{ab} directly in the definite integral expression (3) for the response;

c) Find the response in two steps: first use (9) to get the response $y(t)$ for the physical input $u_a(t)$; this will be valid up to the point $t = b$.

Then, to continue the response for values $t > b$, evaluate $y(b)$ and find the response for $t > b$ to the input 0, with initial condition $y(b)$.

We will follow (c), leaving the first two as exercises.

By (9), the response to the physical input $u_a(t)$ is $y(t) = \begin{cases} 0, & 0 \le t < a; \\ 1 - e^{-k(t-a)}, & t \ge a. \end{cases}$; this is valid up to $t = b$, since $u_{ab}(t) = u_a(t)$ for $t \le b$. Evaluating at b,

$$(11) \qquad y(b) = 1 - e^{-k(b-a)}.$$

Using (2) to find the solution for $t \ge b$, we note first that the steady-state solution will be 0, since $u_{ab} = 0$ for $t > b$; thus by (2) the solution for $t > b$ will have the form

$$(12) \qquad y(t) = 0 + ce^{-kt}$$

where c is determined from the initial value (11). Equating the initial values $y(b)$ from (11) and (12), we get

$$ce^{-kb} = 1 - e^{-kb+ka}$$

from which

$$c = e^{kb} - e^{ka};$$

so by (12),

$$(13) \qquad y(t) = (e^{kb} - e^{ka})e^{-kt}, \quad t \ge b.$$

After combining exponents in (13) to give an alternative form for the response, we assemble the parts, getting the *response to the physical unit-box input u_{ab} :*

$$(14) \qquad y(t) = \begin{cases} 0, & 0 \le t \le a; \\ 1 - e^{-k(t-a)}, & a < t < b; \\ e^{-k(t-b)} - e^{-k(t-a)}, & t \ge b. \end{cases}$$

5. Response to sinusoidal inputs.

Of great importance in the applications is the sinusoidal input, i.e., a pure oscillation like $\cos\omega t$ or $\sin\omega t$, or more generally, $A\cos(\omega t - \phi)$. (The last form includes both of the previous two, as you can see by letting $A = 1$ and $\phi = 0$ or $\pi/2$).

In the temperature model, this could represent the diurnal varying of outside temperature; in the conentration model, the diurnal varying of the level of some hormone in the bloodstream, or the varying concentration in a sewer line of some waste product produced periodically by a manufacturing process.

What follows assumes some familiarity with the vocabulary of pure oscillations: *amplitude, frequency, period, phase lag.* Section 6 following this gives a brief review of these terms plus a few other things that we will need: look at it first, or refer to it as needed when reading this section.

Response of $y' + ky = kq_e$ to the physical inputs $\cos\omega t,\ \sin\omega t$.

This calculation is a good example of how the use of complex exponentials can simplify integrations and lead to a more compact and above all more expressive answer. You should study it very carefully, since the ideas in it will frequently recur.

We begin by complexifying the inputs, the response, and the differential equation:

$$\cos\omega t \;=\; \mathrm{Re}\,(e^{i\omega t}), \qquad \sin\omega t \;=\; \mathrm{Im}\,(e^{i\omega t}); \tag{15}$$

$$\tilde{y}(t) \;=\; y_1(t) + iy_2(t); \tag{16}$$

$$\tilde{y}' + k\tilde{y} \;=\; ke^{i\omega t}. \tag{17}$$

If (16) is a solution to the complex ODE (17), then substituting it into the ODE and using the rule $(u + iv)' = u' + iv'$ for differentiating complex functions (see Notes C, (19)), gives

$$y_1' + iy_2' + k(y_1 + iy_2) \;=\; k(\cos\omega t + i\sin\omega t);$$

equating real and imaginary parts on the two sides gives the two real ODE's

$$y_1' + ky_1 = k\cos\omega t, \qquad\qquad y_2' + ky_2 = k\sin\omega t; \tag{18}$$

this shows that the real and imaginary parts of our complex solution $\tilde{y}(t)$ give us respectively the responses to the physical inputs $\cos\omega t$ and $\sin\omega t$.

It seems wise at this point to illustrate the calculations with an example, before repeating them as a derivation. If you prefer, you can skip the example and proceed directly to the derivation, using the example as a solved exercise to test yourself afterwards.

Example 5. Find the response of $y' + y = 0$ to the input $\cos t$; in other words, find a solution to the equation $y' + y = \cos t$; use complex numbers.

Solution. We follow the above plan and complexify the real ODE, getting

$$\tilde{y}' + \tilde{y} = e^{it}\ .$$

We made the right side e^{it}, since $\cos t = \mathrm{Re}\,(e^{it})$. We will find a complex solution $\tilde{y}(t)$ for the complexified ODE; then $\mathrm{Re}\,(\tilde{y})$ will be a real solution to $y' + y = \cos t$, according to (18) and what precedes it.

The complexified ODE is linear, with the integrating factor e^t; multiplying both sides by this factor, and then following the steps for solving the first order linear ODE, we get

$$\tilde{y}' + \tilde{y} = e^{it} \quad \Rightarrow \quad (\tilde{y}\, e^t)' = e^{(1+i)t} \quad \Rightarrow \quad \tilde{y}\, e^t = \frac{1}{1+i}\, e^{(1+i)t} \quad \Rightarrow \quad \tilde{y} = \frac{1}{1+i}\, e^{it}\ .$$

This gives us our complex solution $\tilde{y}$; the rest of the work is calculating $\mathrm{Re}\,(\tilde{y})$. To do this, we can use either the polar or the Cartesian representations.

Using the polar first, conver $1+i$ to polar form and then use the reciprocal rule (Notes C, (12b)):

$$1+i = \sqrt{2}\, e^{i\pi/4} \quad \Rightarrow \quad \frac{1}{1+i} = \frac{1}{\sqrt{2}}\, e^{-i\pi/4};$$

from which it follows from the multiplication rule (12a) that

$$\tilde{y} = \frac{1}{1+i}\ e^{it} = \frac{1}{\sqrt{2}}\ e^{i(t-\pi/4)}$$

and therefore our solution to $y' + y = \cos t$ is

$$\mathrm{Re}\,(\tilde{y}) = \frac{1}{\sqrt{2}} \cos(t - \pi/4)\ ,$$

the pure oscillation with amplitude $1/\sqrt{2}$, circular frequency 1, and phase lag $\pi/4$.

Repeating the calculation, but using the Cartesian representation, we have

$$\tilde{y} = \frac{1}{1+i}\ e^{it} = \left(\frac{1-i}{2}\right)(\cos t + i \sin t)$$

whose real part is

$$\mathrm{Re}\,(\tilde{y}) = \frac{1}{2}(\cos t + \sin t) = \frac{\sqrt{2}}{2}\cos(t - \pi/4),$$

the last equality following from the sinusoidal identity ((25), section 6).

Instead of converting the Cartesian form to the polar, we could have converted the polar form to the Cartesian form by using the trigonometric addition formula. Since $\cos\pi/4 = \sin\pi/4 = \sqrt{2}/2$, it gives

$$\frac{1}{\sqrt{2}}\cos(t - \pi/4) = \frac{1}{\sqrt{2}}(\cos t \cos\pi/4 + \sin t \sin\pi/4) = \frac{1}{2}(\cos t + \sin t)$$

However, in applications the polar form of the answer is generally preferred, since it gives directly important characteristics of the solution — its amplitude and phase lag, whereas these are not immediately apparent in the Cartesian form of the answer.

Resuming our general solving of the ODE (17) using the circular frequency ω and the constant k, but following the same method as in the example, the integrating factor is e^{kt}; multiplying through by it leads to

$$(\tilde{y}e^{kt})' \;=\; ke^{(k+i\omega)t};$$

integrate both sides, multiply through by e^{-kt}, and scale the coefficient to lump constants and make it look better:

$$\tilde{y} \;=\; \frac{k}{k+i\omega}e^{i\omega t} \;=\; \frac{1}{1+i(\omega/k)}\,e^{i\omega t}. \tag{19}$$

This is our complex solution.

Writing $1+i(\omega/k)$ in polar form, then using the reciprocal rule (Notes C, (12b)), we have

$$1+i(\omega/k) = \sqrt{1+(\omega/k)^2}\;e^{i\phi} \quad\Rightarrow\quad \frac{1}{1+i(\omega/k)} \;=\; \frac{1}{\sqrt{1+(\omega/k)^2}}\;e^{-i\phi}, \tag{20}$$
$$\text{where}\quad \phi = \tan^{-1}(\omega/k).$$

Therefore in polar form,

$$\tilde{y} = \frac{1}{\sqrt{1+(\omega/k)^2}}\;e^{i(\omega t-\phi)}. \tag{21}$$

The real and imaginary parts of this complex response give the responses of the system to respectively the real and imaginary parts of the complex input $e^{i\omega t}$; thus we can summarize our work as follows:

First-order Sinusoidal Input Theorem. *For the equation* $y'+ky=kq_e$ *we have*

$$\textit{physical input:}\;\begin{cases} q_e = \cos\omega t \\ q_e = \sin\omega t\end{cases} \qquad \textit{response:}\;\begin{cases} y_1 = A\cos(\omega t-\phi) \\ y_2 = A\sin(\omega t-\phi)\end{cases}, \tag{22}$$

$$A = \frac{1}{\sqrt{1+(\omega/k)^2}}, \qquad\qquad \phi = \tan^{-1}(\omega/k). \tag{23}$$

The calculation can also be done in Cartesian form as in Example 5, then converted to polar form using the sinusoidal identity. We leave this as an exercise.

The Sinusoidal Input Theorem is more general than it looks; it actually covers any sinusoidal input $f(t)$ having ω as its circular frequency. This is because any such input can be written as a linear combination of $\cos\omega t$ and $\sin\omega t$:

$$f(t) = a\,\cos\omega t + b\,\sin\omega t, \qquad a,\ b \text{ constants}$$

and then it is an easy exercise to show:

$$\textit{physical input:}\;\; q_e = \cos\omega t + b\sin\omega t \qquad \textit{response:}\;\; a\,y_1 + b\,y_2\;. \tag{22'}$$

6. Sinusoidal oscillations: reference.

The work in section 5 uses terms describing a pure, or **sinusoidal** oscillation: analytically, one that can be written in the form

$$A\cos(\omega t-\phi); \tag{24}$$

or geometrically, one whose graph can be obtained from the graph of $\cos t$ by stretching or shrinking the t and y axes by scale factors, then translating the resulting graph along the t-axis.

Terminology. Referring to function (24) whose graph describes the oscillation,

$|A|$ is its **amplitude**: how high its graph rises over the t-axis at its maximum points;

ϕ is its **phase lag**: the smallest non-negative value of ωt for which the graph is at its maximum
(if $\phi = 0$, the graph has the position of $\cos\omega t$; if $\phi = \pi/2$, it has the position of $\sin\omega t$);

ϕ/ω is its **time delay** or **time lag**: how far to the right on the t-axis the graph of $\cos\omega t$ has been moved to make the graph of (24);
(to see this, write $A\cos(\omega t-\phi) = A\cos\omega(t-\phi/\omega)$);

ω is its **circular** or **angular frequency**: the number of complete oscillations it makes in a t-interval of length 2π;

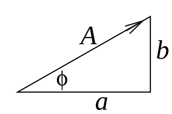
$\omega/2\pi$ (usually written ν) is its **frequency**: the number of complete oscillations the graph makes in a time interval of length 1;

$2\pi/\omega$ or $1/\nu$ is its **period**, the t-interval required for one complete oscillation.

The Sinusoidal Identity. For any real constants a and b,

$$a\cos\omega t + b\sin\omega t \;=\; A\cos(\omega t-\phi), \tag{25}$$

where A, ϕ, a, and b are related as shown in the accompanying right triangle:



$$A=\sqrt{a^2+b^2},\qquad \phi=\tan^{-1}\frac{b}{a},\qquad a=A\cos\phi,\qquad b=A\sin\phi \tag{26}$$

There are at least three ways to verify the sinusoidal identity:

1. Apply the trigonometric addition formula for $\cos(\theta_1-\theta_2)$ to the right side.
(Our good proof, uses only high-school math, but not very satisfying, since it doesn't show where the right side came from.)

2. Observe that the left side is the real part of $(a-bi)(\cos\omega t + i\sin\omega t)$; calculate this product using polar form instead, and take its real part.
(Our better proof, since it starts with the left side, and gives practice with complex numbers to boot.)

3. The left side is the dot product $\langle a,\ b\rangle\cdot\langle\cos\omega t,\ \sin\omega t\rangle$; evaluate the dot product by the geometric formula for it (first day of 18.02).
(Our best proof: starts with the left side, elegant and easy to remember.)

O. Linear Differential Operators

1. Linear differential equations. The general linear ODE of order n is

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y \;=\; q(x). \tag{1}$$

If $q(x) \neq 0$, the equation is **inhomogeneous**. We then call

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y \;=\; 0. \tag{2}$$

the **associated homogeneous** equation or the *reduced* equation.

The theory of the n-th order linear ODE runs parallel to that of the second order equation. In particular, the general solution to the associated homogeneous equation (2) is called the **complementary** function or solution, and it has the form

$$y_c \;=\; c_1y_1 + \ldots + c_ny_n\,, \qquad c_i \;\text{ constants,} \tag{3}$$

where the y_i are n solutions to (2) which are *linearly independent*, meaning that none of them can be expressed as a linear combination of the others, i.e., by a relation of the form (the left side could also be any of the other y_i):

$$y_n \;=\; a_1y_1 + \ldots + a_{n-1}y_{n-1}\,, \qquad a_i \;\text{ constants.}$$

Once the associated homogeneous equation (2) has been solved by finding n independent solutions, the solution to the original ODE (1) can be expressed as

$$y \;=\; y_p + y_c, \tag{4}$$

where y_p is a particular solution to (1), and y_c is as in (3).

2. Linear differential operators with constant coefficients

From now on we will consider only the case where (1) has constant coefficients. This type of ODE can be written as

$$y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny \;=\; q(x)\;; \tag{5}$$

using the differentiation operator D, we can write (5) in the form

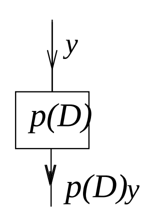
$$(D^n + a_1D^{n-1} + \ldots + a_n)\,y \;=\; q(x) \tag{6}$$

or more simply,

$$p(D)\,y \;=\; q(x)\;,$$

where

$$p(D) = D^n + a_1D^{n-1} + \ldots + a_n\;. \tag{7}$$



We call $p(D)$ a **polynomial differential operator with constant coefficients**. We think of the formal polynomial $p(D)$ as operating on a function $y(x)$, converting it into another function; it is like a black box, in which the function $y(x)$ goes in, and $p(D)y$ (i.e., the left side of (5)) comes out.

Our main goal in this section of the Notes is to develop methods for finding particular solutions to the ODE (5) when $q(x)$ has a special form: an exponential, sine or cosine, x^k, or a product of these. (The function $q(x)$ can also be a sum of such special functions.) These are the most important functions for the standard applications.

The reason for introducing the polynomial operator $p(D)$ is that this allows us to use polynomial algebra to help find the particular solutions. The rest of this chapter of the Notes will illustrate this. Throughout, we let

$$p(D) = D^n + a_1 D^{n-1} + \ldots + a_n \;, \qquad a_i \text{ constants.} \tag{7}$$

3. Operator rules.

Our work with these differential operators will be based on several rules they satisfy. In stating these rules, we will always assume that the functions involved are sufficiently differentiable, so that the operators can be applied to them.

Sum rule. If $p(D)$ and $q(D)$ are polynomial operators, then for any (sufficiently differentiable) function u,

$$[p(D) + q(D)]u = p(D)u + q(D)u \;. \tag{8}$$

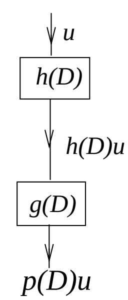
Linearity rule. If u_1 and u_2 are functions, and c_i constants,

$$p(D)\,(c_1 u_1 + c_2 u_2) \;=\; c_1 p(D)\,u_1 + c_2 p(D)\,u_2 \;. \tag{9}$$

The linearity rule is a familiar property of the operator $a\,D^k$; it extends to sums of these operators, using the sum rule above, thus it is true for operators which are polynomials in D. (It is still true if the coefficients a_i in (7) are not constant, but functions of x.)

Multiplication rule. If $p(D) = g(D)\,h(D)$, as polynomials in D, then

$$p(D)\,u \;=\; g(D)\big(h(D)\,u\big) \;. \tag{10}$$



The picture illustrates the meaning of the right side of (10). The property is true when $h(D)$ is the simple operator $a\,D^k$, essentially because

$$D^m(a\,D^k u) \;=\; a\,D^{m+k} u;$$

it extends to general polynomial operators $h(D)$ by linearity. Note that a must be a constant; it's false otherwise.

An important corollary of the multiplication property is that *polynomial operators with constant coefficients commute*; i.e., for every function $u(x)$,

$$g(D)\big(h(D)\,u\big) \;=\; h(D)\big(g(D)\,u\big) \;. \tag{11}$$

For as polynomials, $g(D)h(D) \;=\; h(D)g(D) \;=\; p(D)$, say; therefore by the multiplication rule, both sides of (11) are equal to $p(D)\,u$, and therefore equal to each other.

The remaining two rules are of a different type, and more concrete: they tell us how polynomial operators behave when applied to exponential functions and products involving exponential functions.

Substitution rule.

$$p(D)e^{ax} = p(a)e^{ax} \tag{12}$$

Proof. We have, by repeated differentiation,

$$De^{ax} = ae^{ax}, \quad D^2e^{ax} = a^2e^{ax}, \quad \ldots, \quad D^ke^{ax} = a^ke^{ax};$$

therefore,

$$(D^n + c_1D^{n-1} + \ldots + c_n)\, e^{ax} = (a^n + c_1a^{n-1} + \ldots + c_n)\, e^{ax},$$

which is the substitution rule (12). □

The exponential-shift rule This handles expressions such as x^ke^{ax} and $x^k \sin ax$.

$$p(D)\, e^{ax}u = e^{ax}p(D + a)\, u \ . \tag{13}$$

Proof. We prove it in successive stages. First, it is true when $p(D) = D$, since by the product rule for differentiation,

$$De^{ax}u(x) = e^{ax}Du(x) + ae^{ax}u(x) = e^{ax}(D + a)u(x) \ . \tag{14}$$

To show the rule is true for D^k, we apply (14) to D repeatedly:

$$\begin{aligned} D^2e^{ax}u \ &= \ D(De^{ax}u) \ = \ D(e^{ax}(D + a)u) && \text{by (14);} \\ &= e^{ax}(D + a)\big((D + a)u\big), && \text{by (14);} \\ &= e^{ax}(D + a)^2u \ , && \text{by (10).} \end{aligned}$$

In the same way,

$$\begin{aligned} D^3e^{ax}u \ &= \ D(D^2e^{ax}u) \ = \ D(e^{ax}(D + a)^2u) && \text{by the above;} \\ &= e^{ax}(D + a)\big((D + a)^2u\big), && \text{by (14);} \\ &= e^{ax}(D + a)^3u \ , && \text{by (10),} \end{aligned}$$

and so on. This shows that (13) is true for an operator of the form D^k. To show it is true for a general operator

$$p(D) = D^n + a_1D^{n-1} + \ldots + a_n \ ,$$

we write (13) for each $D^k(e^{ax}u)$, multiply both sides by the coefficient a_k, and add up the resulting equations for the different values of k. □

Remark on complex numbers. By Notes C. (20), the formula

$$D\,(c\,e^{ax}) \ = \ c\,a\,e^{ax} \tag{*}$$

remains true even when c and a are complex numbers; therefore the rules and arguments above remain valid even when the exponents and coefficients are complex. We illustrate.

Example 1. Find $D^3e^{-x}\sin x$.

Solution using the exponential-shift rule. Using (13) and the binomial theorem,

$$\begin{aligned} D^3e^{-x}\sin x \ &= \ e^{-x}(D - 1)^3 \sin x \ = \ e^{-x}(D^3 - 3D^2 + 3D - 1)\sin x \\ &= \ e^{-x}(2\cos x + 2\sin x), \end{aligned}$$

since $D^2 \sin x = -\sin x$, and $D^3 \sin x = -\cos x$.

Solution using the substitution rule. Write $e^{-x}\sin x \ = \ \operatorname{Im} e^{(-1+i)x}$. We have

$$\begin{aligned} D^3e^{(-1+i)x} \ &= \ (-1 + i)^3e^{(-1+i)x}, && \text{by (12) and (*);} \\ &= (2 + 2i)\, e^{-x}(\cos x + i \sin x), \end{aligned}$$

by the binomial theorem and Euler's formula. To get the answer we take the imaginary part: $e^{-x}(2\cos x + 2\sin x)$.

4. Finding particular solutions to inhomogeneous equations.

We begin by using the previous operator rules to find particular solutions to inhomogeneous polynomial ODE's with constant coefficients, where the right hand side is a real or complex exponential; this includes also the case where it is a sine or cosine function.

Exponential-input Theorem. *Let $p(D)$ be a polynomial operator with onstant coefficients, and $p^{(s)}$ its s-th derivative. Then*

$$p(D)y = e^{ax}, \qquad \text{where } a \text{ is real or complex} \tag{15}$$

has the particular solution

$$y_p = \frac{e^{ax}}{p(a)}, \qquad \text{if } p(a) \neq 0; \tag{16}$$

$$y_p = \frac{x^s e^{ax}}{p^{(s)}(a)}, \qquad \text{if } a \text{ is an } s\text{-fold zero}^1 \text{ of } p. \tag{17}$$

Note that (16) is just the special case of (17) when $s = 0$. Before proving the theorem, we give two examples; the first illustrates again the usefulness of complex exponentials.

Example 2. Find a particular solution to $(D^2 - D + 1)\ y = e^{2x}\cos x$.

Solution. We write $e^{2x}\cos x = \mathrm{Re}\,(e^{(2+i)x})$, so the corresponding complex equation is

$$(D^2 - D + 1)\ \tilde{y} = e^{(2+i)x},$$

and our desired y_p will then be $Re(\tilde{y}_p)$. Using (16), we calculate

$$\begin{aligned}
p(2+i) &= (2+i)^2 - (2+i) + 1 = 2 + 3i\ , && \text{from which} \\
\tilde{y}_p &= \frac{1}{2+3i}\ e^{(2+i)x}, && \text{by (16);} \\
&= \frac{2-3i}{13}\ e^{2x}(\cos x + i\sin x)\ ; && \text{thus} \\
Re(\tilde{y}_p) &= \frac{2}{13}\ e^{2x}\cos x + \frac{3}{13}\ e^{2x}\sin x\ , && \text{our desired particular solution.}
\end{aligned}$$

Example 3. Find a particular solution to $y'' + 4y' + 4y \ = \ e^{-2t}$.

Solution. Here $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has -2 as a double root; using (17), we have $p''(-2) = 2$, so that

$$y_p = \frac{t^2 e^{-2t}}{2}.$$

Proof of the Exponential-input Theorem.

That (16) is a particular solution to (15) follows immediately by using the linearity rule (9) and the substitution rule (12):

$$p(D)y_p \ = \ p(D)\frac{e^{ax}}{p(a)} \ = \ \frac{1}{p(a)}p(D)e^{ax} \ = \ \frac{p(a)e^{ax}}{p(a)} \ = \ e^{ax}.$$

[1] John Lewis communicated this useful formula.

For the more general case (17), we begin by noting that to say the polynomial $p(D)$ has the number a as an s-fold zero is the same as saying $p(D)$ has a factorization

$$p(D) = q(D)(D-a)^s, \qquad q(a) \neq 0. \tag{18}$$

We will first prove that (18) implies

$$p^{(s)}(a) = q(a)\, s!\ . \tag{19}$$

To prove this, let k be the degree of $q(D)$, and write it in powers of $(D-a)$:

$$q(D) = q(a) + c_1(D-a) + \ldots + c_k(D-a)^k; \quad \text{then} \tag{20}$$

$$p(D) = q(a)(D-a)^s + c_1(D-a)^{s+1} + \ldots + c_k(D-a)^{s+k};$$

$$p^{(s)}(D) = q(a)\, s! + \text{positive powers of } D-a;$$

substituting a for D on both sides proves (19). □

Using (19), we can now prove (17) easily using the exponential-shift rule (13). We have

$$\begin{aligned} p(D)\frac{e^{ax}x^s}{p^{(s)}(a)} &= \frac{e^{ax}}{p^{(s)}(a)}\, p(D+a)x^s, &&\text{by linearity and (13);}\\ &= \frac{e^{ax}}{p^{(s)}(a)}\, q(D+a)\, D^s x^s, &&\text{by (18);}\\ &= \frac{e^{ax}}{q(a)s!}\, q(D+a)\, s!, &&\text{by (19);}\\ &= \frac{e^{ax}}{q(a)s!}\, q(a)\, s! \;=\; e^{ax}, \end{aligned}$$

where the last line follows from (20), since $s!$ is a constant:

$$q(D+a)s! = (q(a) + c_1 D + \ldots + c_k D^k)\, s! = q(a)s!\ .$$

Polynomial Input: The Method of Undetermined Coefficients.

Let $r(x)$ be a polynomial of degree k; we assume the ODE $p(D)y = q(x)$ has as input

$$q(x) = r(x), \quad p(0) \neq 0; \qquad \text{or more generally,} \quad q(x) = e^{ax}\, r(x), \quad p(a) \neq 0. \tag{21}$$

(Here a can be complex; when $a = 0$ in (21), we get the pure polynomial case on the left.)

The method is to assume a particular solution of the form $y_p = e^{ax}h(x)$, where $h(x)$ is a polynomial of degree k with unknown ("undetermined") coefficients, and then to find the coefficients by substituting y_p into the ODE. It's important to do the work systematically; follow the format given in the following example, and in the solutions to the exercises.

Example 5. Find a particular solution y_p to $y'' + 3y' + 4y = 4x^2 - 2x$.

Solution. Our trial solution is $y_p = Ax^2 + Bx + C$; we format the work as follows. The lines show the successive derivatives; multiply each line by the factor given in the ODE, and add the equations, collecting like powers of x as you go. The fourth line shows the result; the sum on the left takes into account that y_p is supposed to be a particular solution to the given ODE.

$$\begin{aligned} \times 4 \quad y_p &= Ax^2 + Bx + C\\ \times 3 \quad y_p' &= \qquad 2Ax + B\\ y_p'' &= \qquad\qquad 2A\\ 4x^2 - 2x &= (4A)x^2 + (4B + 6A)x + (4C + 3B + 2A). \end{aligned}$$

Equating like powers of x in the last line gives the three equations

$$4A = 4, \qquad 4B + 6A = -2, \qquad 4C + 3B + 2A = 0;$$

solving them in order gives $A = 1$, $B = -2$, $C = 1$, so that $y_p = x^2 - 2x + 1$.

Example 6. Find a particular solution y_p to $y'' + y' - 4y = e^{-x}(1 - 8x^2)$.

Solution. Here the trial solution is $y_p = e^{-x}u_p$, where $u_p = Ax^2 + Bx + C$.

The polynomial operator in the ODE is $p(D) = D^2 + D - 4$; note that $p(-1) \neq 0$, so our choice of trial solution is justified. Substituting y_p into the ODE and using the exponential-shift rule enables us to get rid of the e^{-x} factor:

$$p(D)y_p = p(D)e^{-x}u_p = e^{-x}p(D-1)u_p = e^{-x}(1 - 8x^2),$$

so that after canceling the e^{-x} on both sides, we get the ODE satisfied by u_p:

$$(22) \qquad p(D-1)u_p = 1 - 8x^2; \qquad \text{or} \qquad (D^2 - D - 4)u_p = 1 - 8x^2,$$

since $p(D-1) = (D-1)^2 + (D-1) - 4 = D^2 - D - 4$.

From this point on, finding u_p as a polynomial solution to the ODE on the right of (22) is done just as in Example 5 using the method of undetermined coefficients; the answer is

$$u_p = 2x^2 - x + 1, \qquad \text{so that} \qquad y_p = e^{-x}(2x^2 - x + 1).$$

In the previous examples, $p(a) \neq 0$; if $p(a) = 0$, then the trial solution must be altered by multiplying each term in it by a suitable power of x. The book gives the details; briefly, the terms in the trial solution should all be multiplied by the smallest power x^r for which none of the resulting products occur in the complementary solution y_c, i.e., are solutions of the associated homogeneous ODE. Your book gives examples; we won't take this up here.

5. Higher order homogeneous linear ODE's with constant coefficients.

As before, we write the equation in operator form:

$$(23) \qquad (D^n + a_1 D^{n-1} + \ldots + a_n)\, y \;=\; 0,$$

and define its **characteristic equation** or *auxiliary equation* to be

$$(24) \qquad p(r) = r^n + a_1 r^{n-1} + \ldots + a_n = 0.$$

We investigate to see if e^{rx} is a solution to (23), for some real or complex r. According to the substitution rule (12),

$$p(D)\, e^{rx} = 0 \quad \Leftrightarrow \quad p(r)\, e^{rx} = 0 \quad \Leftrightarrow \quad p(r) = 0\ .$$

Therefore

(25) e^{rx} *is a solution to (7)* $\Leftrightarrow$ r *is a root of its characteristic equation (16).*

Thus, to the real root r_i of (16) corresponds the solution $e^{r_i x}$.

Since the coefficients of $p(r) = 0$ are real, its complex roots occur in pairs which are conjugate complex numbers. Just as for the second-order equation, to the pair of complex conjugate roots $a \pm ib$ correspond the complex solution (we use the root $a + ib$)

$$e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx),$$

whose real and imaginary parts

$$e^{ax} \cos bx \qquad \text{and} \qquad e^{ax} \sin bx \tag{26}$$

are solutions to the ODE (23).

If there are n distinct roots to the characteristic equation $p(r) = 0$, (there cannot be more since it is an equation of degree n), we get according to the above analysis n real solutions $y_1, y_2, \ldots, y_n$ to the ODE (23), and they can be shown to be linearly independent. Thus the the complete solution y_h to the ODE can be written down immediately, in the form:

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \ .$$

Suppose now a real root r_1 of the characteristic equation (24) is a k-fold root, i.e., the characteristic polynomial $p(r)$ can be factored as

$$p(r) = (r - r_1)^k g(r), \qquad \text{where } \ g(r_1) \neq 0 \ . \tag{27}$$

We shall prove in the theorem below that corresponding to this k-fold root there are k linearly independent solutions to the ODE (23), namely:

$$e^{r_1 x}, \ \ x e^{r_1 x}, \ \ x^2 e^{r_1 x}, \ \ldots, \ x^{k-1} e^{r_1 x} \ . \tag{28}$$

(Note the analogy with the second order case that you have studied already.)

Theorem. *If a is a k-fold root of the characteristic equation $p(r) = 0$, then the k functions in (28) are solutions to the differential equation $p(D)\, y = 0$.*

Proof. According to our hypothesis about the characteristic equation, $p(r)$ has $(r-a)^k$ as a factor; denoting by $g(x)$ the other factor, we can write

$$p(r) = g(r)(r - a)^k \ ,$$

which implies that

$$p(D) = g(D)(D - a)^k \ . \tag{29}$$

Therefore, for $i = 0, 1, \ldots, k - 1$, we have

$$\begin{aligned} p(D) x^i e^{ax} &= g(D)(D - a)^k x^i e^{ax} \\ &= g(D)\big((D - a)^k x^i e^{ax}\big), && \text{by the multiplication rule,} \\ &= g(D)\big(e^{ax} D^k x^i\big) \ , && \text{by the exponential-shift rule,} \\ &= g(D)\big(e^{ax} \cdot 0\big), && \text{since } \ D^k x^i = 0 \text{ if } k > i; \\ &= 0, \end{aligned}$$

which shows that all the functions of (20) solve the equation. □

If r_1 is real, the solutions (28) give k linearly independent real solutions to the ODE (23).

In the same way, if $a+ib$ and $a-ib$ are k-fold conjugate complex roots of the characteristic equation, then (28) gives k complex solutions, the real and imaginary parts of which give $2k$ linearly independent solutions to (23):

$$e^{ax}\cos bx,\ e^{ax}\sin bx,\quad xe^{ax}\cos bx,\ xe^{ax}\sin bx,\quad \ldots,\quad x^{k-1}e^{ax}\cos bx,\ x^{k-1}e^{ax}\sin bx\ .$$

Example 6. Write the general solution to $(D+1)(D-2)^2(D^2+2D+2)^2y=0$.

Solution. The characteristic equation is

$$p(r)=(r+1)(r-2)^2(r^2+2r+2)^2\ =\ 0\ .$$

By the quadratic formula, the roots of $r^2+2r+2=0$ are $r=-1\pm i$, so we get

$$y=c_1e^{-x}+c_2e^{2x}+c_3xe^{2x}+e^{-x}(c_4\cos x+c_5\sin x+c_6x\cos x+c_7x\sin x)$$

as the general solution to the differential equation. □

As you can see, if the linear homogeneous ODE has constant coefficients, then the work of solving $p(D)y=0$ is reduced to finding the roots of the characteristic equation. This is "just" a problem in algebra, but a far from trivial one. There are formulas for the roots if the degree $n\le 4$, but of them only the quadratic formula ($n=2$) is practical. Beyond that are various methods for special equations and general techniques for approximating the roots. Calculation of roots is mostly done by computer algebra programs nowadays.

This being said, you should still be able to do the sort of root-finding described in Notes C, as illustrated by the next example.

Example 7. Solve: a) $y^{(4)}+8y''+16y=0$ b) $y^{(4)}-8y''+16y=0$

Solution. The factorizations of the respective characteristic equations are

$$(r^2+4)^2=0 \qquad\text{and}\qquad (r^2-4)^2=(r-2)^2(r+2)^2=0\ .$$

Thus the first equation has the double complex root $2i$, whereas the second has the double real roots 2 and -2. This leads to the respective general solutions

$$y=(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x \qquad\text{and}\qquad y=(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-2x}.$$

6. Justification of the method of undetermined coefficients.

As a last example of the use of these operator methods, we use operators to show where the method of undetermined coefficients comes from. This is the method which assumes the trial particular solution will be a linear combination of certain functions, and finds what the correct coefficients are. It only works when the inhomogeneous term in the ODE (23) (i.e., the term on the right-hand side) is a sum of terms having a special form: each must be the product of an exponential, sin or cos, and a power of x (some of these factors can be missing).

Question: What's so special about these functions?

Answer: They are the sort of functions which appear as solutions to some linear homogeneous ODE with constant coefficients.

With this general principle in mind, it will be easiest to understand why the method of undetermined coefficients works by looking at a typical example.

Example 8. Show that

$$(D-1)(D-2)y \;=\; \sin 2x. \tag{30}$$

has a particular solution of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x \;.$$

Solution. Since $\sin 2x$, the right-hand side of (30), is a solution to $(D^2+4)y=0$, i.e.,

$$(D^2+4)\,\sin 2x \;=\; 0.$$

we operate on both sides of (30) by the operator D^2+4 ; using the multiplication rule for operators with constant coefficients, we get (using y_p instead of y)

$$(D^2+4)(D-1)(D-2)y_p \;=\; 0. \tag{31}$$

This means that y_p is one of the solutions to the *homogeneous* equation (31). But we know its general solution: y_p must be a function of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 e^{2x} \;. \tag{32}$$

Now, in (32) we can drop the last two terms, since they contribute nothing: they are part of the complementary solution to (30), i.e., the solution to the associated homogeneous equation. Therefore they need not be included in the particular solution. Put another way, when the operator $(D-1)(D-2)$ is applied to (32), the last two terms give zero, and therefore don't help in finding a particular solution to (30).

Our conclusion therefore is that there is a particular solution to (30) of the form

$$y_p \;=\; c_1 \cos 2x + c_2 \sin 2x \;.$$

Here is another example, where one of the inhomogeneous terms is a solution to the associated homogeneous equation, i.e., is part of the complementary function.

Example 9. Find the form of a particular solution to

$$(D-1)^2 y_p \;=\; e^x. \tag{33}$$

Solution. Since the right-hand side is a solution to $(D-1)y=0$, we just apply the operator $D-1$ to both sides of (33), getting

$$(D-1)^3 y_p \;=\; 0.$$

Thus y_p must be of the form

$$y_p \;=\; e^x(c_1 + c_2 x + c_3 x^2).$$

But the first two terms can be dropped, since they are already part of the complementary solution to (33); we conclude there must be a particular solution of the form

$$y_p \;=\; c_3 x^2 e^x \;.$$

Exercises: Section 2F

S. Stability

1. The notion of stability.

A system is called *stable* if its long-term behavior does not depend significantly on the initial conditions.

It is an important result of mechanics that any system of masses connected by springs (damped or undamped) is a stable system. In network theory, there is a similar result: any RLC-network gives a stable system. In these notes, we investigate for the simplest such systems why this is so.

In terms of differential equations, the simplest spring-mass system or RLC-circuit is represented by an ODE of the form

$$a_0y'' + a_1y' + a_2y = r(t), \qquad a_i \;\text{ constants}, \quad t = \text{time}. \tag{1}$$

For the spring-mass system, y is the displacement from equilibrium position, and $r(t)$ is the externally applied force.

For the RLC-circuit, y represents the charge on the capacitor, and $r(t)$ is the electromotive force $\mathcal{E}(t)$ applied to the circuit (or else y is the current and $r(t) = \mathcal{E}'$).

By the theory of inhomogeneous equations, the general solution to (1) has the form

$$y = c_1y_1 + c_2y_2 + y_p, \qquad c_1, c_2 \;\text{ arbitrary constants}, \tag{2}$$

where y_p is a *particular solution* to (1), and $c_1y_1 + c_2y_2$ is the *complementary function*, i.e., the general solution to the associated homogeneous equation (the one having $r(t) = 0$).

The initial conditions determine the exact values of c_1 and c_2. So from (2),

$$\begin{array}{c}\textit{the system modeled}\\ \textit{by (1) is stable}\end{array} \iff \begin{array}{l}\textit{for every choice of } c_1, c_2,\\ c_1y_1 + c_2y_2 \to 0 \;\textit{ as } t \to \infty.\end{array} \tag{3}$$

Often one applies the term *stable* to the ODE (1) itself, as well as to the system it models. We shall do this here.

If the ODE (1) is stable, the two parts of the solution (2) are named:

$$y_p \;=\; \textit{steady-state solution} \qquad\qquad c_1y_1 + c_2y_2 \;= \textit{transient}; \tag{4}$$

the whole solution $y(t)$ and the right side $r(t)$ of (1) are described by the terms

$$y(t) = \textit{response} \qquad\qquad r(t) = \textit{input}.$$

From this point of view, the driving force is viewed as the input to the spring-mass system, and the resulting motion of the mass is thought of as the response of the system to the input. So what (2) and (4) are saying is that this response is the sum of two terms: a transient term, which depends on the initial conditions, but whose effects disappear over time; and a steady-state term, which represents more and more closely the response of the system as time goes to ∞, no matter what the initial conditions are.

2. Conditions for stability: second-order equations.

We now ask under what circumstances the ODE (1) will be stable. In view of the definition, together with (2) and (3), we see that stability concerns just the behavior of the solutions to the associated homogeneous equation

$$a_0 y'' + a_1 y' + a_2 y = 0 \; ; \tag{5}$$

the forcing term $r(t)$ plays no role in deciding whether or not (1) is stable.

There are three cases to be considered in studying the stability of (5); they are summarized in the table below, and based on the roots of the characteristic equation

$$a_0 r^2 + a_1 r + a_2 \;=\; 0 \; . \tag{6}$$

roots	**solution to ODE**	**condition for stability**
$r_1 \neq r_2$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 < 0, r_2 < 0$
$r_1 = r_2$	$e^{r_1 t}(c_1 + c_2 t)$	$r_1 < 0$
$a \pm ib$	$e^{at}(c_1 \cos bt + c_2 \sin bt)$	$a < 0$

The first two columns of the table should be familiar, from your work in solving the linear second-order equation (5) with constant coefficients. Let us consider the third column, therefore. In each case, we want to show that if the condition given in the third column holds, then the criterion (3) for stability will be satisfied.

Consider the first case. If $r_1 < 0$ and $r_2 < 0$, then it is immediate that the solution given tends to 0 as $t \to \infty$.

On the other hand, if say $r_1 \geq 0$, then the solution $e^{r_1 t}$ tends to ∞ (or to 1 if $r_1 = 0$). This shows the ODE (5) is not stable, since not all solutions tend to 0 as $t \to \infty$.

In the second case, the reasoning is the same, except that here we are using the limit

$$\lim_{t \to \infty} t e^{rt} = 0 \quad \Leftrightarrow \quad r < 0$$

For the third case, the relevant limits are (assuming $b \neq 0$ for the second limit):

$$\lim_{t \to \infty} e^{at} \cos bt = 0 \quad \Leftrightarrow \quad a < 0, \qquad \lim_{t \to \infty} e^{at} \sin bt = 0 \quad \Leftrightarrow \quad a < 0 \; .$$

The three cases can be summarized conveniently by one statement:

Stability criterion for second-order ODE's — root form

$$a_0 y'' + a_1 y' + a_2 y \;=\; r(t) \;\; \textit{is stable} \quad \Leftrightarrow \quad \begin{array}{c} \textit{all roots of } \; a_0 r^2 + a_1 r + a_2 = 0 \\ \textit{have negative real part.} \end{array} \tag{7}$$

Alternatively, one can phrase the criterion in terms of the coefficients of the ODE; this is convenient, since it doesn't require you to calculate the roots of the characteristic equation.

Stability criterion for second order ODE's — coefficient form. *Assume $a_0 > 0$.*

$$a_0 y'' + a_1 y' + a_2 y = r(t) \;\; \textit{is stable} \quad \Longleftrightarrow \quad a_0, a_1, a_2 > 0 \; . \tag{8}$$

The proof is left for the exercises; it is based on the quadratic formula.

3. Stability of higher-order ODE's.

The stability criterion in the root form (7) also applies to higher-order ODE's with constant coefficients:

$$(9) \qquad (a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n)\ y \ = \ f(t)\ .$$

These model more complicated spring-mass systems and multi-loop RLC circuits. The characteristic equation of the associated homogeneous equation is

$$(10) \qquad a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n \ = \ 0\ .$$

The real and complex roots of the characteristic equation give rise to solutions to the associated homogeneous equation just as they do for second order equations. (For a k-fold repeated root, one gets additional solutions by multiplying by $1, t, t^2, \ldots t^{k-1}$.)

The reasoning which led to the above stability criterion for second-order equations applies to higher-order equations just as well. The end result is the same:

Stability criterion for higher-order ODE's — root form

$$(11) \qquad \textit{ODE } (9) \textit{ is stable} \quad \Longleftrightarrow \quad \textit{all roots of } (10) \textit{ have negative real parts;}$$

that is, all the real roots are negative, and all the complex roots have negative real part.

There is a stability criterion for higher-order ODE's which uses just the coefficients of the equation, but it is not so simple as the one (8) for second-order equations. Without loss of generality, we may assume that $a_0 > 0$. Then it is not hard to prove (see the Exercises) that

$$(12) \qquad \textit{ODE } (9) \textit{ is stable} \quad \Rightarrow \quad a_0, \ldots, a_n > 0\ .$$

The converse is not true (see the exercises). For an implication $\Leftarrow$, the coefficients must satisfy a more complicated set of inequalities, which we give without proof, known as the

Routh-Hurwitz conditions for stability *Assume* $a_0 > 0$; *ODE* (9) *is stable* $\Leftrightarrow$ *in the determinant below, all of the n principal minors (i.e., the subdeterminants in the upper left corner having sizes respectively $1, 2, \ldots, n$) are > 0 when evaluated.*

$$(13) \qquad \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \ldots & \ldots & \ldots & a_n \end{vmatrix}$$

In the determinant, we define $a_k = 0$ if $k > n$; thus for example, the last row always has just one non-zero entry, a_n.

Exercises: Section 2G

I. Impulse Response and Convolution

1. Impulse response. Imagine a mass m at rest on a frictionless track, then given a sharp kick at time $t = 0$. We model the kick as a constant force F applied to the mass over a very short time interval $0 < t < \epsilon$. During the kick the velocity $v(t)$ of the mass rises rapidly from 0 to $v(\epsilon)$; after the kick, it moves with constant velocity $v(\epsilon)$, since no further force is acting on it. We want to express $v(\epsilon)$ in terms of F, ϵ, and m.

By Newton's law, the force F produces a constant acceleration a, and we get

$$(1) \qquad F = ma \quad \Rightarrow \quad v(t) = at, \quad 0 \le t \le \epsilon \quad \Rightarrow \quad v(\epsilon) = a\epsilon = \frac{F\epsilon}{m} \ .$$

If the mass is part of a spring-mass-dashpot system, modeled by the IVP

$$(2) \qquad my'' + cy' + ky = f(t), \qquad y(0) = 0, \ y'(0^-) = 0,$$

to determine the motion $y(t)$ of the mass, we should solve (2), taking the driving force $f(t)$ to be a constant F over the time interval $[0, \epsilon]$ and 0 afterwards. But this will take work and the answer will need interpretation.

Instead, we can both save work and get quick insight by solving the problem approximately, as follows. Assume the time interval ϵ is negligible compared to the other parameters. Then according to (1), the kick should impart the instantaneous velocity $F\epsilon/m$ to the mass, after which its motion $y(t)$ will be the appropriate solution to the homogeneous ODE associated with (2). That is, if the time interval ϵ for the initial kick is very small, the motion is approximately given (for $t \ge 0$) by the solution $y(t)$ to the IVP

$$(3) \qquad my'' + cy' + ky = 0, \qquad y(0) = 0, \quad y'(0) = \frac{F\epsilon}{m} \ .$$

Instead of worrying about the constants, assume for the moment that $F\epsilon/m = 1$; then the IVP (3) becomes

$$(4) \qquad my'' + cy' + ky = 0, \qquad y(0) = 0, \quad y'(0) = 1 \ ;$$

its solution for $t > 0$ will be called $w(t)$, and in view of the physical problem, we define $w(t) = 0$ for $t < 0$.

Comparing (3) and (4), we see that we can write the solution to (3) in terms of $w(t)$ as

$$(5) \qquad y(t) = \frac{F\epsilon}{m}\, w(t) \ ,$$

for since the ODE (4) is linear, multiplying the initial values $y(0)$ and $y'(0)$ by the same factor $F\epsilon/m$ multiplies the solution by this factor.

The solution $w(t)$ to (4) is of fundamental importance for the system (2); it is often called in engineering the **weight function** for the ODE in (4). A longer but more expressive name for it is the **unit impulse response** of the system: the quantity $F\epsilon$ is called in physics the *impulse* of the force, as is $F\epsilon/m$ (more properly, the *impulse/unit mass*), so that if $F\epsilon/m = 1$, the function $w(t)$ is the response of the system to a unit impulse at time $t = 0$.

Example 1. Find the unit impulse response to an undamped spring-mass system having (circular) frequency ω_0.

Solution. Taking $m = 1$, the IVP (4) is $y'' + \omega_0^2 y = 0$, $y(0) = 0$, $y'(0) = 1$, so that $y_c = a\cos\omega_0 t + b\sin\omega_0 t$; substituting in the initial conditions, we find

$$w(t) = \begin{cases} \frac{1}{\omega_0}\sin\omega_0 t, & t > 0; \\ 0, & t < 0. \end{cases}$$

Example 2. Find the unit impulse response to a critically damped spring-mass-dashpot system having e^{-pt} in its complementary function.

Solution. Since it is critically damped, it has a repeated characteristic root $-p$, and the complementary function is $y_c = e^{-pt}(c_1 + c_2 t)$. The function in this family satisfying $y(0) = 0$, $y'(0) = 1$ must have $c_1 = 0$; it is $t\,e^{-pt}$, either by differentiation or by observing that its power series expansion starts $t(1 - pt + ...) \approx t$.

2. Superposition.

We now return to the general second-order linear ODE with constant coefficients

$$(6) \qquad my'' + cy' + ky = f(t), \qquad \text{or} \qquad L(y) = f(t), \quad \text{where} \;\; L = mD^2 + cD + k.$$

We shall continue to interpret (6) as modeling a spring-mass-dashpot system, this time with an arbitrary driving force $f(t)$.

Since we know how to solve the associated homogeneous ODE, i.e., find the complementary solution y_c, our problem with (6) is to find a particular solution y_p. We can do this if $f(t)$ is "special" — sums of products of polynomials, exponentials, and sines and cosines. If $f(t)$ is periodic, or we are interested in it only on a finite interval, we can try expanding it into a Fourier series over that interval , and obtaining the particular solution y_p as a Fourier series. But what can we do for a general $f(t)$?

We use the *linearity* of the ODE (6), which allows us to make use of a

Superposition principle

If $f(t) = f_1(t) + \ldots + f_n(t)$ and y_i are corresponding particular solutions:

$$L(y_i) = f_i(t), \qquad i = 1, \ldots, n,$$

then $y_p = y_1 + \ldots + y_n$ is a particular solution to (2).

Proof. Using the linearity of the polynomial operator L, the proof takes one line:

$$L(y_p) \;=\; L(y_1) + \ldots + L(y_n) \;=\; f_1(t) + \ldots + f_n(t) \;=\; f(t).$$

Of course a general $f(t)$ is not the sum of a finite number of simpler functions. But over a finite interval we can approximate $f(t)$ by a sum of such functions.

Let the time interval be $0 < t < x$; we want to find the value of the particular solution $y_p(t)$ to (6) at the time $t = x$. We divide the time interval $[0, x]$ into n equal small intervals of length Δt:

$$0 = t_0,\; t_1,\; t_2,\; \ldots\,,\; t_n = x, \qquad t_{i+1} - t_i = \Delta t.$$

Over the time interval $[t_i, t_{i+1}]$ we have approximately $f(t) \approx f(t_i)$, and therefore if we set

$$f_i(t) = \begin{cases} f(t_i), & t_i \le t < t_{i+1}; \\ 0, & \text{elsewhere}, \end{cases} \qquad i = 0, 1, \ldots, n-1 \ ,$$

we will have approximately

$$f(t) \approx f_0(t) + \ldots + f_{n-1}(t), \qquad 0 < t < x \ . \tag{7}$$

We now apply our superposition principle. Since $w(t)$ is the response of the system in (6) to a unit impulse at time $t = 0$, then the response of (6) to the impulse given by $f_i(t)$ (in other words, the particular solution to (6) corresponding to $f_i(t)$) will be

$$f(t_i)w(t - t_i)\Delta t; \tag{8}$$

we translated $w(t)$ to the right by t_i units since the impulse is delivered at time t_i rather than at $t = 0$; we multiplied it by the constant $f(t_i)\Delta t$ since this is the actual impulse: the force $f_i(t)$ has magnitude $f(t_i)$ and is applied over a time interval Δt.

Since (7) breaks up $f(t)$, and (8) gives the response to each $f_i(t)$, the superposition principle tells us that the particular solution is approximated by the sum:

$$y_p(t) = \sum_0^{n-1} f(t_i)w(t - t_i)\Delta t, \qquad 0 \le t \le x,$$

so that at the time $t = x$,

$$y_p(x) \approx \sum_0^{n-1} f(t_i)w(x - t_i)\Delta t \ . \tag{9}$$

We recognize the sum in (9) as the sum which approximates a definite integral; if we pass to the limit as $n \to \infty$, i.e., as $\Delta t \to 0$, in the limit the sum becomes the definite integral and the approximation becomes an equality:

$$y_p(x) = \int_0^x f(t)\, w(x - t)\, dt \qquad \textbf{system response to } f(t) \tag{10}$$

> In effect, we are imagining the driving force $f(t)$ to be made up of an infinite succession of infinitely close kicks $f_i(t)$; by the superposition principle, the response of the system can then be obtained by adding up (via integration) the responses of the system to each of these kicks.

Which particular solution does (10) give? The answer is:

$$\text{for } y_p \text{ as in (10)}, \qquad y_p(0) = 0, \quad y'(0) = 0. \tag{11}$$

The first equation is clear from (10); we will derive the second in the next section.

The formula (10) is a very remarkable one. It expresses a particular solution to a second-order differential equation directly as a definite integral, whose integrand consists of two parts: a factor $w(x - t)$ depending only on the left-hand-side of (6) — that is, only on the spring-mass-dashpot system itself, not on how it is being driven — and a factor $f(t)$ depending only on the external driving force. For example, this means that once the unit impulse response $w(t)$ is calculated for the system, one only has to put in the different driving forces to determine the responses of the system to each.

The formula (10) makes superposition clear: to the sum of driving forces corresponds the sum of the corresponding particular solutions.

Still another advantage of our formula (10) is that it allows the driving force to have discontinuities, as long as they are isolated, since such functions can always be integrated. For instance, $f(t)$ could be a step function or the square wave function.

Let us check out the formula (10) in some simple cases where we can find the particular solution y_p also by the method of undetermined coefficients.

Example 3. Find the particular solution given by (10) to $y''+y=A$, where $D=d/dx$.

Solution. From Example 1, we have $w(t)=\sin t$. Therefore for $x\geq 0$, we have

$$y_p(x)=\int_0^x A\sin(x-t)\,dt = A\cos(x-t)\Big]_0^x = A(1-\cos x).$$

Here the method of undetermined coefficients would produce $y_p=A$; however, $A-A\cos x$ is also a particular solution, since $-A\cos x$ is in the complementary function y_c. Note that the above y_p satisfies (11), whereas $y_p=A$ does not.

Example 4. Find the particular solution for $x\geq 0$ given by (10) to $y''+y=f(x)$, where $f(x)=1$ if $0\leq x\leq\pi$, and $f(x)=0$ elsewhere.

Solution. Here the method of Example 3 leads to two cases: $0\leq x\leq\pi$ and $x\geq\pi$:

$$y_p=\int_0^x f(t)\sin(x-t)\,dt=\begin{cases}\displaystyle\int_0^x \sin(x-t)\,dt=\cos(x-t)\Big]_0^x = 1-\cos x, & 0\leq x\leq\pi;\\ \displaystyle\int_0^\pi \sin(x-t)\,dt=\cos(x-t)\Big]_0^\pi = -2\cos x, & x\geq\pi\end{cases}$$

3. Leibniz' Formula. To gain further confidence in our formula (10), which was obtained as a limit of approximations of varying degrees of shadiness, we want to check that it satisfies the ODE (6), with the initial conditions $y(0)=y'(0)=0$.

To do this, we will have to differentiate the right side of (10) with respect to x. The following theorem tells us how to do this.

Theorem. *If the integrand $g(x,t)$ and its partial derivative $g_x(x,t)$ are continuous, then*

$$\frac{d}{dx}\int_a^b g(x,t)\,dt \;=\; \int_a^b g_x(x,t)\,dt. \tag{12}$$

When we try to apply the theorem to differentiating the integral in (10), a difficulty arises because x occurs not just in the integrand, but as one of the limits as well. The best way to handle this is to give these two x's different names: u and v, and write

$$F(u,v) \;=\; \int_0^u g(v,t)\,dt, \qquad u=x,\quad v=x.$$

Using the 18.02 chain rule $\dfrac{d}{dx}F(u,v) \;=\; \dfrac{\partial F}{\partial u}\dfrac{du}{dx} + \dfrac{\partial F}{\partial v}\dfrac{dv}{dx}$, we get

$$\frac{d}{dx}\int_0^u g(v,t)\,dt \;=\; g(v,u) + \int_0^u \frac{\partial}{\partial v}g(v,t)\,dt\ ,$$

by the Second Fundamental Theorem of calculus and the preceding Theorem; then if we substitute $u = x$ and $v = x$, we get

$$(13) \qquad \frac{d}{dx}\int_0^x g(x,t)\,dt \;=\; g(x,x) + \int_0^x \frac{\partial}{\partial x}g(x,t)\,dt \qquad\qquad \textbf{Leibniz' Formula}.$$

We can now use Leibniz' formula to show that (10) satisfies the ODE (6); we have

$$\begin{aligned} y_p &= \int_0^x f(t)w(x-t)\,dt \\ y_p' &= f(x)w(x-x) + \int_0^x f(t)w'(x-t)\,dt; \end{aligned}$$

the first term on the right is 0 since $w(0) = 0$ by (5); using Leibniz' formula once more:

$$y_p'' \;=\; f(x)w'(x-x) + \int_0^x f(t)w''(x-t)\,dt;$$

again, the first term on the right is $f(x)$ since $w'(0) = 1$ by (5); multipying each of the three preceding equations by the appropriate coefficient of the ODE and then adding the three equations gives

$$\begin{aligned} y_p'' + ay_p' + by_p &= f(x) \;+\; \int_0^x f(t)\,[w''(x-t) + aw'(x-t) + bw(x-t)]\,dt \\ &= f(x), \end{aligned}$$

since for any independent variable, $w''(u) + aw'(u) + bw(u) = 0$, and therefore the same is true if u is replaced by $x - t$.

This shows that the integral in (10) satisfies the ODE; as for the initial conditions, we have $y_p(0) = 0$ from the definition of the integral in the equation for y_p above, and $y_p'(0) = 0$ from the equation for y_p' and the fact that $w(0) = 0$.

4. Convolution.

Integrals of the form

$$\int_0^x f(t)w(x-t)\,dt$$

occur widely in applications; they are called "convolutions" and a special symbol is used for them. Since w and f have a special meaning in these notes related to second-order ODE's and their associated spring-mass-dashpot systems, we give the definition of convolution using fresh symbols.

Definition. The **convolution** of $u(x)$ and $v(x)$ is the function of x defined by

$$u * v \;=\; \int_0^x u(t)v(x-t)\,dt. \tag{14}$$

The form of the convolution of two functions is not really predictable from the functions. Two simple and useful ones are worth remembering:

$$e^{ax} * e^{bx} = \frac{e^{ax} - e^{bx}}{a-b}, \quad a \neq b; \qquad\qquad e^{ax} * e^{ax} = x\,e^{ax} \tag{15}$$

Proof. We do the first; the second is similar. If $a \neq b$,

$$e^{ax} * e^{bx} = \int_0^x e^{a(t)}e^{b(x-t)}dt = e^{bx}\int_0^x e^{(a-b)t}dt = e^{bx}\frac{e^{(a-b)t}}{a-b}\bigg]_0^x = e^{bx}\frac{e^{(a-b)x}-1}{a-b} = \frac{e^{ax}-e^{bx}}{a-b}.$$

Properties of the convolution.

(*linearity*)
$$(u_1 + u_2) * v = u_1 * v + u_2 * v, \qquad (cu) * v = c(u * v);$$
$$u * (v_1 + v_2) = u * v_1 + u * v_2, \qquad u * (cv) = c(u * v);$$

which follow immediately from the corresponding properties of the definite integral.

(*commutativity*)
$$u * v = v * u$$

Since the definition (14) of convolution does not treat u and v symmetrically, the commutativity is not obvious. We will prove the commutativity later using the Laplace transform. One can also prove it directly, by making a change of variable in the convolution integral. As an example, the formula in (15) shows that $e^{ax} * e^{bx} = e^{bx} * e^{ax}$.

5. Examples of using convolution.

Higher-order linear ODE's. The formula $y_p(x) = f(x) * w(x)$ given in (10) also holds for the n-th order ODE $p(D)y = f(t)$ analogous to (6); the weight function $w(t)$ is defined to be the unique solution to the IVP

$$p(D)y = 0, \qquad w(0) = w'(0) = \dots w^{(n-2)}(0) = 0, \quad w^{(n-1)}(0) = 1. \tag{16}$$

As in the second-order case, $w(t)$ is the response of the system to the driving force $f(t)$ given by a unit impulse at time $t = 0$.

Example 5. Verify $y_p = f(x) * w(x)$ for the solution to the first-order IVP

$$y' + ay = f(x); \quad y(0) = 0. \tag{17}$$

Solution. According to (16), the weight function $w(t)$ should be the solution of the associated homogeneous equation satisfying $w(0) = 1$; it is therefore $w = e^{-at}$. Using the standard integrating factor e^{ax} to solve (17), and a definite integral to express the solution,

$$\begin{aligned}
(y\,e^{ax})' &= f(x)e^{ax} \\
y_p\,e^{ax} &= \int_0^x f(t)e^{at}\,dt \qquad t \text{ is a dummy variable} \\
y_p &= \int_0^x f(t)e^{-a(x-t)}\,dt \\
y_p &= f(x) * e^{-ax}.
\end{aligned}$$

Example 6. Radioactive dumping. A radioactive substance decays exponentially:

$$R = R_0 e^{-at}, \tag{18}$$

where R_0 is the initial amount, $R(t)$ the amount at time t, and a the decay constant.

A factory produces this substance as a waste by-product, and it is dumped daily on a waste site. Let $f(t)$ be the rate of dumping; this means that in a relatively small time period $[t_0, t_0 + \Delta t]$, approximately $f(t_0)\Delta t$ grams of the substance is dumped.

The dumping starts at time $t = 0$.

Find a formula for the amount of radioactive waste in the dump site at time x, and express it as a convolution.

Solution. Divide up the time interval $[0, x]$ into n equal intervals of length Δt, using the times

$$t_0 = 0,\ t_1,\ \dots,\ t_n = x.$$

$$\text{amount dumped in the interval } [t_i, t_{i+1}] \ \approx\ f(t_i)\Delta t\ ;$$

By time x, it will have decayed for approximately the length of time $x - t_i$; therefore, according to (18), at time x the amount of waste coming from what was dumped in the time interval $[t_i, t_{i+1}]$ is approximately

$$f(t_i)\Delta t \cdot e^{-a(x-t_i)};$$

this shows that the

$$\text{total amount at time } x \ \approx\ \sum_{0}^{n-1} f(t_i) e^{-a(x-t_i)} \Delta t\ .$$

As $n \to \infty$ and $\Delta t \to 0$, the sum approaches the corresponding definite integral and the approximation becomes an equality. So we conclude that

$$\text{total amount at time } x \ =\ \int_0^x f(t) e^{-a(x-t)}\, dt \ =\ f(x) * e^{-ax};$$

i.e., the amount of waste at time x is the convolution of the dumping rate and the decay function.

Example 7. Bank interest. On a savings account, a bank pays the continuous interest rate r, meaning that a sum A_0 deposited at time $t = 0$ will by time t grow to the amount $A_0 e^{rt}$.

Suppose that starting at day $t = 0$ a Harvard square juggler deposits every day his take, with deposit rate $d(t)$ — i.e., over a relatively small time interval $[t_0, t_0 + \Delta t]$, he deposits approximately $d(t_0)\Delta t$ dollars in his account. Assuming that he makes no withdrawals and the interest rate doesn't change, give with reasoning an approximate expression (involving a convolution) for the amount of money in his account at time $t = x$.

Solution. Similar to Example 6, and left as an exercise.

Exercises: Section 2H

H. Heaviside's Cover-up Method

The cover-up method was introduced by Oliver Heaviside as a fast way to do a decomposition into partial fractions. This is an essential step in using the Laplace transform to solve differential equations, and this was more or less Heaviside's original motivation.

The cover-up method can be used to make a partial fractions decomposition of a rational function $\dfrac{p(x)}{q(x)}$ *whenever the denominator can be factored into distinct linear factors.*

We first show how the method works on a simple example, and then show why it works.

Example 1. Decompose $\dfrac{x-7}{(x-1)(x+2)}$ into partial fractions.

Solution. We know the answer will have the form

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}\ . \tag{1}$$

To determine A by the cover-up method, on the left-hand side we mentally remove (or cover up with a finger) the factor $x-1$ associated with A, and substitute $x=1$ into what's left; this gives A:

$$\left.\frac{x-7}{\qquad\ (x+2)}\right|_{x=1} = \frac{1-7}{1+2} = -2 = A\ . \tag{2}$$

Similarly, B is found by covering up the factor $x+2$ on the left, and substituting $x=-2$ into what's left. This gives

$$\left.\frac{x-7}{(x-1)\qquad\ }\right|_{x=-2} = \frac{-2-7}{-2-1} = 3 = B\ .$$

Thus, our answer is

$$\frac{x-7}{(x-1)(x+2)} = \frac{-2}{x-1} + \frac{3}{x+2}\ . \tag{3}$$

Why does the method work? The reason is simple. The "right" way to determine A from equation (1) would be to multiply both sides by $(x-1)$; this would give

$$\frac{x-7}{\qquad\ (x+2)} = A + \frac{B}{x+2}(x-1)\ . \tag{4}$$

Now if we substitute $x=1$, what we get is exactly equation (2), since the term on the right disappears. The cover-up method therefore is just any easy way of doing the calculation without going to the fuss of writing (4) — it's unnecessary to write the term containing B since it will become 0 .

In general, if the denominator of the rational function factors into the product of distinct linear factors:

$$\frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_r)} = \frac{A_1}{x-a_1} + \ldots + \frac{A_r}{x-a_r}\ , \quad a_i \neq a_j\ ,$$

then A_i is found by covering up the factor $x - a_i$ on the left, and setting $x = a_i$ in the rest of the expression.

Example 2. Decompose $\dfrac{1}{x^3-x}$ into partial fractions.

Solution. Factoring, $x^3 - x = x(x^2-1) = x(x-1)(x+1)$. By the cover-up method,

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}\ .$$

To be honest, the real difficulty in all of the partial fractions methods (the cover-up method being no exception) is in factoring the denominator. Even the programs which do symbolic integration, like Macsyma, or Maple, can only factor polynomials whose factors have integer coefficients, or "easy coefficients" like $\sqrt{2}$. and therefore they can only integrate rational functions with "easily-factored" denominators.

Heaviside's cover-up method also can be used even when the denominator doesn't factor into distinct linear factors. To be sure, it gives only partial results, but these can often be a big help. We illustrate.

Example 3. Decompose $\dfrac{5x+6}{(x^2+4)(x-2)}$.

Solution. We write

(5) $$\frac{5x+6}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}\ .$$

We first determine C by the cover-up method, getting $C = 2$. Then A and B can be found by the method of undetermined coefficients; the work is greatly reduced since we need to solve only two simultaneous equations to find A and B, not three.

Following this plan, using $C = 2$, we combine terms on the right of (5) so that both sides have the same denominator. The numerators must then also be equal, which gives us

(6) $$5x + 6 = (Ax + B)(x - 2) + 2(x^2 + 4)\ .$$

Comparing the coefficients say of x^2 and of the constant terms on both sides of (6) then gives respectively the two equations

$$0 = A + 2 \qquad \text{and} \qquad 6 = -2B + 8,$$

from which $A = -2$ and $B = 1$.

In using (6), one could have instead compared the coefficients of x, getting $5 = -2A + B$; this provides a valuable check on the correctness of our values for A and B.

In Example 3, an alternative to undetermined coefficients would be to substitute two numerical values for x into the original equation (5), say $x = 0$ and $x = 1$ (any values other than $x = 2$ are usable). Again one gets two simultaneous equations for A and B. This method requires addition of fractions, and is usually better when only one coefficient remains to be determined (as in Example 4 below).

Still another method would be to factor the denominator completely into linear factors, using complex coefficients, and then use the cover-up method, but with complex numbers. At the end, conjugate complex terms have to be combined in pairs to produce real summands. The calculations are sometimes longer, and require skill with complex numbers.

The cover-up method can also be used if a linear factor is repeated, but there too it gives just partial results. It applies only to the *highest power of the linear factor*. Once again, we illustrate.

Example 4. Decompose $\dfrac{1}{(x-1)^2(x+2)}$.

Solution. We write

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}\ . \tag{7}$$

To find A cover up $(x-1)^2$ and set $x = 1$; you get $A = 1/3$. To find C, cover up $x+2$, and set $x = -2$; you get $C = 1/9$.

This leaves B which cannot be found by the cover-up method. But since A and C are already known in (7), B can be found by substituting any numerical value (other than 1 or -2) for x in equation (7). For instance, if we put $x = 0$ and remember that $A = 1/3$ and $C = 1/9$, we get

$$\frac{1}{2} = \frac{1/3}{1} + \frac{B}{-1} + \frac{1/9}{2}\ ,$$

from which we see that $B = -1/9$.

B could also be found by applying the method of undetermined coefficients to the equation (7); note that since A and C are known, it is enough to get a single linear equation in order to determine B — simultaneous equations are no longer needed.

The fact that the cover-up method works for just the *highest power* of the repeated linear factor can be seen just as before. In the above example for instance, the cover-up method for finding A is just a short way of multiplying equation (5) through by $(x-1)^2$ and then substituting $x = 1$ into the resulting equation.

LT. Laplace Transform

1. Translation formula. The usual L.T. formula for translation on the t-axis is

(1) $$\mathcal{L}\big(u(t-a)f(t-a)\big) \;=\; e^{-as}F(s), \qquad \text{where } \; F(s)=\mathcal{L}\big(f(t)\big)\;, \quad a>0.$$

This formula is useful for computing the inverse Laplace transform of $e^{-as}F(s)$, for example. On the other hand, as written above it is not immediately applicable to computing the L.T. of functions having the form $u(t-a)f(t)$. For this you should use instead this form of (1):

(2) $$\mathcal{L}\big(u(t-a)f(t)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big), \quad a>0.$$

Example 1. Calculate the Laplace transform of $u(t-1)(t^2+2t)$.

Solution. Here $f(t)=t^2+2t$, so (check this!) $f(t+1)=t^2+4t+3$. So by (2),
$$\mathcal{L}\big(u(t-1)(t^2+2t)\big) \;=\; e^{-s}\mathcal{L}(t^2+4t+3) \;=\; e^{-s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}\right)\;.$$

Example 2. Find $\mathcal{L}\big(u(t-\frac{\pi}{2})\sin t\big)$.

Solution.
$$\begin{aligned}\mathcal{L}\big(u(t-\tfrac{\pi}{2})\sin t\big) \;&=\; e^{-\pi s/2}\mathcal{L}\big(\sin(t+\tfrac{\pi}{2})\big)\\ &=\; e^{-\pi s/2}\mathcal{L}(\cos t) \;=\; e^{-\pi s/2}\frac{s}{s^2+1}\;.\end{aligned}$$

Proof of formula (2). According to (1), for any $g(t)$ we have
$$\mathcal{L}\big(u(t-a)g(t-a)\big) \;=\; e^{-as}\mathcal{L}\big(g(t)\big);$$
this says that to get the factor on the right side involving g, we should replace $t-a$ by t in the function $g(t-a)$ on the left, and then take its Laplace transform.

Apply this procedure to the function $f(t)$, written in the form $f(t)=f((t-a)+a)$; we get ("replacing $t-a$ by t and then taking the Laplace Transform")
$$\mathcal{L}\big(u(t-a)f((t-a)+a)\big) \;=\; e^{-as}\mathcal{L}\big(f(t+a)\big),$$
exactly the formula (2) that we wanted to prove. $\square$

Exercises. Find: a) $\mathcal{L}\big(u(t-a)e^t\big)$ b) $\mathcal{L}\big(u(t-\pi)\cos t\big)$ c) $\mathcal{L}\big(u(t-2)te^{-t}\big)$

Solutions. a) $e^{-as}\dfrac{e^a}{s-1}$ b) $-e^{-\pi s}\dfrac{s}{s^2+1}$ c) $e^{-2s}\dfrac{e^{-2}(2s+3)}{(s+1)^2}$

CG. Convolution and Green's Formula

1. Convolution. A peculiar-looking integral involving two functions $f(t)$ and $g(t)$ occurs widely in applications; it has a special name and a special symbol is used for it.

Definition. The **convolution** of $f(t)$ and $g(t)$ is the function $f * g$ of t defined by

$$[f * g](t) \;=\; \int_0^t f(u)g(t-u)\,du. \tag{1}$$

Example 1 below calculates two useful convolutions from the definition (1). As you can see, the form of $f * g$ is not very predictable from the form of f and g.

Example 1. Show that

$$e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a-b}, \quad a \neq b; \qquad\qquad e^{at} * e^{at} = t\,e^{at} \tag{2}$$

Solution. We do the first; the second is similar. If $a \neq b$,

$$e^{at} * e^{bt} = \int_0^t e^{au}e^{b(t-u)}du = e^{bt}\int_0^t e^{(a-b)u}du = e^{bt}\,\frac{e^{(a-b)u}}{a-b}\Bigg]_0^t = e^{bt}\,\frac{e^{(a-b)t}-1}{a-b} = \frac{e^{at}-e^{bt}}{a-b}.$$

The convolution gives us an expressive formula for a particular solution y_p to an inhomogeneous linear ODE. The next example illustrates this for the first-order equation.

Example 2. Express as a convolution the solution to the first-order constant-coefficient linear IVP (cf. Notes IR (3))

$$y' + ky = q(t); \quad y(0) = 0. \tag{3}$$

Solution. The integrating factor is e^{kt}; multiplying both sides by it gives

$$(y\,e^{kt})' \;=\; q(t)e^{kt}.$$

Integrate both sides from 0 to t, and apply the Fundamental Theorem of Calculus to the left side; since the particular solution y_p we want satisfies $y(0) = 0$, we get

$$y_p\,e^{kt} \;=\; \int_0^t q(u)e^{ku}\,du\ ; \qquad\qquad (u \text{ is a dummy variable.})$$

Moving the e^{kt} to the right side and placing it under the integral sign gives

$$\begin{aligned} y_p &= \int_0^t q(u)e^{-k(t-u)}\,du \\ y_p &= q(t) * e^{-kt}. \end{aligned}$$

We see that the solution is the convolution of the input $q(t)$ with the solution to the IVP (3) where $q = 0$ and the initial value is $y(0) = 1$. This is the simplest case of **Green's formula**, which does the same thing for higher-order linear ODE's. We will describe it in section 3.

2. Physical applications of the convolution. The convolution comes up naturally in a variety of physical sitations. Here are two typical examples.

Example 3. Radioactive dumping. A radioactive substance decays exponentially:

$$R = R_0 e^{-kt}, \tag{4}$$

where R_0 is the initial amount, $R(t)$ the amount at time t, and k the decay constant.

A factory produces this substance as a waste by-product, and it is dumped daily on a waste site. Let $f(t)$ be the rate of dumping; this means that in a relatively small time period $[t_0, t_0 + \Delta t]$, approximately $f(t_0)\Delta t$ grams of the substance is dumped.

Find a formula for the amount of radioactive waste in the dump site at time t, and express it as a convolution. Assume the dumping starts at time $t = 0$.

Solution. Divide up the time interval $[0, t]$ into n equal intervals of length Δu, using the times

$$u_0 = 0,\ u_1,\ u_2,\ \dots,\ u_n = t.$$

$$\text{amount dumped in the interval } [u_i, u_{i+1}] \ \approx\ f(u_i)\Delta u\ ;$$

by time t, this amount will have decayed for approximately the length of time $t - u_i$; therefore, according to (4), at time t the amount of waste coming from what was dumped in the time interval $[u_i, u_{i+1}]$ is approximately

$$f(u_i)\Delta u \cdot e^{-k(t-u_i)}.$$

Adding up the radioactive material in the pile coming from the dumping over each time interval, we get

$$\text{total amount at time } t \ \approx\ \sum_{0}^{n-1} f(u_i)e^{-k(t-u_i)}\Delta u\ .$$

As $n \to \infty$ and $\Delta u \to 0$, the sum approaches the corresponding definite integral and the approximation becomes an equality. So we conclude that

$$\text{total amount at time } t \ =\ \int_0^t f(u)e^{-k(t-u)}\,du \ =\ f(t) * e^{-kt};$$

i.e., the amount of waste still radioactive at time t is the convolution of the dumping rate and the decay function.

Example 4. Bank interest. On a savings account, a bank pays the continuous interest rate r, meaning that a sum A_0 deposited at time $u = 0$ will by time $u = t$ grow to the amount $A_0 e^{rt}$.

Suppose that starting at day $t = 0$ a Harvard square juggler deposits every day his take, with deposit rate $d(t)$ — i.e., over a relatively small time interval $[u_0, u_0 + \Delta u]$, he deposits approximately $d(u_0)\Delta u$ dollars in his account. Assuming that he makes no withdrawals and the interest rate doesn't change, give with reasoning an approximate expression (involving a convolution) for the amount of money in his account at time $u = t$.

Solution. Similar to Example 3, and left as an exercise.

3. Weight and transfer functions and Green's formula.

In Example 2 we expressed the solution to the IVP $y' + ky = q(t), \quad y(0) = 0$ as a convolution. We can do the same with higher order ODE's which are linear with constant coefficients. We will illustrate using the second order equation,

$$y'' + ay' + by \;=\; f(t), \qquad y(0) = 0,\; y'(0) = 0. \tag{5}$$

The Laplace transform of this IVP, with the usual notation, is

$$s^2Y + asY + bY \;=\; F(s)\;;$$

solving as usual for $Y = \mathcal{L}(y)$, we get

$$Y \;=\; F(s)\,\frac{1}{s^2 + as + b}\;;$$

using the convolution operator to take the inverse transform, we get the solution in the form called **Green's formula** (the function $w(t)$ is defined below):

$$y \;=\; f(t) * w(t) \;=\; \int_0^t f(u)w(t-u)\,du\;. \tag{6}$$

In connection with this form of the solution, the following terminology is often used. Let $p(D) = D^2 + aD + b$ be the differential operator; then we write

$$\begin{aligned} W(s) &= \frac{1}{s^2+as+b} && \text{the \textbf{transfer function} for } p(D),\\ w(t) &= \mathcal{L}^{-1}\big(W(s)\big) && \text{the \textbf{weight function} for } p(D),\\ G(t,u) &= w(t-u) && \text{the \textbf{Green's function} for } p(D). \end{aligned}$$

The important thing to note is that each of these functions depends only on the operator, not on the input $f(t)$; once they are calculated, the solution (6) to the IVP can be written down immediately by Green's formula, and used for a variety of different inputs $f(t)$.

The weight $w(t)$ is the unique solution to either of the IVP's

$$y'' + ay' + by \;=\; 0; \qquad y(0) = 0, \quad y'(0) = 1; \tag{7}$$

$$y'' + ay' + by \;=\; \delta(t); \qquad y(0) = 0, \quad y'(0^-) = 0; \tag{8}$$

in (8), the $\delta(t)$ is the Dirac delta function. It is an easy Laplace transform exercise to show that $w(t)$ is the solution to (7) and to (8). In the next section, we will give a physical interpretation for the weight function and Green's formula.

Let us check out Green's formula (6) in some simple cases where we can find the particular solution y_p also by another method.

Example 5. Find the particular solution given by (6) to $y'' + y = A, \quad y(0) = 0$.

Solution. From (7), we see that $w(t) = \sin t$. Therefore for $t \ge 0$, we have

$$y_p(t) = \int_0^t A\sin(t-u)\,du = A\cos(t-u)\Big]_0^t = A(1-\cos t).$$

Here the exponential response formula or the method of undetermined coefficients would produce the particular solution $y_p = A$; however, $A - A\cos t$ is also a particular solution, since $-A\cos t$ is in the complementary function y_c; the extra cosine term is required to satisfy $y(0) = 0$.

Example 6. Find the particular solution for $t \geq 0$ given by (6) to $y'' + y = f(t)$, where $f(t) = 1$ if $0 \leq t \leq \pi$, and $f(t) = 0$ elsewhere.

Solution. Here the method of Example 5 leads to two cases: $0 \leq t \leq \pi$ and $t \geq \pi$:

$$y_p = \int_0^t f(u)\sin(t-u)\,du = \begin{cases} \displaystyle\int_0^t \sin(t-u)\,du = \cos(t-u)\Big]_0^t = 1 - \cos t, & 0 \leq t \leq \pi; \\ \displaystyle\int_0^\pi \sin(t-u)\,du = \cos(t-u)\Big]_0^\pi = -2\cos t, & t \geq \pi\ . \end{cases}$$

Terminology and results analogous to (6) hold for the higher-order linear IVP's with constant coefficients (here $p(D)$ is any polynomial in D)

$$p(D)y = f(t), \quad y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0;$$

Green's formula for the solution is once again (6), where the weight function $w(t)$ is defined to be the unique solution to the IVP

$$p(D)y = 0, \qquad w(0) = w'(0) = \ldots w^{(n-2)}(0) = 0, \quad w^{(n-1)}(0) = 1. \tag{9}$$

Equivalently, it can be defined as the unique solution to the analogue of (8).

4. Impulse-response; interpretation of Green's formula.

We obtained Green's formula (6) by using the Laplace transform; our aim now is to interpret it physically, to see the "why" of it. This will give us further insight into the weight function and the convolution operation.

We know the weight function $w(t)$ is the solution to the IVP

$$y'' + ay' + by = 0, \qquad y(0) = 0, \quad y'(0) = 1. \tag{7}$$

We think of this as modeling the motion of the mass in a spring-mass-dashpot system (we will take the mass $m = 1$, for simplicity). The system is initially at rest, but at time $t = 0$ the mass is given a kick in the positive direction, which imparts to it unit velocity:

$$y'(0^+) = 1.$$

According to physical mechanics, to impart this unit velocity to a unit mass, the kick must have unit impulse, which is defined for a constant force F to be

$$\text{Impulse} = (\text{force } F)(\text{length of time } F \text{ is applied})\ . \tag{10}$$

A kick is modeled as a constant force F applied over a very short time interval $0 \leq t \leq \Delta u$; according to (10), for the force to impart a unit impulse over this time interval, it must have magnitude $1/\Delta u$:

$$1 = (1/\Delta u)(\Delta u)\ .$$

Input: force $(1/\Delta u)$ applied over time interval $[0, \Delta u]$ Response: $w(t)$

If the kick is applied instead over the time interval $u \le t \le u + \Delta u$, the response is the same as before, except that it starts at time u:

Input: force $(1/\Delta u)$ applied over time interval $[u, u + \Delta u]$ Response: $w(t-u)$

Finally, if the force is not a constant $(1/\Delta u)$, but varies with time: $F = f(u)$, by (10) the impulse it imparts over the time interval $[u, u + \Delta u]$ is approximately $f(u)\Delta u$, instead of 1, so the response must be multiplied by this factor; our final result therefore is

(11) Input: force $f(u)$ applied over $[u, u + \Delta u]$ Response: $f(u)\Delta u \cdot w(t-u)$.

From this last it is but a couple of steps to the physical interpretation of Green's formula. We use the

Superposition principle for the IVP (5): *if $f(t) = f_1(t) + \dots f_n(t)$, and $y_i(t)$ is the solution corresponding to $f_i(t)$, then $y_1 + \dots y_n$ is the solution corresponding to $f(t)$.*

In other words, the response to a sum of inputs is the sum of the corresponding responses to each separate input.

Of course, the input force $f(t)$ to our spring-mass-dashpot system is not the sum of simpler functions, but it can be approximated by such a sum. To do this, divide the time interval from $u = 0$ to $u = t$ into n equal subintervals, of length Δu:

$$0 = u_0, u_1, \dots, u_n = t, \qquad u_{i+1} - u_i = \Delta u\ .$$

Assuming $f(t)$ is continuous,

$$f(t) \approx f(u_i) \quad \text{over the time interval } [u_i, u_{i+1}]$$

Therefore if we set

$$f_i(t) = \begin{cases} f(u_i), & u_i \le t < u_{i+1}; \\ 0, & \text{elsewhere}, \end{cases} \qquad i = 0, 1, \dots, n-1\ , \tag{12}$$

we will have approximately

$$f(t) \ \approx\ f_0(t) + \dots + f_{n-1}(t), \qquad 0 < u < t\ . \tag{13}$$

We now apply our superposition principle. According to (10), the response of the system to the input $f_i(t)$ described in (12) will be approximately

$$f(u_i)w(t-u_i)\Delta u; \tag{14}$$

Applying the superposition principle to (13), we find the response $y_p(t)$ of the system to the input $f(t) = \sum f_i(t)$ is given approximately by

$$y_p(t) \ \approx\ \sum_0^{n-1} f(u_i)w(t-u_i)\Delta u\ . \tag{15}$$

We recognize the sum in (15) as the sum which approximates a definite integral; if we pass to the limit as $n \to \infty$, i.e., as $\Delta u \to 0$, in the limit the sum becomes the definite integral and the approximation becomes an equality and we get Green's formula:

$$(16) \qquad y_p(t) = \int_0^t f(u)\, w(t-u)\, du \qquad \textit{system response to } f(t)$$

> In effect, we are imagining the driving force $f(t)$ to be made up of an infinite succession of infinitely close kicks $f_i(t)$; by the superposition principle, the response of the system can then be obtained by adding up (via integration) the responses of the system to each of these kicks.

Green's formula is a very remarkable one. It expresses a particular solution to a second-order differential equation directly as a definite integral, whose integrand consists of two parts: a factor $w(t-u)$ depending only on the left-hand-side of (5) — that is, only on the spring-mass-dashpot system itself, not on how it is being driven — and a factor $f(t)$ depending only on the external driving force. For example, this means that once the unit impulse response $w(t)$ is calculated for the system, one only has to put in the different driving forces to determine the responses of the system to each.

Green's formula makes the superposition principle clear: to the sum of input forces corresponds the sum of the corresponding particular solutions.

Still another advantage of Green's formula is that it allows the input force to have discontinuities, as long as they are isolated, since such functions can always be integrated. For instance, $f(t)$ could be a step function or the square wave function.

Exercises: Section 2H

LS. LINEAR SYSTEMS

LS.1 Review of Linear Algebra

In these notes, we will investigate a way of handling a linear system of ODE's directly, instead of using elimination to reduce it to a single higher-order equation. This gives important new insights into such systems, and it is usually a more convenient and faster way of solving them.

The method makes use of some elementary ideas about linear algebra and matrices, which we will assume you know from your work in multivariable calculus. Your textbook contains a section (5.3) reviewing most of these facts, with numerical examples. Another source is the 18.02 Supplementary Notes, which contains a beginning section on linear algebra covering approximately the right material.

For your convenience, what you need to know is summarized briefly in this section. Consult the above references for more details and for numerical examples.

1. Vectors. A **vector** (or *n-vector*) is an n-tuple of numbers; they are usually real numbers, but we will sometimes allow them to be complex numbers, and all the rules and operations below apply just as well to n-tuples of complex numbers. (In the context of vectors, a single real or complex number, i.e., a constant, is called a **scalar**.)

The n-tuple can be written horizontally as a **row vector** or vertically as a **column vector**. In these notes it will almost always be a column. To save space, we will sometimes write the column vector as shown below; the small T stands for **transpose**, and means: change the row to a column.

$\mathbf{a} = (a_1, \ldots, a_n)$ *row vector* $\qquad$ $\mathbf{a} = (a_1, \ldots, a_n)^T$ *column vector*

These notes use boldface for vectors; in handwriting, place an arrow $\vec{a}$ over the letter.

Vector operations. The three standard operations on n-vectors are:

addition: $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$.

multiplication by a scalar: $c\,(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n)$

scalar product: $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \ldots + a_n b_n$.

2. Matrices and Determinants. An $m \times n$ **matrix** A is a rectangular array of numbers (real or complex) having m rows and n columns. The element in the i-th row and j-th column is called the *ij-th entry* and written a_{ij}. The matrix itself is sometimes written (a_{ij}), i.e., by giving its generic entry, inside the matrix parentheses.

A $1 \times n$ matrix is a row vector; an $n \times 1$ matrix is a column vector.

Matrix operations. These are

addition: if A and B are both $m \times n$ matrices, they are added by adding the corresponding entries; i.e., if $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$.

multiplication by a scalar: to get cA, multiply every entry of A by the scalar c; i.e., if $A = (a_{ij})$, then $cA = (ca_{ij})$.

matrix multiplication: if A is an $m \times n$ matrix and B is an $n \times k$ matrix, their product AB is an $m \times k$ matrix, defined by using the scalar product operation:

$$ij\text{-}th\ entry\ of\ AB = (i\text{-}th\ row\ of\ A) \cdot (j\text{-}th\ column\ of\ B)^T \ .$$

The definition makes sense since both vectors on the right are n-vectors. In what follows, the most important cases of matrix multiplication will be

(i) *A and B are square matrices of the same size*, i.e., both A and B are $n \times n$ matrices. In this case, multiplication is always possible, and the product AB is again an $n \times n$ matrix.

(ii) *A is an $n \times n$ matrix and $B = \mathbf{b}$, a column n-vector.* In this case, the matrix product $A\mathbf{b}$ is again a column n-vector.

Laws satisfied by the matrix operations.

For any matrices for which the products and sums below are defined, we have

$$(A\,B)\,C = A\,(B\,C) \qquad \textit{(associative law)}$$
$$A\,(B+C) = A\,B + A\,C, \quad (A+B)\,C = A\,B + A\,C \qquad \textit{(distributive laws)}$$
$$A\,B \neq B\,A \qquad \textit{(commutative law fails in general)}$$

Identity matrix. We denote by I_n the $n \times n$ matrix with 1's on the main diagonal (upper left to bottom right), and 0's elsewhere. If A is an arbitrary $n \times n$ matrix, it is easy to check from the definition of matrix multiplication that

$$A\,I_n = A \qquad \text{and} \qquad I_n A = A\ .$$

I_n is called the **identity matrix** of order n; the subscript n is often omitted.

Determinants. Associated with every *square* matrix A is a number, written $|A|$ or $\det A$, and called the **determinant** of A. For these notes, it will be enough if you can calculate the determinant of 2×2 and 3×3 matrices, by any method you like.

Theoretically, the determinant should not be confused with the matrix itself; the determinant is a *number*, the matrix is the *square array*. But everyone puts vertical lines on either side of the matrix to indicate its determinant, and then uses phrases like "the first row of the determinant", meaning the first row of the corresponding matrix.

An important formula which everyone uses and no one can prove is

(1) $$\det(A\,B) = \det A \cdot \det B\ .$$

Inverse matrix. A square matrix A is called **nonsingular** or **invertible** if $\det A \neq 0$.

If A is nonsingular, there is a unique square matrix B of the same size, called the **inverse** to A, having the property that

$$B\,A = I, \qquad \text{and} \qquad A\,B = I\ .$$

This matrix B is denoted by A^{-1}. To confirm that a given matrix B is the inverse to A, you only have to check one of the above equations; the other is then automatically true.

Different ways of calculating A^{-1} are given in the references. However, if A is a 2×2 matrix, as it usually will be in the notes, it is easiest simply to use the formula for it:

$$(2)\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Remember this as a procedure, rather than as a formula: switch the entries on the main diagonal, change the sign of the other two entries, and divide every entry by the determinant. (Often it is better for subsequent calculations to leave the determinant factor outside, rather than to divide all the terms in the matrix by det$A$.) As an example of (2),

$$\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}^{-1} = \tfrac{1}{2}\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

To calculate the inverse of a nonsingular 3×3 matrix, see for example the 18.02 notes.

3. Square systems of linear equations. Matrices and determinants were originally invented to handle in an efficient way the solution of a system of simultaneous linear equations. This is still one of their most important uses. We give a brief account of what you need to know. This is not in your textbook, but can be found in the 18.02 Notes. We will restrict ourselves to *square* systems — those having as many equations as they have variables (or "unknowns", as they are frequently called). Our notation will be:

$$\begin{aligned} A &= (a_{ij}), \quad \textit{a square } n \times n \textit{ matrix of constants,} \\ \mathbf{x} &= (x_1, \dots, x_n)^T, \quad \textit{a column vector of unknowns,} \\ \mathbf{b} &= (b_1, \dots, b_n)^T, \quad \textit{a column vector of constants;} \end{aligned}$$

then the square system

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be abbreviated by the matrix equation

$$(3)\qquad A\mathbf{x} = \mathbf{b}.$$

If $\mathbf{b} = \mathbf{0} = (0, \dots, 0)^T$, the system (3) is called **homogeneous**; if this is not assumed, it is called **inhomogeneous**. The distinction between the two kinds of system is significant. There are two important theorems about solving square systems: an easy one about inhomogeneous systems, and a more subtle one about homogeneous systems.

Theorem about square inhomogeneous systems.

If A is nonsingular, the system (3) has a unique solution, given by

$$(4)\qquad \mathbf{x} = A^{-1}\mathbf{b}.$$

Proof. Suppose $\mathbf{x}$ represents a solution to (3). We have

$$\begin{aligned} A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b}, \\ \Rightarrow (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b}, \qquad \text{by associativity;} \\ \Rightarrow \quad I\mathbf{x} &= A^{-1}\mathbf{b}, \qquad \text{definition of inverse;} \\ \Rightarrow \quad \mathbf{x} &= A^{-1}\mathbf{b}, \qquad \text{definition of } I. \end{aligned}$$

This gives a formula for the solution, and therefore shows it is unique if it exists. It does exist, since it is easy to check that $A^{-1}\mathbf{b}$ is a solution to (3). $\square$

The situation with respect to a homogeneous square system $A\mathbf{x} = \mathbf{0}$ is different. This always has the solution $\mathbf{x} = \mathbf{0}$, which we call the *trivial* solution; the question is: when does it have a nontrivial solution?

Theorem about square homogeneous systems. *Let A be a square matrix.*

(5) $\quad A\mathbf{x} = \mathbf{0}$ *has a nontrivial solution* $\Leftrightarrow$ $\det A = 0$ *(i.e., A is singular).*

Proof. The direction $\Rightarrow$ follows from (4), since if A is nonsingular, (4) tells us that $A\mathbf{x} = \mathbf{0}$ can have only the trivial solution $\mathbf{x} = \mathbf{0}$.

The direction $\Leftarrow$ follows from the criterion for linear independence below, which we are not going to prove. But in 18.03, you will always be able to show by calculation that the system has a nontrivial solution if A is singular.

4. Linear independence of vectors.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, x_k$ be a set of n-vectors. We say they are **linearly dependent** (or simply, *dependent*) if there is a non-zero relation connecting them:

(6) $$c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0}, \qquad (c_i \text{ constants, not all } 0).$$

If there is no such relation, they are called **linearly independent** (or simply, *independent*). This is usually phrased in a positive way: the vectors are *linearly independent* if the only relation among them is the zero relation, i.e.,

(7) $$c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0} \quad \Rightarrow \quad c_i = 0 \quad \text{for all } i.$$

We will use this definition mostly for just two or three vectors, so it is useful to see what it says in these low-dimensional cases. For $k = 2$, it says

(8) $\quad \mathbf{x}_1$ *and* $\mathbf{x}_2$ *are dependent* $\Leftrightarrow$ *one is a constant multiple of the other.*

For if say $\mathbf{x}_2 = c\mathbf{x}_1$, then $c\mathbf{x}_1 - x_2 = \mathbf{0}$ is a non-zero relation; while conversely, if we have non-zero relation $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, with say $c_2 \neq 0$, then $\mathbf{x}_2 = -(c_1/c_2)\,\mathbf{x}_1$.

By similar reasoning, one can show that

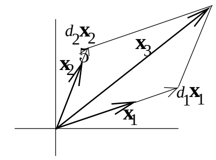
(9) $\quad \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ *are dependent* $\Leftrightarrow$ *one of them is a linear combination of the other two.*

Here by a **linear combination** of vectors we mean a sum of scalar multiples of them, i.e., an expression like that on the left side of (6). If we think of the three vectors as origin vectors in three space, the geometric interpretation of (9) is

(10) $\quad$ *three origin vectors in 3-space are dependent* $\Leftrightarrow$ *they lie in the same plane.*

For if they are dependent, say $\mathbf{x}_3 = d_1\mathbf{x}_1 + d_2\mathbf{x}_2$, then (thinking of them as origin vectors) the parallelogram law for vector addition shows that $\mathbf{x}_3$ lies in the plane of $\mathbf{x}_1$ and $\mathbf{x}_2$ — see the figure.

Conversely, the same figure shows that if the vectors lie in the same plane and say $\mathbf{x}_1$ and $\mathbf{x}_2$ span the plane (i.e., don't lie on a line), then by completing the



parallelogram, $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. (If they all lie on a line, they are scalar multiples of each other and therefore dependent.)

Linear independence and determinants. We can use (10) to see that

(11) $\qquad$ *the rows of a* 3×3 *matrix* A *are dependent* $\Leftrightarrow$ *det* $A = \mathbf{0}$.

Proof. If we denote the rows by $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$, then from 18.02,

$$\begin{array}{c}\text{volume of the parallelepiped} \\ \text{spanned by } \mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\end{array} = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = \det A,$$

so that

$$\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3 \quad \text{lie in a plane} \quad \Leftrightarrow \quad \det A = 0.$$

The above statement (11) generalizes to an $n \times n$ matrix A ; we rephrase it in the statement below by changing both sides to their negatives. (We will not prove it, however.)

Determinantal criterion for linear independence

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *be* n*-vectors, and* A *the square matrix having these vectors for its rows (or columns). Then*

(12) $\qquad \mathbf{a}_1, \ldots, \mathbf{a}_n$ *are linearly independent* $\Leftrightarrow$ *det*$A \neq 0$.

Remark. The theorem on square homogeneous systems (5) follows from the criterion (12), for if we let $\mathbf{x}$ be the column vector of n variables, and A the matrix whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then

(13)
$$A\mathbf{x} = (\mathbf{a}_1 \ldots \mathbf{a}_n) \begin{pmatrix} x_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} = \mathbf{a}_1 x_1 + \ldots + \mathbf{a}_n x_n$$

and therefore

$$\begin{aligned} A\mathbf{x} = \mathbf{0} \quad & \text{has only the solution } \mathbf{x} = \mathbf{0} \\ \Leftrightarrow \quad & \mathbf{a}_1 x_1 + \ldots + \mathbf{a}_n x_n = \mathbf{0} \quad \text{has only the solution } \mathbf{x} = \mathbf{0}, \quad \text{by (13)}; \\ \Leftrightarrow \quad & \mathbf{a}_1, \ldots, \mathbf{a}_n \quad \text{are linearly independent, by (7)}; \\ \Leftrightarrow \quad & \det A \neq 0, \quad \text{by the criterion (12).} \end{aligned}$$

Exercises: Section 4A

LS.2 Homogeneous Linear Systems with Constant Coefficients

1. Using matrices to solve linear systems.

The naive way to solve a linear system of ODE's with constant coefficients is by eliminating variables, so as to change it into a single higher-order equation. For instance, if

$$(1)\qquad \begin{aligned} x' &= x + 3y \\ y' &= x - y \end{aligned}$$

we can eliminate x by solving the second equation for x, getting $x = y + y'$, then replacing x everywhere by $y + y'$ in the first equation. This gives

$$y'' - 4y = 0 \;;$$

the characteristic equation is $(r-2)(r+2) = 0$, so the general solution for y is

$$y = c_1 e^{2t} + c_2 e^{-2t} \;.$$

From this we get x from the equation $x = y + y'$ originally used to eliminate x; the whole solution to the system is then

$$(2)\qquad \begin{aligned} x &= 3c_1 e^{2t} - c_2 e^{-2t} \\ y &= c_1 e^{2t} + c_2 e^{-2t} \;. \end{aligned}$$

We now want to introduce linear algebra and matrices into the study of systems like the one above. Our first task is to see how the above equations look when written using matrices and matrix multiplication.

When we do this, the system (1) and its general solution (2) take the forms

$$(4)\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

$$(5)\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3c_1 e^{2t} - c_2 e^{-2t} \\ c_1 e^{2t} + c_2 e^{-2t} \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \;.$$

Study the above until it is clear to you how the matrices and column vectors are being used to write the system (1) and its solution (2). Note that when we multiply the column vectors by scalars or scalar functions, it does not matter whether we write them behind or in front of the column vector; the way it is written above on the right of (5) is the one usually used, since it is easiest to read and interpret.

We are now going to show a new method of solving the system (1), which makes use of the matrix form (4) for writing it. We begin by noting from (5) that two particular solutions to the system (4) are

$$(6)\qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} \qquad \text{and} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \;.$$

Based on this, our new method is to look for solutions to (4) of the form

$$(7) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} ,$$

where a_1, a_2 and λ are unknown constants. We substitute (7) into the system (4) to determine what these unknown constants should be. This gives

$$(8) \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \lambda e^{\lambda t} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} .$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$(9) \qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

We have to solve the matrix equation (9) for the three constants. It is not very clear how to do this. When faced with equations in unfamiliar notation, a reasonable strategy is to rewrite them in more familiar notation. If we try this, (9) becomes the pair of equations

$$(10) \qquad \begin{aligned} \lambda a_1 &= a_1 + 3a_2 \\ \lambda a_2 &= a_1 - \ a_2 . \end{aligned}$$

Technically speaking, these are a pair of non-linear equations in three variables. The trick in solving them is to look at them as a pair of linear equations in the unknowns a_i, with λ viewed as a parameter. If we think of them this way, it immediately suggests writing them in standard form

$$(11) \qquad \begin{aligned} (1-\lambda)a_1 + 3a_2 &= 0 \\ a_1 + (-1-\lambda)a_2 &= 0 . \end{aligned}$$

In this form, we recognize them as forming a square system of homogeneous linear equations. According to the theorem on square systems (LS.1, (5)), they have a non-zero solution for the a's if and only if the determinant of coefficients is zero:

$$(12) \qquad \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 ,$$

which after calculation of the determinant becomes the equation

$$(13) \qquad \lambda^2 - 4 = 0 .$$

The roots of this equation are 2 and -2; what the argument shows is that the equations (10) or (11) (and therefore also (8)) have non-trivial solutions for the a's exactly when $\lambda = 2$ or $\lambda = -2$.

To complete the work, we see that for these values of the parameter λ, the system (11) becomes respectively

$$(14) \qquad \begin{aligned} -a_1 + 3a_2 &= 0 \\ a_1 - 3a_2 &= 0 \end{aligned} \quad (\text{for } \lambda = 2) \qquad\qquad \begin{aligned} 3a_1 + 3a_2 &= 0 \\ a_1 + a_2 &= 0 \end{aligned} \quad (\text{for } \lambda = -2)$$

It is of course no accident that in each case the two equations of the system become dependent, i.e., one is a constant multiple of the other. If this were not so, the two equations would have only the trivial solution $(0,0)$. All of our effort has been to locate the two values of λ for which this will *not* be so. The dependency of the two equations is thus a check on the correctness of the value of λ.

To conclude, we solve the two systems in (14). This is best done by assigning the value 1 to one of the unknowns, and solving for the other. We get

$$(15) \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda = 2\ ; \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = -2\ ,$$

which thus gives us, in view of (7), essentially the two solutions (6) we had found previously by the method of elimination. Note that the solutions (6) could be multiplied by an arbitrary non-zero constant without changing the validity of the general solution (5); this corresponds in the new method to selecting an arbitrary value of one of the a's, and then solving for the other value.

One final point before we discuss this method in general. Is there some way of passing from (9) (the point at which we were temporarily stuck) to (11) or (12) by using matrices, without writing out the equations separately? The temptation in (9) is to try to combine the two column vectors $\mathbf{a}$ by subtraction, but this is impossible as the matrix equation stands. If we rewrite it however as

$$(9') \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\ ,$$

it now makes sense to subtract the left side from the right; using the distributive law for matrix multiplication, the matrix equation (9′) then becomes

$$(11') \qquad \begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ ,$$

which is just the matrix form for (11). Now if we apply the theorem on square homogeneous systems, we see that (11′) has a nontrivial solution for the $\mathbf{a}$ if and only if its coefficient determinant is zero, and this is precisely (12). The trick therefore was in (9) to replace the scalar λ by the diagonal matrix λI .

2. Eigenvalues and eigenvectors.

With the experience of the preceding example behind us, we are now ready to consider the general case of a homogeneous linear 2×2 system of ODE's with constant coefficients:

$$(16) \qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy\ , \end{aligned}$$

where the a, b, c, d are constants. We write this system in matrix form as

$$(17) \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\ .$$

We look for solutions to (17) having the form

$$(18) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix}\ ,$$

where a_1, a_2, and λ are unkinown constants. We substitute (18) into the system (17) to determine these unknown constants. Since $D(ae^{\lambda t}) = \lambda ae^{\lambda t}$, we arrive at

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} . \tag{19}$$

We can cancel the factor $e^{\lambda t}$ from both sides, getting

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \tag{20}$$

As the equation (20) stands, we cannot combine the two sides by subtraction, since the scalar λ cannot be subtracted from the square matrix on the right. As in the previously worked example however (9′), the trick is to replace the scalar λ by the diagonal matrix λI; then (20) becomes

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} . \tag{21}$$

If we now proceed as we did in the example, subtracting the left side of (6) from the right side and using the distributive law for matrix addition and multiplication, we get a 2×2 homogeneous linear system of equations:

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{22}$$

Written out without using matrices, the equations are

$$\begin{aligned} (a-\lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda)a_2 &= 0 . \end{aligned} \tag{23}$$

According to the theorem on square homogeneous systems, this system has a non-zero solution for the a's if and only if the determinant of the coefficients is zero:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 . \tag{24}$$

The equation (24) is a quadratic equation in λ, evaluating the determinant, we see that it can be written

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 . \tag{25}$$

Definition. *The equation (24) or (25) is called the* **characteristic equation** *of the matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{26}$$

Its roots λ_1 and λ_2 are called the **eigenvalues** *or characteristic values of the matrix A.*

There are now various cases to consider, according to whether the eigenvalues of the matrix A are two distinct real numbers, a single repeated real number, or a pair of conjugate complex numbers. We begin with the first case: *we assume for the rest of this chapter that the eigenvalues are two distinct real numbers λ_1 and λ_2.*

To complete our work, we have to find the solutions to the system (23) corresponding to the eigenvalues λ_1 and λ_2. Formally, the systems become

$$(27) \qquad \begin{aligned} (a-\lambda_1)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda_1)a_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} (a-\lambda_2)a_1 + ba_2 &= 0 \\ ca_1 + (d-\lambda_2)a_2 &= 0 \end{aligned}$$

The solutions to these two systems are column vectors, for which we will use Greek letters rather than boldface.

Definition. *The respective solutions* $\mathbf{a} = \vec{\alpha}_1$ *and* $\mathbf{a} = \vec{\alpha}_2$ *to the systems (27) are called the* **eigenvectors** *(or characteristic vectors) corresponding to the eigenvalues* λ_1 *and* λ_2*.*

If the work has been done correctly, in each of the two systems in (27), the two equations will be dependent, i.e., one will be a constant multiple of the other. Namely, the two values of λ have been selected so that in each case the coefficient determinant of the system will be zero, which means the equations will be dependent. The solution $\vec{\alpha}$ is determined only up to an arbitrary non-zero constant factor. A convenient way of finding the eigenvector $\vec{\alpha}$ is to assign the value 1 to one of the a_i, then use the equation to solve for the corresponding value of the other a_i.

Once the eigenvalues and their corresponding eigenvectors have been found, we have two independent solutions to the system (16); According to (19), they are

$$(28) \qquad \mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Then the general solution to the system (16) is

$$(29) \qquad \mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}.$$

At this point, you should stop and work another example, like the one we did earlier. Try **5.4** Example 1 in your book; work it out yourself, using the book's solution to check your work. Note that the book uses $\mathbf{v}$ instead of $\vec{\alpha}$ for an eigenvector, and v_i or a, b instead of a_i for its components.

We are still not done with the general case; without changing any of the preceding work, you still need to see how it appears when written out using an even more abridged notation. Once you get used to it (and it is important to do so), the compact notation makes the essential ideas stand out very clearly.

As before, we let A denote the matrix of constants, as in (26). Below, on the left side of each line, we will give the compact matrix notation, and on the right, the expanded version. The equation numbers are the same as the ones above.

We start with the system (16), written in matrix form, with A as in (26):

$$(17') \qquad \mathbf{x}' = A\,\mathbf{x} \qquad\qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We use as the trial solution

$$(18') \qquad \mathbf{x} = \mathbf{a}\,e^{\lambda t} \qquad\qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}.$$

We substitute this expression for $\mathbf{x}$ into the system (17′), using $\mathbf{x}' = \lambda\mathbf{a}e^{\lambda t}$:

$$(19') \qquad \lambda\,\mathbf{a}\,e^{\lambda t} \;=\; A\,\mathbf{a}\,e^{\lambda t} \qquad\qquad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \;.$$

Cancel the exponential factor from both sides, and replace λ by λI, where I is the identity matrix:

$$(21') \qquad \lambda I\mathbf{a} \;=\; A\mathbf{a} \qquad\qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \;.$$

Subtract the left side from the right and combine terms, getting

$$(22') \qquad (A-\lambda I)\mathbf{a} \;=\; 0 \qquad\qquad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;.$$

This square homogeneous system has a non-trivial solution if and only if the coefficient determinant is zero:

$$(24') \qquad |A-\lambda I| \;=\; 0 \qquad\qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \;=\; 0 \;.$$

Definition. *Let A be a square matrix of constants, Then by definition*

(i) $|A - \lambda\, I| = 0$ *is the* **characteristic equation** *of A ;*

(ii) its roots λ_i are the **eigenvalues** *(or characteristic values) of A ;*

(iii) for each eigenvalue λ_i, the corresonding solution $\vec{\alpha}_i$ to (22′) is the **eigenvector** *(or characteristic vector) associated with λ_i .*

If the eigenvalues are distinct and real, as we are assuming in this chapter, we obtain in this way two independent solutions to the system (17′):

$$(28) \qquad \mathbf{x}_1 \;=\; \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x}_2 \;=\; \vec{\alpha}_2 e^{\lambda_2 t}, \qquad \text{where } \mathbf{x}_i \;=\; \begin{pmatrix} x_i \\ y_i \end{pmatrix} \;.$$

Then the general solution to the system (16) is

$$(29) \qquad \mathbf{x} \;=\; c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \;=\; c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t} \;.$$

The matrix notation on the left above in (17′) to (24′) is compact to write, makes the derivation look simpler. Moreover, when written in matrix notation, *the derivation applies to square systems of any size*: $n \times n$ just as well as 2×2. This goes for the subsequent definition as well: it defines *characteristic equation, eigenvalue* and *associated eigenvector* for a square matrix of any size.

The chief disadvantage of the matrix notation on the left is that for beginners it is very abridged. Practice writing the sequence of matrix equations so you get some skill in using the notation. Until you acquire some confidence, keep referring to the written-out form on the right above, so you are sure you understand what the abridged form is actually saying.

Since in the compact notation, the definitions and derivations are valid for square systems of any size, you now know for example how to solve a 3×3 system, if its eigenvalues turn out to be real and distinct; **5.4** Example 2 in your book is such a system. First however read the following remarks which are meant to be helpful in doing calculations: remember and *use* them.

Remark 1. Calculating the characteristic equation.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its characteristic equation is given by (cf. (24) and (25)):

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \qquad \text{or} \qquad \lambda^2 - (a+d)\lambda + (ad-bc) = 0\ . \tag{30}$$

Since you will be calculating the characteristic equation frequently, learn to do it using the second form given in (30). The two coefficients have analogs for any square matrix:

$$ad - bc = \det A \qquad\qquad a + d = \operatorname{tr} A \quad (\textit{trace } A)$$

where the **trace** of a square matrix A is the sum of the elements on the main diagonal. Using this, the characteristic equation (30) for a 2×2 matrix A can be written

$$\lambda^2 - (\operatorname{tr} A)\,\lambda + \det A = 0\ . \tag{31}$$

In this form, the characteristic equation of A can be written down by inspection; you don't need the intermediate step of writing down $|A - \lambda I| = 0$. For an $n \times n$ matrix, the characteristic equation reads in part (watch the signs!)

$$|A - \lambda I| = (-\lambda)^n + \operatorname{tr} A(-\lambda)^{n-1} + \ldots + \det A = 0\ . \tag{32}$$

In one of the exercises you are asked to derive the two coefficients specified.

Equation (32) shows that the characteristic polynomial $|A - \lambda I|$ of an $n \times n$ matrix A is a polynomial of degree n, so that such a matrix has at most n real eigenvalues. The trace and determinant of A give two of the coefficients of the polynomial. Even for $n = 3$ however this is not enough, and you will have to calculate the characteristic equation by expanding out $|A - \lambda I|$. Nonetheless, (32) is still very valuable, as it enables you to get an independent check on your work. Use it whenever $n > 2$.

Remark 2. Calculating the eigenvectors.

This is a matter of solving a homogeneous system of linear equations (22′).

For $n = 2$, there will be just one equation (the other will be a multiple of it); give one of the a_i's the value 1 (or any other convenient non-zero value), and solve for the other a_i.

For $n = 3$, two of the equations will usually be independent (i.e., neither a multiple of the other). Using just these two equations, give one of the a's a convenient value (say 1), and solve for the other two a's. (The case where the three equations are all multiples of a single one occurs less often and will be dealt with later.)

Remark 3. Normal modes.

When the eigenvalues of A are all real and distinct, the corresponding solutions (28)

$$\mathbf{x}_i = \vec{\alpha}_i e^{\lambda_i t}, \qquad i = 1, \ldots, n,$$

are usually called the *normal modes* in science and engineering applications. They often have physical interpretations, which sometimes makes it possible to find them just by inspection of the physical problem. The exercises will illustrate this.

Exercises: Section 4C

LS.3 Complex and Repeated Eigenvalues

1. Complex eigenvalues.

In the previous chapter, we obtained the solutions to a homogeneous linear system with constant coefficients

$$A\mathbf{x} = \mathbf{0}$$

under the assumption that the roots of its characteristic equation $|A - \lambda I| = 0$ — i.e., the eigenvalues of A — were *real* and *distinct.*

In this section we consider what to do if there are complex eigenvalues. Since the chanracteristic equation has real coefficients, its complex roots must occur in conjugate pairs:

$$\lambda = a + bi, \qquad \bar{\lambda} = a - bi \ .$$

Let's start with the eigenvalue $a + bi$. According to the solution method described in Chapter LS.2, the next step would be to find the corresponding eigenvector $\vec{\alpha}$, by solving the equations (LS.2, (27))

$$\begin{aligned} (a - \lambda)a_1 + ba_2 &= 0 \\ ca_1 + (d - \lambda)a_2 &= 0 \end{aligned}$$

for its components a_1 and a_2. Since λ is complex, the a_i will also be complex, and therefore the eigenvector $\vec{\alpha}$ corresponding to λ will have complex components.

Putting together the eigenvalue and eigenvector gives us formally the complex solution

$$\mathbf{x} = \vec{\alpha}\, e^{(a+bi)t} \ . \tag{1}$$

Naturally, we want real solutions to the system, since it was real to start with. To get them, the following theorem tells us to just take the real and imaginary parts of (1).

Theorem 3.1 *Given a system* $\mathbf{x}' = A\mathbf{x}$, *where* A *is a real matrix. If* $\mathbf{x} = \mathbf{x}_1 + i\,\mathbf{x}_2$ *is a complex solution, then its real and imaginary parts* $\mathbf{x}_1, \mathbf{x}_2$ *are also solutions to the system.*

Proof. Since $\mathbf{x}_1 + i\,\mathbf{x}_2$ is a solution, we have

$$(\mathbf{x} + i\,\mathbf{x}_2)' = A\,(\mathbf{x} + i\,\mathbf{x}_2) \ ;$$

equating real and imaginary parts of this equation,

$$\mathbf{x}_1' = A\mathbf{x}_1 \ , \qquad \mathbf{x}_2' = A\mathbf{x}_2 \ ,$$

which shows that the real vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $\mathbf{x}' = A\mathbf{x}$. □

Example 1. Find the corresponding two real solutions to $\mathbf{x}' = A\mathbf{x}$ if a complex eigenvalue and corresponding eigenvector are

$$\lambda = -1 + 2i \ , \qquad \vec{\alpha} = \begin{pmatrix} i \\ 2 - 2i \end{pmatrix} \ .$$

Solution. First write $\vec{\alpha}$ in terms of its real and imaginary parts:

$$\vec{\alpha} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ -2 \end{pmatrix} \ .$$

The corresponding complex solution $\mathbf{x} = \vec{\alpha}\, e^{\lambda t}$ to the system can then be written

$$\mathbf{x} \;=\; \left(\begin{pmatrix}0\\2\end{pmatrix} + i\begin{pmatrix}1\\-2\end{pmatrix}\right)e^{-t}\big(\cos 2t + i\sin 2t\big)\ ,$$

so that we get respectively for the real and imaginary parts of $\mathbf{x}$

$$x_1 \;=\; e^{-t}\left(\begin{pmatrix}0\\2\end{pmatrix}\cos 2t - i\begin{pmatrix}1\\-2\end{pmatrix}\sin 2t\right) \;=\; e^{-t}\begin{pmatrix}-\sin 2t\\ 2\cos 2t + 2\sin 2t\end{pmatrix}\ ,$$

$$x_2 \;=\; e^{-t}\left(\begin{pmatrix}1\\-2\end{pmatrix}\cos 2t - i\begin{pmatrix}0\\2\end{pmatrix}\sin 2t\right) \;=\; e^{-t}\begin{pmatrix}-\cos 2t\\ -2\cos 2t + 2\sin 2t\end{pmatrix}\ ;$$

these are the two real solutions to the system. □

In general, if the complex eigenvalue is $a + bi$, to get the real solutions to the system, we write the corresponding complex eigenvector $\vec{\alpha}$ in terms of its real and imaginary part:

$$\vec{\alpha} \;=\; \vec{\alpha}_1 + i\,\vec{\alpha}_2, \qquad \vec{\alpha}_i \quad \text{real vectors;}$$

(study carefully in the above example how this is done in practice). Then we substitute into (1) and calculate as in the example:

$$\mathbf{x} \;=\; (\vec{\alpha}_1 + i\vec{\alpha}_2)\, e^a t(\cos bt + i\sin bt),$$

so that the real and imaginary parts of $\mathbf{x}$ give respectively the two real solutions

$$\mathbf{x}_1 \;=\; e^{at}(\vec{\alpha}_1\cos bt - \vec{\alpha}_2\sin bt)\ ,$$
$$\mathbf{x}_2 \;=\; e^{at}(\vec{\alpha}_1\sin bt + \vec{\alpha}_2\cos bt)\ . \tag{3}$$

These solutions are linearly independent if $n = 2$. If $n > 2$, that portion of the general solution corresonding to the eigenvalues $a \pm bi$ will be

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2\ .$$

Note that, as for second-order ODE's, the complex conjugate eigenvalue $a - bi$ gives up to sign the same two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$.

The expression (3) was not written down for you to memorize, learn, or even use; the point was just for you to get some practice in seeing how a calculation like that in Example 1 looks when written out in general. To actually solve ODE systems having complex eigenvalues, imitate the procedure in Example 1.

Stop at this point, and practice on an example (try Example 3, p. 377).

2. Repeated eigenvalues.

Again we start with the real $n \times n$ system

$$\mathbf{x}' \;=\; A\mathbf{x}\ . \tag{4}$$

We say an eigenvalue λ_1 of A is **repeated** if it is a multiple root of the characteristic equation of A—in other words, the characteristic polynomial $|A - \lambda I|$ has $(\lambda - \lambda_1)^2$ as a factor. Let's suppose that λ_1 is a double root; then we need to find *two* linearly independent solutions to the system (4) corresponding to λ_1.

One solution we can get: we find in the usual way an eigenvector $\vec{\alpha}_1$ corresponding to λ_1 by solving the system

$$(A - \lambda_1 I)\,\mathbf{a} \;=\; \mathbf{0}\ . \tag{5}$$

This gives the solution $\mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}$ to the system (4). Our problem is to find a second solution. To do this we have to distinguish two cases. The first one is easy.

A. The complete case.

Still assuming λ_1 is a real double root of the characteristic equation of A, we say λ_1 is a **complete** eigenvalue if there are two linearly independent eigenvectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ corresponding to λ_1; i.e., if these two vectors are two linearly independent solutions to the system (5). Using them we get two independent solutions to (4), namely

$$\mathbf{x}_1 = \vec{\alpha}_1 e^{\lambda_1 t}, \qquad \text{and} \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{\lambda_1 t}\ . \tag{6}$$

Naturally we would like to see an example of this for $n = 2$, but the following theorem explains why there aren't any good examples: if the matrix A has a repeated eigenvalue, it is so simple that no one would solve the system (4) by fussing with eigenvectors!

Theorem 3.2 *Let A be a 2×2 matrix and λ_1 an eigenvalue. Then*

$$\lambda_1 \text{ is repeated and complete} \quad \Leftrightarrow \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Proof. Let $\vec{\alpha}_1$ and $\vec{\alpha}_2$ be two independent solutions to (5). Then any 2-vector $\vec{\alpha}$ is a solution to (5), for by using the parallelogram law of addition, $\vec{\alpha}$ can be written in the form

$$\vec{\alpha} = c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2\ ,$$

and this shows it is a solution to (5), since $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are:

$$(A - \lambda_1 I)\vec{\alpha} = c_1(A - \lambda_1 I)\vec{\alpha}_1 + c_2(A - \lambda_1 I)\vec{\alpha}_2 = 0 + 0\ .$$

In particular, the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy (5); letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a = \lambda_1 \\ c = 0 \end{cases};$$

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} b = 0 \\ d = \lambda_1 \end{cases}.$$

This proves the theorem in the direction $\Rightarrow$; in the other direction, one sees immediately that the characteristic polynomial is $(\lambda - \lambda_1)^2$, so that λ_1 is a repeated eigenvalue; it is complete since the matrix $A - \lambda_1 I$ has 0 for all its entries, and therefore every 2-vector $\vec{\alpha}$ is a solution to (5).

For $n = 3$ the situation is more interesting. Still assuming λ_1 is a double root of the characteristic equation, it will be a complete eigenvalue when the system (5) has two independent solutions; this will happen when the system (5) has essentially only one equation: the other two equations are constant multiples of it (or identically 0). You can then find two independent solutions to the system just by inspection.

Example 2. If the system (5) turns out to be three equations, each of which is a constant multiple of say

$$2a_1 - a_2 + a_3 = 0\ ,$$

we can give a_1 and a_2 arbitrary values, and then a_3 will be determined by the above equation. Hence two independent solutions (eigenvectors) would be the column 3-vectors

$$(1,0,2)^T \qquad \text{and} \qquad (0,1,1)^T\ .$$

In general, if an eigenvalue λ_1 of A is k-tuply repeated, meaning the polynomial $A-\lambda I$ has the power $(\lambda - \lambda_1)^k$ as a factor, but no higher power, the eigenvalue is called **complete** if it has k independent associated eigenvectors, i.e., if the system (5) has k linearly independent solutions. These then produce k solutions to the ODE system (4).

There is an important theorem in linear algebra (it usually comes at the very end of a linear algebra course) which guarantees that all the eigenvalues of A will be complete, regardless of what their multiplicity is:

Theorem 3.3 *If the real square matrix A is symmetric, meaning $A^T = A$, then all its eigenvalues are real and complete.*

B. The defective case.

If the eigenvalue λ is a double root of the characteristic equation, but the system (5) has only one non-zero solution $\vec{\alpha}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective**, and no second eigenvector exists. In this case, the second solution to the system (4) has a different form. It is

$$\mathbf{x} \;=\; (\vec{\beta} + \vec{\alpha}_1 t)e^{\lambda_1 t}\ , \tag{7}$$

where $\vec{\beta}$ is an unknown vector which must be found. This may be done by substituting (7) into the system (4), and using the fact that $\vec{\alpha}_1$ is an eigenvector, i.e., a solution to (5) when $\lambda = \lambda_1$. When this is done, we find that $\vec{\beta}$ must be a solution to the system

$$(A - \lambda_1 I)\,\vec{\beta} \;=\; \vec{\alpha}_1\ . \tag{8}$$

This is an inhomogeneous system of equations for determining β. It is guaranteed to have a solution, provided that the eigenvalue λ_1 really is defective.

> Notice that (8) doesn't look very solvable, because the matrix of coefficients has determinant zero! So you won't solve it by finding the inverse matrix or by using Cramer's rule. It has to be solved by elimination.

Some people do not bother with (7) or (8); when they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (4) by elimination.

Example. Try Example 2 (section 5.6) in your book; do it by using (7) and (8) above to find β, then check your answer by instead using elimination to solve the ODE system.

Proof of (7) for $n = 2$. Let A be a 2×2 matrix.

Since λ_1 is to be a double root of the characteristic equation, that equation must be

$$(\lambda - \lambda_1)^2 \;=\; 0, \qquad i.e., \qquad \lambda^2 - 2\lambda_1\lambda + \lambda_1^2 \;=\; 0.$$

From the known form of the characteristic equation (LS.2, (25)), we see that

$$\text{trace}\, A \;=\; 2\lambda_1, \qquad \det A \;=\; \lambda_1^2.$$

A convenient way to write two numbers whose sum is $2\lambda_1$ is: $\lambda_1 \pm a$; doing this, we see that our matrix A takes the form

$$A \;=\; \begin{pmatrix} \lambda_1 + a & b \\ c & \lambda_1 - a \end{pmatrix}, \qquad \text{where } bc = -a^2, \quad (\text{since } \det A = \lambda_1^2). \tag{9}$$

Now calculate the eigenvectors of such a matrix A. Note that b and c are not both zero, for if they were, $a = 0$ by (9), and the eigenvalue would be complete. If say $b \neq 0$, we may choose as the eigenvector

$$\vec{\alpha}_1 = \begin{pmatrix} b \\ -a \end{pmatrix},$$

and then by (8), we get

$$\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Exercises: Section 4D

LS.4 Decoupling Systems

1. Changing variables.

A common way of handling mathematical models of scientific or engineering problems is to look for a change of coordinates or a change of variables which simplifies the problem. We handled some types of first-order ODE's — the Bernouilli equation and the homogeneous equation, for instance — by making a change of dependent variable which converted them into equations we already knew how to solve. Another example would be the use of polar or spherical coordinates when a problem has a center of symmetry.

An example from physics is the description of the acceleration of a particle moving in the plane: to get insight into the acceleration vector, a new coordinate system is introduced whose basis vectors are $\mathbf{t}$ and $\mathbf{n}$ (the unit tangent and normal to the motion), with the result that $\mathbf{F} = m\mathbf{a}$ becomes simpler to handle.

We are going to do something like that here. Starting with a homogeneous linear system with constant coefficients, we want to make a linear change of coordinates which simplifies the system. We will work with $n = 2$, though what we say will be true for $n > 2$ also.

How would a simple system look? The simplest system is one with a diagonal matrix: written first in matrix form and then in equation form, it is

$$(1) \qquad \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} u' &= \lambda_1 u \\ v' &= \lambda_2 v \end{aligned} \; .$$

As you can see, if the coefficient matrix has only diagonal entries, the resulting "system" really consists of a set of first-order ODE's, side-by-side as it were, each involving only its own variable. Such a system is said to be **decoupled** since the variables do not interact with each other; each variable can be solved for independently, without knowing anything about the others. Thus, solving the system on the right of (1) gives

$$(2) \qquad \begin{aligned} u &= c_1 e^{\lambda_1 t} \\ v &= c_2 e^{\lambda_2 t} \end{aligned} \; , \qquad \text{or} \qquad \mathbf{u} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 t}.$$

So we start with a 2×2 homogeneous system with constant coefficients,

$$(3) \qquad \mathbf{x}' = A\,\mathbf{x}\,,$$

and we want to introduce new dependent variables u and v, related to x and y by a linear change of coordinates, i.e., one of the form (we write it three ways):

$$(4) \qquad \mathbf{u} = D\,\mathbf{x}\,, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned} \; .$$

We call D the **decoupling matrix**. After the change of variables, we want the system to be decoupled, i.e., to look like the system (1). What should we choose as D?

The matrix D will define the new variables u and v in terms of the old ones x and y. But in order to substitute into the system (3), it is really the inverse to D that we need; we shall denote it by E :

$$(5) \qquad \mathbf{u} = D\mathbf{x}, \qquad \mathbf{x} = E\,\mathbf{u}\,, \qquad E = D^{-1}\,.$$

In the decoupling, we first produce E ; then D is calculated as its inverse. We need both matrices: D to define the new variables, E to do the substitutions.

We are now going to assume that the ODE system $\mathbf{x}' = A\mathbf{x}$ has *two real and distinct eigenvalues*; with their associated eigenvectors, they are denoted as usual in these notes by

$$(6) \qquad \lambda_1, \quad \vec{\alpha}_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \;; \qquad \lambda_2, \quad \vec{\alpha}_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\,.$$

The idea now is the following. Since these eigenvectors are somehow "special" to the system, let us choose the new coordinates so that the eigenvectors become the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the uv-system. To do this, we make the eigenvectors the two columns of the matrix E ; that is, we make the change of coordinates

$$(7) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad E = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\,.$$

With this choice for the matrix E,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the uv-system correspond in the xy-system respectively to the first and second columns of E, as you can see from (7).

We now have to show that this change to the uv-system decouples the ODE system $\mathbf{x}' = A\mathbf{x}$. This rests on the following very important equation connecting a matrix A, one of its eigenvalues λ, and a corresponding eigenvector $\vec{\alpha}$:

$$(8) \qquad A\,\vec{\alpha} = \lambda\,\vec{\alpha}\,,$$

which follows immediately from the equation used to calculate the eigenvector:

$$(A - \lambda I)\,\vec{\alpha} = 0 \qquad \Rightarrow \qquad A\,\vec{\alpha} = (\lambda I)\,\vec{\alpha} = \lambda(I\,\vec{\alpha}) = \lambda\,\vec{\alpha}\,.$$

> The equation (8) is often used as the definition of eigenvector and eigenvalue: an *eigenvector of* A is a vector which changes by some scalar factor λ when multiplied by A; the factor λ is the *eigenvalue* associated with the vector.

As it stands, (8) deals with only one eigenvector at a time. We recast it into the standard form in which it deals with both eigenvectors simultaneously. Namely, (8) says that

$$A\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \qquad A\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\,.$$

These two equations can be combined into the single matrix equation

$$(9) \quad A\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \text{or} \qquad A\,E = E\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

as is easily checked. Note that the diagonal matrix of λ's must be placed on the right in order to multiply the *columns* by the λ's; if we had placed it on the left, it would have multiplied the *rows* by the λ's, which is not what we wanted.

From this point on, the rest is easy. We want to show that the change of variables $\mathbf{x} = E\,\mathbf{u}$ decouples the system $\mathbf{x}' = A\,\mathbf{x}$, where E is defined by (7). We have, substituting $\mathbf{x} = E\,\mathbf{u}$ into the system, the successive equations

$$\begin{aligned} \mathbf{x}' &= A\,\mathbf{x} \\ E\,\mathbf{u}' &= A\,E\,\mathbf{u} \\ E\,\mathbf{u}' &= E\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{u}, \qquad \text{by (9);} \end{aligned}$$

multiplying both sides on the left by $D = E^{-1}$ then shows the system is decoupled:

$$\mathbf{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{u} \quad .$$

Definition. *For a matrix A with two real and distinct eigenvalues, the matrix E in (7) whose columns are the eigenvectors of A is called an* **eigenvector matrix** *for A, and the matrix $D = E^{-1}$ is called the* **decoupling matrix** *for the system $\mathbf{x}' = A\,\mathbf{x}$; the new variables u, v in (7) are called the* **canonical variables**.

> One can alter the matrices by switching the columns, or multiplying a column by a non-zero scalar, with a corresponding alteration in the new variables; apart from that, they are unique.

Example 1. For the system

$$\begin{aligned} x' &= x - y \\ y' &= 2x + 4y \end{aligned}$$

make a linear change of coordinates which decouples the system; verify by direct substitution that the system becomes decoupled.

Solution. In matrix form the system is $\mathbf{x}' = A\,\mathbf{x}$, where $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

We calculate first E, as defined by (7); for this we need the eigenvectors. The characteristic polynomial of A is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)\ ;$$

the eigenvalues and corresponding eigenvectors are, by the usual calculation,

$$\lambda_1 = 2, \quad \vec{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\ ; \qquad \lambda_2 = 3, \quad \vec{\alpha}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The matrix E has the eigenvectors as its columns; then $D = E^{-1}$. We get (cf. LS.1, (2) to calculate the inverse matrix to E)

$$E = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

By (4), the new variables are defined by

$$\mathbf{u} = D\,\mathbf{x}\ , \qquad \begin{aligned} u &= 2x + y \\ v &= -x - y\ . \end{aligned}$$

To substitute these into the system and check they they decouple we use

$$\mathbf{x} = E\,\mathbf{u}\,, \qquad \begin{aligned} x &= u+v \\ y &= -u-2v \end{aligned}\,.$$

Substituting these into the original system (on the left below) gives us the pair of equations on the right:

$$\begin{aligned} x' &= x-y \\ y' &= 2x+4y \end{aligned} \qquad \begin{aligned} u'+v' &= 2u+3v \\ -u'-2v' &= -2u-6v \end{aligned}\;;$$

adding the equations eliminates u; multiplying the top equation by 2 and adding eliminates v, giving the system

$$\begin{aligned} u' &= 2u \\ v' &= 3v \end{aligned}$$

which shows that in the new coordinates the system is decoupled.

The work up to this point assumes that $n = 2$ and the eigenvalues are real and distinct. What if this is not so?

If the eigenvalues are complex, the corresponding eigenvectors will also be complex, i.e., have complex components. All of the above remains formally true, provided we allow all the matrices to have complex entries. This means the new variables u and v will be expressed in terms of x and y using complex coefficients, and the decoupled system will have complex coefficients.

In some branches of science and engineering, this is all perfectly acceptable, and one gets in this way a complex decoupling. If one insists on using real variables only, a decoupling is not possible.

If there is only one (repeated) eigenvalue, there are two cases, as discussed in LS.3 . In the complete case, there are two independent eigenvalues, but as pointed out there (Theorem 3.2), the system will be be automatically decoupled, i.e. A will be a diagonal matrix. In the incomplete case, there is only one eigenvector, and decoupling is impossible (since in the decoupled system, both $\mathbf{i}$ and $\mathbf{j}$ would be eigenvectors).

For $n \geq 3$, real decoupling requires us to find n linearly independent real eigenvectors, to form the columns of the nonsingular matrix E. This is possible if

a) all the eigenvalues are real and distinct, or

b) all the eigenvalues are real, and each repeated eigenvalue is complete.

Repeating the end of LS.3, we note again the important theorem in linear algebra which guarantees decoupling is possible:

Theorem. *If the matrix A is real and symmetric, i.e., $A^T = A$, all its eigenvalues will be real and complete, so that the system $\mathbf{x}' = A\mathbf{x}$ can always be decoupled.*

Exercises: Section 4E

LS.5 Theory of Linear Systems

1. General linear ODE systems and independent solutions.

We have studied the homogeneous system of ODE's with constant coefficients,

$$\mathbf{x}' = A\mathbf{x}\,, \tag{1}$$

where A is an $n \times n$ matrix of constants ($n = 2, 3$). We described how to calculate the eigenvalues and corresponding eigenvectors for the matrix A, and how to use them to find n independent solutions to the system (1).

With this concrete experience solving low-order systems with constant coefficients, what can be said in general when the coefficients are not constant, but functions of the independent variable t? We can still write the linear system in the matrix form (1), but now the matrix entries will be functions of t:

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned}\,, \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

or in more abridged notation, valid for $n \times n$ linear homogeneous systems,

$$\mathbf{x}' = A(t)\,\mathbf{x}\,. \tag{3}$$

Note how the matrix becomes a function of t — we call it a "matrix-valued function" of t, since to each value of t the function rule assigns a matrix:

$$t_0 \;\to\; A(t_0) = \begin{pmatrix} a(t_0) & b(t_0) \\ c(t_0) & d(t_0) \end{pmatrix}$$

In the rest of this chapter we will often not write the variable t explicitly, but it is always understood that the matrix entries are functions of t.

We will sometimes use $n = 2$ or 3 in the statements and examples in order to simplify the exposition, but the definitions, results, and the arguments which prove them are essentially the same for higher values of n.

Definition 5.1 Solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ to (3) are called **linearly dependent** if there are constants c_i, not all of which are 0, such that

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0, \qquad \text{for all } t. \tag{4}$$

If there is no such relation, i.e., if

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) = 0 \quad \text{for all } t \quad \Rightarrow \quad \text{all } c_i = 0, \tag{5}$$

the solutions are called **linearly independent**, or simply *independent*.

The phrase "for all t" is often in practice omitted, as being understood. This can lead to ambiguity; to avoid it, we will use the symbol $\equiv 0$ for **identically 0**, meaning: "zero for all t"; the symbol $\not\equiv 0$ means "not identically 0", i.e., there is some t-value for which it is not zero. For example, (4) would be written

$$c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) \ \equiv\ 0\ .$$

Theorem 5.1 *If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a linearly independent set of solutions to the $n \times n$ system $\mathbf{x}' = A(t)\mathbf{x}$, then the general solution to the system is*

$$\mathbf{x} \ =\ c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n \tag{6}$$

Such a linearly independent set is called a **fundamental** *set of solutions.*

This theorem is the reason for expending so much effort in LS.2 and LS.3 on finding two independent solutions, when $n = 2$ and A is a constant matrix. In this chapter, the matrix A is not constant; nevertheless, (6) is still true.

Proof. There are two things to prove:

(a) All vector functions of the form (6) really are solutions to $\mathbf{x}' \ =\ A\,\mathbf{x}$.

This is the *superposition principle* for solutions of the system; it's true because the system is *linear*. The matrix notation makes it really easy to prove. We have

$$\begin{aligned}
(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n)' \ &=\ c_1\mathbf{x}_1' + \ldots + c_n\mathbf{x}_n' \\
&=\ c_1 A\,\mathbf{x}_1 + \ldots + c_n A\,\mathbf{x}_n, \qquad \text{since } \mathbf{x}_i' = A\,\mathbf{x}_i\ ; \\
&=\ A\,(c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n), \qquad \text{by the distributive law (see LS.1).}
\end{aligned}$$

(b) All solutions to the system are of the form (6).

This is harder to prove, and will be the main result of the next section.

2. The existence and uniqueness theorem for linear systems.

For simplicity, we stick with $n = 2$, but the results here are true for all n. There are two questions that need answering about the general linear system

$$\begin{aligned} x' \ &=\ a(t)x + b(t)y \\ y' \ &=\ c(t)x + d(t)y \end{aligned}\ ; \qquad \text{in matrix form,} \quad \begin{pmatrix} x \\ y \end{pmatrix}' \ =\ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\ . \tag{2}$$

The first is from the previous section: to show that all solutions are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + x_2\mathbf{x}_2,$$

where the $\mathbf{x}_i$ form a fundamental set (i.e., neither is a constant multiple of the other). (The fact that we can write down *all* solutions to a linear system in this way is one of the main reasons why such systems are so important.)

An even more basic question for the system (2) is, how do we know that *has* two linearly independent solutions? For systems with a constant coefficient matrix A, we showed in the previous chapters how to solve them explicitly to get two independent solutions. But the general non-constant linear system (2) does not have solutions given by explicit formulas or procedures.

The answers to these questions are based on following theorem.

Theorem 5.2 Existence and uniqueness theorem for linear systems.

If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0, then the initial value problem

$$\mathbf{x}' = A(t)\,\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

has one and only one solution $\mathbf{x}(t)$ on the interval I.

The proof is difficult, and we shall not attempt it. More important is to see how it is used. The three theorems following answer the questions posed, for the 2×2 system (2). They are true for $n > 2$ as well, and the proofs are analogous.

In the theorems, *we assume the entries of $A(t)$ are continuous on an open interval I*; then the conclusions are valid on the interval I. (For example, I could be the whole t-axis.)

Theorem 5.2A Linear independence theorem.

Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions to (2) on the interval I, such that at some point t_0 in I, the vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly independent. Then

a) the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent on I, and

b) the vectors $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_1)$ are linearly independent at every point t_1 of I.

Proof. a) By contradiction. If they were dependent on I, one would be a constant multiple of the other, say $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$; then $\mathbf{x}_2(t_0) = c_1\mathbf{x}_1(t_0)$, showing them dependent at t_0. □

b) By contradiction. If there were a point t_1 on I where they were dependent, say $\mathbf{x}_2(t_1) = c_1\mathbf{x}_1(t_1)$, then $\mathbf{x}_2(t)$ and $c_1\mathbf{x}_1(t)$ would be solutions to (2) which agreed at t_1, hence by the uniqueness statement in Theorem 5.2, $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$ on all of I, showing them linearly dependent on I. □

Theorem 5.2B General solution theorem.

a) The system (2) has two linearly independent solutions.

b) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two linearly independent solutions, then every solution $\mathbf{x}$ can be written in the form (8), for some choice of c_1 and c_2:

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2; \tag{8}$$

Proof. Choose a point $t = t_0$ in the interval I.

a) According to Theorem 5.2, there are two solutions $\mathbf{x}_1$, $\mathbf{x}_2$ to (3), satisfying respectively the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{i}, \qquad \mathbf{x}_2(t_0) = \mathbf{j}, \tag{9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors in the xy-plane. Since the two solutions are linearly independent when $t = t_0$, they are linearly independent on I, by Theorem 5.2A.

b) Let $\mathbf{u}(t)$ be a solution to (2) on I. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent at t_0 by Theorem 5.2, using the parallelogram law of addition we can find constants c_1' and c_2' such that

$$\mathbf{u}(t_0) = c_1'\mathbf{x}_1(t_0) + c_2'\mathbf{x}_2(t_0). \tag{10}$$

The vector equation (10) shows that the solutions $\mathbf{u}(t)$ and $c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t)$ agree at t_0; therefore by the uniqueness statement in Theorem 5.2, they are equal on all of I, that is,

$$\mathbf{u}(t) = c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t) \qquad \text{on } I.$$

3. The Wronskian

We saw in chapter LS.1 that a standard way of testing whether a set of n n-vectors are linearly independent is to see if the $n \times n$ determinant having them as its rows or columns is non-zero. This is also an important method when the n-vectors are solutions to a system; the determinant is given a special name. (Again, we will assume $n = 2$, but the definitions and results generalize to any n.)

Definition 5.3 *Let* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *be two 2-vector functions. We define their* **Wronskian** *to be the determinant*

$$W(\mathbf{x}_1, \mathbf{x}_2)(t) \;=\; \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \tag{11}$$

whose columns are the two vector functions.

The independence of the two vector functions should be connected with their Wronskian not being zero. At least for points, the relationship is clear; using the result mentioned above, we can say

$$W(\mathbf{x}_1, \mathbf{x}_2)(t_0) \;=\; \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} \;=\; 0 \quad \Leftrightarrow \quad \mathbf{x}_1(t_0) \text{ and } \mathbf{x}_2(t_0) \textit{ are dependent.} \tag{12}$$

However for vector functions, the relationship is clear-cut *only when* $\mathbf{x}_1$ *and* x_2 *are solutions to a well-behaved ODE sysem (2).* The theorem is:

Theorem 5.3 Wronskian vanishing theorem.

On an interval I where the entries of $A(t)$ are continuous, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solutions to (2), and $W(t)$ their Wronskian (11). Then either

a) $W(t) \equiv 0$ on I, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent on I, or

b) $W(t)$ is never 0 on I, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I.

Proof. Using (12), there are just two possibilities.

a) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent on I; say $\mathbf{x}_2 = c_1\mathbf{x}_1$. In this case they are dependent at each point of I, and $W(t) \equiv 0$ on I, by (12);

b) $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on I, in which case by Theorem 5.2A they are linearly independent at each point of I, and so $W(t)$ is never zero on I, by (12). □

Exercises: Section 4E

LS.6 Solution Matrices

In the literature, solutions to linear systems often are expressed using square matrices rather than vectors. You need to get used to the terminology. As before, we state the definitions and results for a 2×2 system, but they generalize immediately to $n \times n$ systems.

1. Fundamental matrices. We return to the system

(1) $$\mathbf{x}' = A(t)\,\mathbf{x} ,$$

with the general solution

(2) $$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) ,$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two independent solutions to (1), and c_1 and c_2 are arbitrary constants.

We form the matrix whose columns are the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$:

(3) $$X(t) = (\,\mathbf{x}_1 \quad \mathbf{x}_2\,) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} .$$

Since the solutions are linearly independent, we called them in LS.5 a *fundamental* set of solutions, and therefore we call the matrix in (3) a **fundamental matrix** for the system (1).

Writing the general solution using X(t). As a first application of $X(t)$, we can use it to write the general solution (2) efficiently. For according to (2), it is

$$\mathbf{x} = c_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} ,$$

which becomes using the fundamental matrix

(4) $$\mathbf{x} = X(t)\,\mathbf{c} \qquad \text{where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} , \qquad (\textit{general solution to (1)}).$$

Note that the vector $\mathbf{c}$ must be written on the right, even though the c's are usually written on the left when they are the coefficients of the solutions $\mathbf{x}_i$.

Solving the IVP using X(t). We can now write down the solution to the IVP

(5) $$\mathbf{x}' = A(t)\,\mathbf{x} , \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Starting from the general solution (4), we have to choose the $\mathbf{c}$ so that the initial condition in (6) is satisfied. Substituting t_0 into (5) gives us the matrix equation for $\mathbf{c}$:

$$X(t_0)\,\mathbf{c} = \mathbf{x}_0 .$$

Since the determinant $|X(t_0)|$ is the value at t_0 of the Wronskian of $\mathbf{x}_1$ amd $\mathbf{x}_2$, it is non-zero since the two solutions are linearly independent (Theorem 5.2C). Therefore the inverse matrix exists (by LS.1), and the matrix equation above can be solved for $\mathbf{c}$:

$$\mathbf{c} = X(t_0)^{-1}\mathbf{x}_0 ;$$

using the above value of $\mathbf{c}$ in (4), the solution to the IVP (1) can now be written

(6) $$\mathbf{x} = X(t)X(t_0)^{-1}\mathbf{x}_0 .$$

Note that when the solution is written in this form, it's "obvious" that $\mathbf{x}(t_0) = \mathbf{x}_0$, i.e., that the initial condition in (5) is satisfied.

An equation for fundamental matrices We have been saying "a" rather than "the" fundamental matrix since the system (1) doesn't have a unique fundamental matrix: thare are many different ways to pick two independent solutions of $\mathbf{x}' = A\,\mathbf{x}$ to form the columns of X. It is therefore useful to have a way of recognizing a fundamental matrix when you see one. The following theorem is good for this; we'll need it shortly.

Theorem 6.1 *$X(t)$ is a fundamental matrix for the system (1) if its determinant $|X(t)|$ is non-zero and it satisfies the matrix equation*

$$X' \;=\; A\,X\;, \tag{7}$$

where X' means that each entry of X has been differentiated.

Proof. Since $|X| \not\equiv 0$, its columns $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, by section LS.5. And writing $X \;=\; (\,\mathbf{x}_1 \quad \mathbf{x}_2\,)$, (7) becomes, according to the rules for matrix multiplication,

$$(\,\mathbf{x}_1' \quad \mathbf{x}_2'\,) \;=\; A\;(\,\mathbf{x}_1 \quad \mathbf{x}_2\,) \;=\; (\,A\mathbf{x}_1 \quad A\mathbf{x}_2\,)\,,$$

which shows that

$$\mathbf{x}_1' \;=\; A\,x_1 \qquad \text{and} \qquad \mathbf{x}_2' \;=\; A\,\mathbf{x}_2\;;$$

this last line says that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the system (1). □

2. The normalized fundamental matrix.

Is there a "best" choice for fundamental matrix?

There are two common choices, each with its advantages. If the ODE system has constant coefficients, and its eigenvalues are real and distinct, then a natural choice for the fundamental matrix would be the one whose columns are the normal modes — the solutions of the form

$$\mathbf{x}_i \;=\; \vec{\alpha}_i e^{\lambda_i t}\;, \qquad i = 1, 2.$$

There is another choice however which is suggested by (6) and which is particularly useful in showing how the solution depends on the initial conditions. Suppose we pick $X(t)$ so that

$$X(t_0) \;=\; I \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\;. \tag{8}$$

Referring to the definition (3), this means the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are picked so

$$\mathbf{x}_1(t_0) \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;, \qquad \mathbf{x}_2(t_0) \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}\;. \tag{8$'$}$$

Since the $\mathbf{x}_i(t)$ are uniquely determined by these initial conditions, the fundamental matrix $X(t)$ satisfying (8) is also unique; we give it a name.

Definition 6.2 The unique matrix $\widetilde{X}_{t_0}(t)$ satisfying

$$\widetilde{X}_{t_0}' \;=\; A\,\widetilde{X}_{t_0}\;, \qquad \widetilde{X}_{t_0}(t_0) \;=\; I \tag{9}$$

is called the **normalized fundamental matrix** at t_0 for A.

For convenience in use, the definition uses Theorem 6.1 to guarantee $\widetilde{X}_{t_0}$ will actually be a fundamental matrix; the condition $|\widetilde{X}_{t_0}(t)| \neq 0$ in Theorem 6.1 is satisfied, since the definition implies $|\widetilde{X}_{t_0}(t_0)| = 1$.

To keep the notation simple, we will assume in the rest of this section that $t_0 = 0$, as it almost always is; then $\widetilde{X}_0$ is the normalized fundamental matrix. Since $\widetilde{X}_0(0) = I$, we get from (6) the matrix form for the solution to an IVP:

(10) *The solution to the IVP* $\quad \mathbf{x}' = A(t)\,\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad$ *is* $\quad \mathbf{x}(t) = \widetilde{X}_0(t)\mathbf{x}_0.$

Calculating $\widetilde{X}_0$. One way is to find the two solutions in $(8')$, and use them as the columns of $\widetilde{X}_0$. This is fine if the two solutions can be determined by inspection.

If not, a simpler method is this: find any fundamental matrix $X(t)$; then

$$\widetilde{X}_0(t) \;=\; X(t)\,X(0)^{-1}. \tag{11}$$

To verify this, we have to see that the matrix on the right of (11) satisfies the two conditions in Definition 6.2. The second is trivial; the first is easy using the rule for matrix differentiation:

If $M = M(t)$ and $B,\ C$ are constant matrices, then $\quad (BM)' = BM', \quad (MC)' = M'C,$

from which we see that since X is a fundamental matrix,

$$(X(t)X(0)^{-1})' = X(t)'X(0)^{-1} = AX(t)X(0)^{-1} = A(X(t)X(0)^{-1}),$$

showing that $X(t)X(0)^{-1}$ also satisfies the first condition in Definition 6.2. □

Example 6.2A Find the solution to the IVP: $\mathbf{x}' \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}\,, \quad \mathbf{x}(0) = \mathbf{x}_0\,.$

Solution Since the system is $x' = y, \;\; y' = -x$, we can find by inspection the fundamental set of solutions satisfying $(8')$:

$$\begin{array}{l} x = \;\;\cos t \\ y = -\sin t \end{array} \quad \text{and} \quad \begin{array}{l} x = \sin t \\ y = \cos t \end{array}\;.$$

Thus by (10) the normalized fundamental matrix at 0 and solution to the IVP is

$$\mathbf{x} \;=\; \widetilde{X}\,\mathbf{x}_0 \;=\; \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \;=\; x_0 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\;.$$

Example 6.2B Give the normalized fundamental matrix at 0 for $\mathbf{x}' \;=\; \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{x}\,.$

Solution. This time the solutions $(8')$ cannot be obtained by inspection, so we use the second method. We calculated the normal modes for this sytem at the beginning of LS.2; using them as the columns of a fundamental matrix gives us

$$X(t) \;=\; \begin{pmatrix} 3e^{2t} & -e^{-2t} \\ e^{2t} & e^{-2t} \end{pmatrix}\;.$$

Using (11) and the formula for calculating the inverse matrix given in LS.1, we get

$$X(0) \;=\; \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \qquad X(0)^{-1} \;=\; \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

so that

$$\widetilde{X}(t) \;=\; \tfrac{1}{4}\begin{pmatrix} 3e^{2t} & -e^{-2t} \\ e^{2t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \;=\; \tfrac{1}{4}\begin{pmatrix} 3e^{2t}+e^{2t} & 3e^{2t}-3e^{-2t} \\ e^{2t}-e^{-2t} & e^{2t}+3e^{-2t} \end{pmatrix}.$$

6.3 The Exponential matrix.

The work in the preceding section with fundamental matrices was valid for any linear homogeneous square system of ODE's,

$$\mathbf{x}' \;=\; A(t)\,\mathbf{x}\,.$$

However, if the system has *constant coefficients*, i.e., the matrix A is a constant matrix, the results are usually expressed by using the exponential matrix, which we now define.

Recall that if x is any real number, then

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,. \tag{12}$$

Definition 6.3 Given an $n \times n$ constant matrix A, the **exponential matrix** e^A is the $n \times n$ matrix defined by

$$e^A \;=\; I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots\,. \tag{13}$$

> Each term on the right side of (13) is an $n \times n$ matrix; adding up the ij-th entry of each of these matrices gives you an infinite series whose sum is the ij-th entry of e^A. (The series always converges.)

In the applications, an independent variable t is usually included:

$$e^{At} \;=\; I + A\,t + A^2\,\frac{t^2}{2!} + \ldots + A^n\,\frac{t^n}{n!} + \ldots\,. \tag{14}$$

This is not a new definition, it's just (13) above applied to the matrix $A\,t$ in which every element of A has been multiplied by t, since for example

$$(At)^2 \;=\; At \cdot At \;=\; A \cdot A \cdot t^2 \;=\; A^2t^2.$$

Try out (13) and (14) on these two examples; the first is worked out in your book (Example 2, p. 417); the second is easy, since it is not an infinite series.

Example 6.3A Let $A \;=\; \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, show: $e^A \;=\; \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$; $e^{At} \;=\; \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

Example 6.3B Let $A \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show: $e^A \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $e^{At} \;=\; \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

What's the point of the exponential matrix? The answer is given by the theorem below, which says that the exponential matrix provides a royal road to the solution of a square system with constant coefficients: no eigenvectors, no eigenvalues, you just write down the answer!

Theorem 6.3 *Let A be a square constant matrix. Then*

(15) *(a)* $e^{At} = \widetilde{X}_0(t)$, *the normalized fundamental matrix at 0;*

(16) *(b) the unique solution to the IVP* $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ *is* $\mathbf{x} = e^{At}\mathbf{x}_0$.

Proof. Statement (16) follows immediately from (15), in view of (10).

We prove (15) is true, by using the description of a normalized fundamental matrix given in Definition 6.2: letting $X = e^{At}$, we must show $X' = AX$ and $X(0) = I$.

The second of these follows from substituting $t = 0$ into the infinite series definition (14) for e^{At}.

To show $X' = AX$, we assume that we can differentiate the series (14) term-by-term; then we have for the individual terms

$$\frac{d}{dt}A^n\frac{t^n}{n!} = A^n \cdot \frac{t^{n-1}}{(n-1)!},$$

since A^n is a constant matrix. Differentiating (14) term-by-term then gives

$$\begin{aligned} \frac{dX}{dt} = \frac{d}{dt}e^{At} &= A + A^2t + \ldots + A^n\frac{t^{n-1}}{(n-1)!} + \ldots \\ &= A\,e^{At} = A\,X . \end{aligned} \tag{18}$$

Calculation of e^{At}.

The main use of the exponential matrix is in (16) — writing down explicitly the solution to an IVP. If e^{At} has to be actually calculated for a specific system, several techniques are available.

a) In simple cases, it can be calculated directly as an infinite series of matrices.

b) It can always be calculated, according to Theorem 6.3, as the normalized fundamental matrix $\widetilde{X}_0(t)$, using (11): $\widetilde{X}_0(t) = X(t)X(0)^{-1}$.

c) A third technique uses the exponential law

$$e^{(B+C)t} = e^{Bt}e^{Ct}, \qquad \textit{valid if } BC = CB. \tag{19}$$

To use it, one looks for constant matrices B and C such that

$$A = B + C, \qquad BC = CB, \qquad e^{Bt} \text{ and } e^{Ct} \text{ are computable;} \tag{20}$$

then

$$e^{At} = e^{B\,t}e^{C\,t}. \tag{21}$$

Example 6.3C Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Solve $\mathbf{x}' = A\,\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, using e^{At}.

Solution. We set $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then (20) is satisfied, and

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

by (21) and Examples 6.3A and 6.3B. Therefore, by (16), we get

$$\mathbf{x} = e^{At}\,\mathbf{x}_0 = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{2t} \begin{pmatrix} 1+2t \\ 2 \end{pmatrix} .$$

Exercises: Sections 4G,H

GS. Graphing ODE Systems

1. The phase plane.

Up to now we have handled systems analytically, concentrating on a procedure for solving linear systems with constant coefficients. In this chapter, we consider methods for sketching graphs of the solutions. The emphasis is on the workd *sketching.* Computers do the work of drawing reasonably accurate graphs. Here we want to see how to get quick qualitative information about the graph, without having to actually calculate points on it.

First some terminology. The sort of system for which we will be trying to sketch the solutions is one which can be written in the form

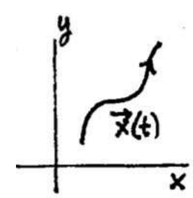
$$(1) \qquad \begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \; .$$

Such a system is called **autonomous**, meaning the independent variable (which we understand to be t) does not appear explicitly on the right, though of course it lurks in the derivatives on the left. The system (1) is a *first-order autonomous system*; it is in *standard form* — the derivatives on the left, the functions on the right.

A **solution** of such a system has the form (we write it two ways):

$$(2) \qquad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \; .$$

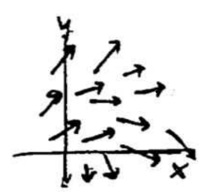
It is a vector function of t, whose components satisfy the system (1) when they are substituted in for x and y. In general, you learned in 18.02 and physics that such a vector function describes a motion in the xy-plane; the equations in (2) tell how the point (x,y) moves in the xy-plane as the time t varies. The moving point traces out a curve called the **trajectory** of the solution (2). The xy-plane itself is called the **phase plane** for the system (1), when used in this way to picture the trajectories of its solutions.



That is how we can picture the solutions (2) to the system; how can we picture the system (1) itself? We can think of the derivative of a solution

$$(3) \qquad \mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

as representing the **velocity vector** of the point (x,y) as it moves according to (2). From this viewpoint, we can interpret geometrically the system (1) as prescribing for each point (x_0, y_0) in the xy-plane a velocity vector having its tail at (x_0, y_0):



$$(4) \qquad \mathbf{x}' = \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} = f(x_0,y_0)\,\mathbf{i} + g(x_0,y_0)\,\mathbf{j} \; .$$

The system (1) is thus represented geometrically as a vector field, the **velocity field**. A solution (2) of the system is a point moving in the xy-plane so that at each point of its trajectory, it has the velocity prescribed by the field. The trajectory itself will be a curve which at each point has the direction of the velocity vector at that point. (An arrowhead is put on the trajectory to show the sense in which t is increasing.)

Sketching trajectories ought to remind you a lot of the work you did drawing integral curves for direction fields to first-order ODE's. What is the relation?

2. First-order autonomous ODE systems and first-order ODE's.

We can eliminate t from the first-order system (1) by dividing one equation by the other. Since by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

we get after the division a single first-order ODE in x and y :

$$(5) \qquad \begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \longrightarrow \qquad \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}.$$

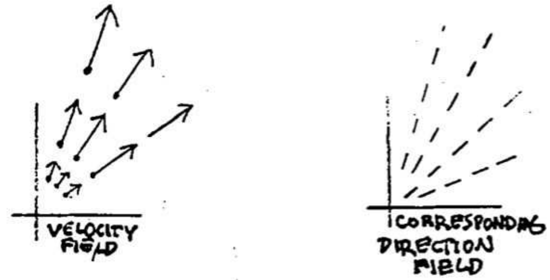
If the first order equation on the right is solvable, this is an important way of getting information about the solutions to the system on the left. Indeed, in the older literature, little distinction was made between the system and the single equation — "solving" meant to solve either one.

There is however a difference between them: the system involves time, whereas the single ODE does not. Consider how their respective solutions are related:

$$(6) \qquad \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \qquad \longrightarrow \qquad F(x,y) = 0\ ,$$

where the equation on the right is the result of eliminating t from the pair of equations on the left. Geometrically, $F(x,y) = 0$ is the equation for the *trajectory* of the solution $\mathbf{x}(t)$ on the left. The trajectory in other words is the path traced out by the moving point $\big(x(t), y(t)\big)$; it doesn't contain any record of how fast the point was moving; it is only the track (or *trace*, as one sometimes says) of its motion.

In the same way, we have the difference between the *velocity field*, which represents the left side of (5), and the *direction field*, which represents the right side. The velocity vectors have magnitude and sense, whereas the line segments that make up the direction field only have slope. The passage from the left side of (5) to the right side is represented geometrically by changing each of the velocity vectors to a line segment of standard length. Even the arrowhead is dropped, since it represents the direction of increasing time, and time has been eliminated; only the slope of the vector is retained.



In considering how to sketch trajectories of the system (1), the first thing to consider are the *critical points* (sometimes called *stationary points*.

Definition 2.1 *A point (x_0, y_0) is a* **critical point** *of the system (1) if*

$$f(x_0, y_0) = 0, \quad \textit{and} \quad g(x_0, y_0) = 0 \tag{7a}$$

or equivalently, if

$$x = x_0, \quad y = y_0 \qquad \textit{is a solution to (1).} \tag{7b}$$

The equations of the system (1) show that (7a) and (7b) are equivalent — either implies the other.

If we adopt the geometric viewpoint, thinking of the system as represented by a velocity vector field, then a critical point is one where the velocity vector is zero. Such a point is a trajectory all by itself, since by not moving it satisfies the equations (1) of the system (this explains the alternative designation "stationary point").

The critical points represent the simplest possible solutions to (1), so you begin by finding them; by (7a), this is done by solving the pair of simultaneous equations

$$\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned} \tag{8}$$

Next, you can try the strategy indicated in (5) of passing to the associated first-order ODE and trying to solve that and sketch the solutions; or you can try to locate some sketchable solutions to (1) and draw them in. None of this is likely to work if the functions $f(x, y)$ and $g(x, y)$ on the right side of the system (1) aren't simple, but for linear equations with constant coefficients, both procedures are helpful as we shall see in the next section.

A principle that was important in sketching integral curves for direction fields applies also to sketching trajectories of the system (1): assuming the functions $f(x, y)$ and $g(x, y)$ are smooth (i.e., have continuous partial derivatives), we have the

(9) **Sketching principle.** *Two trajectories of (1) cannot intersect.*

The sketching principle is a consequence of the existence and uniqueness theorem for systems of the form (1), which implies that in a region where where the partial derivatives of f and g are continuous, through any point passes one and only one trajectory.

3. Sketching some basic linear systems.

We use the above ideas to sketch a few of the simplest linear systems, so as to get an idea of the various possibilities for their trajectories, and introduce the terminology used to describe the resulting geometric pictures.

Example 3.1 Let's consider the linear system on the left below. Its characteristic equation is $\lambda^2 - 1 = 0$, so the eigenvalues are ± 1, and it is easy to see its general solution is the one on the right below:

$$\begin{cases} x' = y \\ y' = x \end{cases} ; \qquad \mathbf{x} \ = \ c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \ . \tag{10}$$

$$(10) \qquad \begin{cases} x' = y \\ y' = x \end{cases} ; \qquad \mathbf{x} \;=\; c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \;.$$

By (8), the only critical point of the system is $(0,0)$. We try the strategy in (5); this converts the system to the first-order ODE below, whose general solution (on the right) is found by separation of variables:

$$(11) \qquad \frac{dy}{dx} \;=\; \frac{x}{y} \; ; \qquad \text{general solution:} \quad y^2 - x^2 \;=\; c \;.$$

Plotted, these are the family of hyperbolas having the diagonal lines $y = \pm x$ as asymptotes; in addition there are the two lines themselves, corresponding to $c = 0$; see fig. 1.

This shows us what the trajectories of (7) must look like, though it does not tell us what the direction of motion is. A further difficulty is that the two lines cross at the origin, which seems to violate the sketching principle (9) above.

We turn therefore to another strategy: plotting simple trajectories that we know. Looking at the general solution in (10), we see that by giving one of the c's the value 0 and the other one the value 1 or -1, we get four easy solutions:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \;, \quad -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \;, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \;, \quad -\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \;.$$

These four solutions give four trajectories which are easy to plot. Consider the first, for example. When $t = 0$, the point is at $(1,1)$. As t increases, the point moves outward along the line $y = x$; as t decreases through negative values, the point moves inwards along the line, toward $(0,0)$. Since t is always understood to be increasing on the trajectory, the whole trajectory consists of the ray $y = x$ in the first quadrant, excluding the origin (which is not reached in finite negative time), the direction of motion being outward.

A similar analysis can be made for the other three solutions; see fig. 2 below.

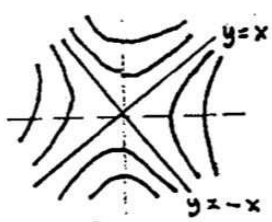


fig. 1

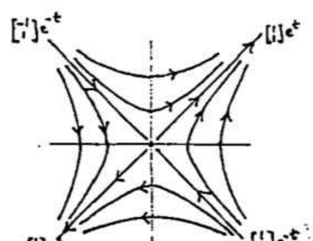


fig 2

As you can see, each of the four solutions has as its trajectory one of the four rays, with the indicated direction of motion, outward or inward according to whether the exponential factor increases or decreases as t increases. There is even a fifth trajectory: the origin itself, which is a stationary point, i.e., a solution all by itself. So the paradox of the intersecting diagonal trajectories is resolved: the two lines are actually five trajectories, no two of which intersect.

Once we know the motion along the four rays, we can put arrowheads on them to indicate the direction of motion along the hyperbolas as t increases, since it must be compatible with the motion along the rays — for by continuity, nearby trajectories must have arrowheads pointing in similar directions. The only possibility therefore is the one shown in fig. 2.

A linear system whose trajectories show the general features of those in fig. 2 is said to be an **unstable saddle**. It is called *unstable* because the trajectories go off to infinity as t increases (there are three exceptions: what are they?); it is called a *saddle* because of its general resemblance to the level curves of a saddle-shaped surface in 3-space.

Example 3.2 This time we consider the linear system below — since it is decoupled, its general solution (on the right) can be obtained easily by inspection:

$$(12) \qquad \begin{cases} x' = -x \\ y' = -2y \end{cases} \qquad \mathbf{x} \;=\; c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \;.$$

Converting it as in (5) to a single first order ODE and solving it by separating variables gives as the general solutions (on the right below) a family of parabolas:

$$\frac{dy}{dx} \;=\; \frac{2y}{x}; \qquad\qquad y = cx^2 \;.$$

Following the same plan as in Example 3.1, we single out the four solutions

$$(13) \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \;, \quad -\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \;, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \;, \quad -\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \;.$$

Their trajectories are the four rays along the coordinate axes, the motion being always inward as t increases. Put compatible arrowheads on the parabolas and you get fig. 3.

A linear system whose trajectories have the general shape of those in fig. 3 is called an **asymptotically stable node** or a **sink node**. The word *node* is used when the trajectories have a roughly parabolic shape (or exceptionally, they are rays); *asymptotically stable* or *sink* means that all the trajectories approach the critical point as t increases.

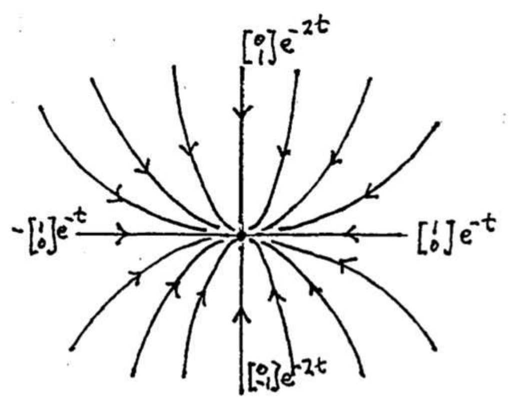


fig. 3

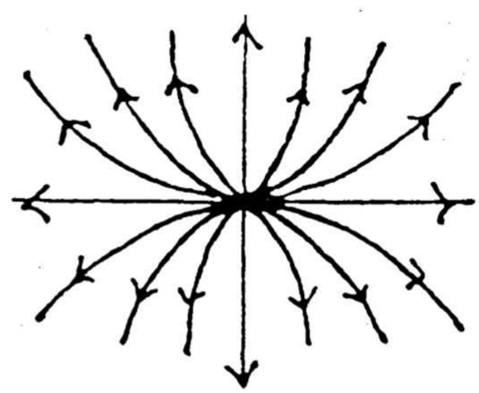


fig. 4

Example 3.3 This is the same as Example 3.2, except that the signs are reversed:

$$(12) \qquad \begin{cases} x' = x \\ y' = 2y \end{cases} \qquad \mathbf{x} \;=\; c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \;.$$

The first-order differential equation remains the same; we get the same parabolas. The only difference in the work is that the exponentials now have positive exponents; the picture remains exactly the same except that now the trajectories are all traversed in the opposite direction — away from the origin — as t increases. The resulting picture is fig. 4, which we call an **unstable node** or **source node**.

Example 3.4 A different type of simple system (eigenvalues $\pm i$) and its solution is

$$(14) \qquad \begin{cases} x' = y \\ y' = -x \end{cases} ; \qquad \mathbf{x} \;=\; c_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} .$$

Converting to a first-order ODE by (5) and solving by separation of variables gives

$$\frac{dy}{dx} \;=\; -\frac{x}{y}, \qquad\qquad x^2 + y^2 = c \;;$$

the trajectories are the family of circles centered at the origin. To determine the direction of motion, look at the solution in (14) for which $c_1 = 0,\ c_2 = 1$; it is the reflection in the y-axis of the usual (counterclockwise) parametrization of the circle; hence the motion is *clockwise* around the circle. An even simpler procedure is to determine a single vector in the velocity field — that's enough to determine all of the directions. For example, the velocity vector at $(1,0)$ is $< 0, -1 >= -\mathbf{j}$, again showing the motion is clockwise. (The vector is drawn in on fig. 5, which illustrates the trajectories.)

This type of linear system is called a **stable center** . The word *stable* signifies that any trajectory stays within a bounded region of the phase plane as t increases or decreases indefinitely. (We cannot use "asymptotically stable," since the trajectories do not approach the critical point $(0,0)$ as t increases. The word *center* describes the geometric configuration: it would be used also if the curves were ellipses having the origin as center.

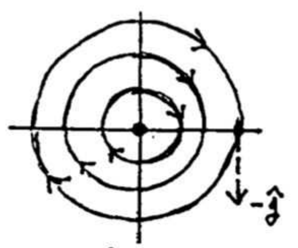


fig. 5


fig. 6

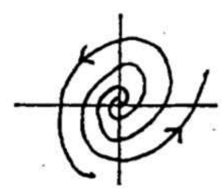


fig. 7

Example 3.5 As a last example, a system having a complex eigenvalue $\lambda = -1 + i$ is, with its general solution,

$$(15) \qquad \begin{cases} x' = -x + y \\ y' = -x - y \end{cases} \qquad \mathbf{x} \;=\; c_1 e^{-t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} .$$

The two fundamental solutions (using $c_1 = 0$ and $c_1 = 1$, and vice-versa) are typical. They are like the solutions in Example 3.4, but multiplied by e^{-t}. Their trajectories are therefore traced out by the tip of an origin vector that rotates clockwise at a constant rate, while its magnitude shrinks exponentially to 0: the trajectories spiral in toward the origin as t increases. We call this pattern an **asymptotically stable spiral** or a *sink spiral*; see fig. 6. (An older terminology uses *focus* instead of spiral.)

To determine the direction of motion, it is simplest to do what we did in the previous example: determine from the ODE system a single vector of the velocity field: for instance, the system (15) has at $(1,0)$ the velocity vector $-\mathbf{i} - \mathbf{j}$, which shows the motion is clockwise.

For the system $\begin{cases} x' = \ \ x + y \\ y' = -x + y \end{cases}$, an eigenvalue is $\lambda = 1 + i$, and in (15) e^t replaces e^{-t}; the magnitude of the rotating vector increases as t increases, giving as pattern an **unstable spiral**, or *source spiral*, as in fig. 7.

4. Sketching more general linear systems.

In the preceding section we sketched trajectories for some particular linear systems; they were chosen to illustrate the different possible geometric pictures. Based on that experience, we can now describe how to sketch the general system

$$\mathbf{x}' = A\mathbf{x}, \qquad A = 2 \times 2 \text{ constant matrix.}$$

The geometric picture is largely determined by the eigenvalues and eigenvectors of A, so there are several cases.

For the first group of cases, we suppose the eigenvalues λ_1 and λ_2 are *real* and *distinct*.

Case 1. *The λ_i have opposite signs:* $\lambda_1 > 0$. $\lambda_2 < 0$; **unstable saddle**.

Suppose the corresponding eigenvectors are $\vec{\alpha}_1$ and $\vec{\alpha}_2$, respectively. Then four solutions to the system are

$$\mathbf{x} = \pm\vec{\alpha}_1 e^{\lambda_1 t}, \qquad \mathbf{x} = \pm\vec{\alpha}_2 e^{\lambda_2 t} . \tag{16}$$

How do the trajectories of these four solutions look?

In fig. 8 below, the four vectors $\pm\vec{\alpha}_1$ and $\pm\vec{\alpha}_2$ are drawn as origin vectors; in fig. 9, the corresponding four trajectories are shown as solid lines, with the direction of motion as t increases shown by arrows on the lines. The reasoning behind this is the following.

Look first at $\mathbf{x} = \vec{\alpha}_1 e^{\lambda_1 t}$. We think of $e^{\lambda_1 t}$ as a scalar factor changing the length of $\mathbf{x}$; as t increases from $-\infty$ to ∞, this scalar factor increases from 0 to ∞, since $\lambda_1 > 0$. The tip of this lengthening vector represents the trajectory of the solution $\mathbf{x} = \vec{\alpha}_1 e^{\lambda_1 t}$, which is therefore a ray going out from the origin in the direction of the vector $\vec{\alpha}_1$.

Similarly, the trajectory of $\mathbf{x} = -\vec{\alpha}_1 e^{\lambda_1 t}$ is a ray going out from the origin in the opposite direction: that of the vector $-\vec{\alpha}_1$.

The trajectories of the other two solutions $\mathbf{x} = \pm\vec{\alpha}_2 e^{\lambda_2 t}$ will be similar, except that since $\lambda_2 < 0$, the scalar factor $e^{\lambda_2 t}$ decreases as t increases; thus the solution vector will be shrinking as t increases, so the trajectory traced out by its tip will be a ray having the direction of $\vec{\alpha}_2$ or $-\vec{\alpha}_2$, but traversed toward the origin as t increases, getting arbitrarily close but never reaching it in finite time.

To complete the picture, we sketch in some nearby trajectories; these will be smooth curves generally following the directions of the four rays described above. In Example 3.1 they were hyperbolas; in general they are not, but they look something like hyperbolas, and they do have the rays as asymptotes. They are the trajectories of the solutions

$$\mathbf{x} = c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}, \tag{17}$$

for different values of the constants c_1 and c_2.

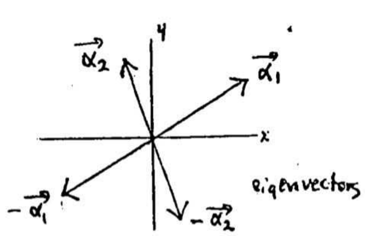


fig. 8

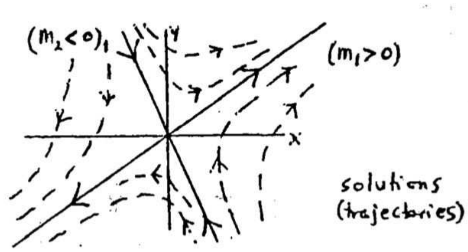


fig. 9

Case 2. λ_1 and λ_2 are distinct and negative: say $\lambda_1 < \lambda_2 < 0$;

asymptotically stable *(sink)* **node**

Formally, the solutions (16) are written the same way, and we draw their trajectories just as before. The only difference is that now all four trajectories are represented by rays coming in towards the origin as t increases, because both of the λ_i are negative. The four trajectories are represented as solid lines in figure 10, on the next page.

The trajectories of the other solutions (17) will be smooth curves which generally follow the four rays. In the corresponding Example 3.2, they were parabolas; here too they will be parabola-like, but this does not tell us how to draw them, and a little more thought is needed. The parabolic curves will certainly come in to the origin as t increases, but tangent to which of the rays? Briefly, the answer is this:

Node-sketching principle. *Near the origin, the trajectories follow the ray attached to the λ_i nearer to zero; far from the origin, they follow (i.e. are roughly parallel to) the ray attached to the λ_i further from zero.*

You need not memorize the above; instead learn the reasoning on which it is based, since this type of argument will be used over and over in science and engineering work having nothing to do with differential equations.

Since we are assuming $\lambda_1 < \lambda_2 < 0$, it is λ_2 which is closer to 0. We want to know the behavior of the solutions near the origin and far from the origin. Since all solutions are approaching the origin,

near the origin corresponds to large positive t (we write $t \gg 1$):
far from the origin corresponds to large negative t (written $t \ll -1$).

As before, the general solution has the form

$$\mathbf{x} \;=\; c_1\vec{\alpha}_1 e^{\lambda_1 t} + c_2\vec{\alpha}_2 e^{\lambda_2 t}, \qquad \lambda_1 < \lambda_2 < 0. \tag{18}$$

If $t \gg 1$, then $\mathbf{x}$ is near the origin, since both terms in (18) are small; however, the first term is negligible compared with the second: for since $\lambda_1 - \lambda_2 < 0$, we have

$$\frac{e^{\lambda_1 t}}{e^{\lambda_2 t}} \;=\; e^{(\lambda_1-\lambda_2)t} \;\approx\; 0, \qquad t \gg 1\ . \tag{19}$$

Thus if $\lambda_1 < \lambda_2 < 0$ and $t \gg 1$, we can neglect the first term of (18), getting

$$\mathbf{x} \;\sim\; c_2\vec{\alpha}_2 e^{\lambda_2 t}. \qquad \text{for } \ t \gg 1 \quad (\mathbf{x} \ \text{ near the origin}),$$

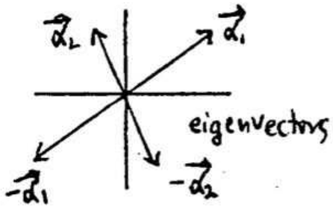
which shows that $\mathbf{x}(t)$ follows the ray corresponding to the the eigenvalue λ_2 closer to zero.

Similarly, if $t \ll -1$, then $\mathbf{x}$ is far from the origin since both terms in (18) are large. This time the ratio in (19) is large, so that it is the first term in (18) that dominates the expression, which tells us that

$$\mathbf{x} \;\sim\; c_1\vec{\alpha}_1 e^{\lambda_1} t. \qquad \text{for } \ t \ll -1 \quad (\mathbf{x} \ \text{ far from the origin}).$$

This explains the reasoning behind the node-sketching principle in this case.

Some of the trajectories of the solutions (18) are sketched in dashed lines in figure 10, using the node-sketching principle, and assuming $\lambda_1 < \lambda_2 < 0$.



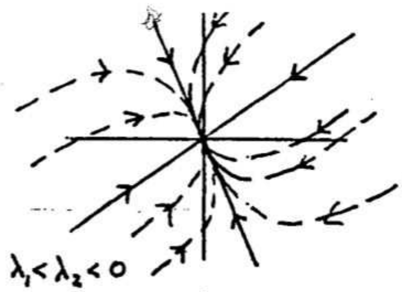


fig. 10

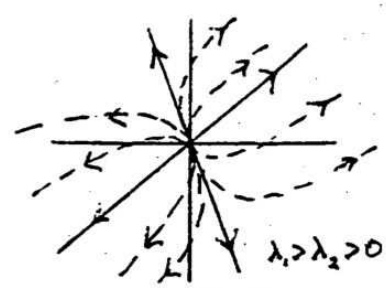


fig. 11

Case 3. λ_1 and λ_2 are distinct and positive: say $\lambda_1 > \lambda_2 > 0$ **unstable** *(source)* **node**

The analysis is like the one we gave above. The direction of motion on the four rays coming from the origin is outwards, since the $\lambda_i > 0$. The node-sketching principle is still valid, and the reasoning for it is like the reasoning in case 2. The resulting sketch looks like the one in fig. 11.

Case 4. Eigenvalues pure imaginary: $\lambda = \pm bi, \quad b > 0$ **stable center**

Here the solutions to the linear system have the form

$$\mathbf{x} = \mathbf{c}_1 \cos bt + \mathbf{c}_2 \sin bt, \qquad \mathbf{c}_1, \mathbf{c}_2 \text{ constant vectors .} \tag{20}$$

(There is no exponential factor since the real part of λ is zero.) Since every solution (20) is periodic, with period $2\pi/b$, the moving point representing it retraces its path at intervals of $2\pi/b$. The trajectories therefore are closed curves; ellipses, in fact; see fig. 12.

Sketching the ellipse is a little troublesome, since the vectors $\mathbf{c}_i$ do not have any simple relation to the major and minor axes of the ellipse. For this course, it will be enough if you determine whether the motion is clockwise or counterclockwise. As in Example 3.4, this can be done by using the system $\mathbf{x}' = A\mathbf{x}$ to calculate a single velocity vector $\mathbf{x}'$ of the velocity field; from this the sense of motion can be determined by inspection.

> The word *stable* means that each trajectory stays for all time within some circle centered at the critical point; *asymptotically stable* is a stronger requirement: each trajectory must approache the critical point (here, the origin) as $t \to \infty$.

Case 5. The eigenvalues are complex, but not purely imaginary; there are two cases:

$a \pm bi, \quad a < 0, \ b > 0;$ **asymptotically stable** *(sink)* **spiral**;

$a \pm bi, \quad a > 0, \ b > 0;$ **unstable** *(source)* **spiral**;

Here the solutions to the linear system have the form

$$\mathbf{x} = e^{at}(\mathbf{c}_1 \cos bt + \mathbf{c}_2 \sin bt), \qquad \mathbf{c}_1, \mathbf{c}_2 \text{ constant vectors .} \tag{21}$$

They look like the solutions (20), except for a scalar factor e^{at} which either

decreases towards 0 as $t \to \infty \quad (a < 0)$, *or*

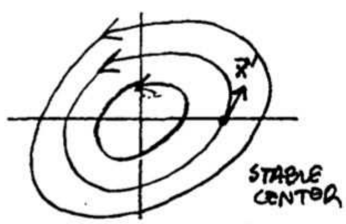
increases towards ∞ *as* $t \to \infty \quad (a > 0)$.

Thus the point $\mathbf{x}$ travels in a trajectory which is like an ellipse, except that the distance from the origin keeps steadily shrinking or expanding. The result is a trajectory which does one of the following:

spirals steadily towards the origin, *(asymptotically stable spiral)* : $a < 0$

spirals steadily away from the origin. *(unstable spiral)*; $a > 0$

The exact shape of the spiral, is not obvious and perhaps best left to computers; but you should determine the direction of motion, by calculating from the linear system $\mathbf{x}' = A\mathbf{x}$ a single velocity vector $\mathbf{x}'$ near the origin. Typical spirals are pictured (figs. 12, 13).



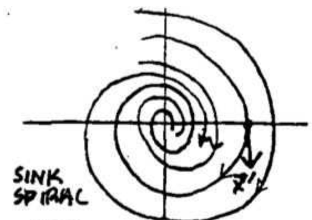


fig. 12

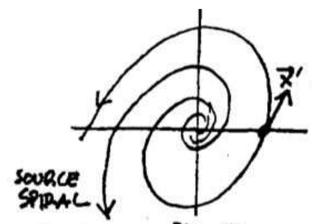


fig. 13

Other cases.

Repeated real eigenvalue $\lambda \neq 0$, defective (incomplete: one independent eigenvector)
defective node; *unstable* if $\lambda > 0$; *asymptotically stable* if $\lambda < 0$ (fig. 14)

Repeated real eigenvalue $\lambda \neq 0$, complete (two independent eigenvectors)
star node; *unstable* if $\lambda > 0$; *asymptotically stable* if $\lambda > 0$. (fig. 15)

One eigenvalue $\lambda = 0$. (Picture left for exercises and problem sets.)

5. Summary

To sum up, the procedure of sketching trajectories of the 2×2 linear homogeneous system $\mathbf{x}' = A\mathbf{x}$, where A is a constant matrix, is this. Begin by finding the eigenvalues of A.

1. If they are real, distinct, and non-zero:

a) find the corresponding eigenvectors;

b) draw in the corresponding solutions whose trajectories are rays; use the sign of the eigenvalue to determine the direction of motion as t increases; indicate it with an arrowhead on the ray;

c) draw in some nearby smooth curves, with arrowheads indicating the direction of motion:

(i) if the eigenvalues have opposite signs, this is easy;

(ii) if the eigenvalues have the same sign, determine which is the dominant term in the solution for $t \gg 1$ and $t \ll -1$, and use this to to determine which rays the trajectories are tangent to, near the origin, and which rays they are parallel to, away from the origin. (Or use the node-sketching principle.)

2. If the eigenvalues are complex: $a \pm bi$, the trajectories will be
ellipses if $a = 0$,
spirals if $a \neq 0$: inward if $a < 0$, outward if $a > 0$;

in all cases, determine the direction of motion by using the system $\mathbf{x}' = A\mathbf{x}$ to find one velocity vector.

3. The details in the other cases (eigenvalues repeated, or zero) will be left as exercises using the reasoning in this section.

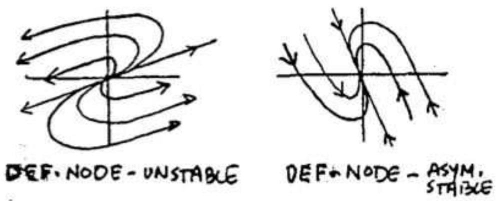


fig. 14

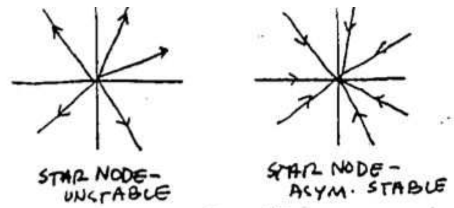


fig. 15

6. Sketching non-linear systems

In sections 3, 4, and 5, we described how to sketch the trajectories of a linear system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad a, b, c, d \;\; \text{constants.}$$

We now return to the general (i.e., non-linear) 2×2 autonomous system discussed at the beginning of this chapter, in sections 1 and 2:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \quad ; \tag{22}$$

it is represented geometrically as a vector field, and its trajectories — the solution curves — are the curves which at each point have the direction prescribed by the vector field. Our goal is to see how one can get information about the trajectories of (22), without determining them analytically or using a computer to plot them numerically.

Linearizing at the origin. To illustrate the general idea, let's suppose that $(0,0)$ is a critical point of the system (22), i.e.,

$$f(0,0) = 0, \qquad g(0,0) = 0, \tag{23}$$

Then if f and g are sufficiently differentiable, we can approximate them near $(0,0)$ (the approximation will have no constant term by (23)):

$$\begin{aligned} f(x,y) &= a_1 x + b_1 y + \;\; \text{higher order terms in } x \text{ and } y \\ g(x,y) &= a_2 x + b_2 y + \;\; \text{higher order terms in } x \text{ and } y. \end{aligned}$$

If (x,y) is close to $(0,0)$, then x and y will be small and we can neglect the higher order terms. Then the non-linear system (23) is approximated near $(0,0)$ by a linear system, the **linearization** of (23) at (0,0):

$$\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned} \quad , \tag{24}$$

and near (0,0), the solutions of (22) — about which we know nothing — will be like the solutions to (24), about which we know a great deal from our work in the previous sections.

Example 6.1 Linearize the system $\begin{cases} x' = y\cos x \\ y' = x(1+y)^2 \end{cases}$ at the critical point $(0,0)$.

Solution We have $\begin{cases} x' \approx y(1-\frac{1}{2}x^2) \\ y' = x(1+2y+y^2) \end{cases}$ so the linearization is $\begin{cases} x' = y \\ y' = x \end{cases}$.

Linearising at a general point More generally, suppose now the critical point of (22) is (x_0, y_0), so that

$$f(x_0,y_0) = 0, \qquad g(x_0,y_0) = 0.$$

One way this can be handled is to make the change of variable

$$x_1 = x - x_0, \qquad y_1 = y - y_0; \tag{24}$$

in the x_1y_1-coordinate system, the critical point is $(0,0)$, and we can proceed as before.

Example 6.2 Linearize $\begin{cases} x' = x - x^2 - 2xy \\ y' = y - y^2 - \dfrac{3}{2}xy \end{cases}$ at its critical points on the x-axis.

Solution. When $y = 0$, the functions on the right are zero when $x = 0$ and $x = 1$, so the critical points on the x-axis are $(0,0)$ and $(1,0)$.

The linearization at $(0,0)$ is $x' = x, \quad y' = y$.

To find the linearization at $(1,0)$ we change of variable as in (24): $x_1 = x - 1, \quad y_1 = y$; substituting for x and y in the system and keeping just the linear terms on the right gives us as the linearization:

$$\begin{aligned} x_1' &= (x_1+1) - (x_1+1)^2 - 2(x_1+1)y_1 &&\approx -x_1 - 2y_1 \\ y_1' &= y_1 - y_1^2 - \tfrac{3}{2}(x_1+1)y_1 &&\approx -\tfrac{1}{2}y_1 \ . \end{aligned}$$

Linearization using the Jacobian matrix

Though the above techniques are usable if the right sides are very simple, it is generally faster to find the linearization by using the Jacobian matrix, especially if there are several critical points, or the functions on the right are not simple polynomials. We derive the procedure.

We need to approximate f and g near (x_0, y_0). While this can sometimes be done by changing variable, a more basic method is to use the main approximation theorem of multivariable calculus. For this we use the notation

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta f = f(x,y) - f(x_0,y_0) \tag{25}$$

and we have then the basic approximation formula

$$\Delta f \approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y, \qquad \text{or}$$

$$f(x,y) \approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y \ , \tag{26}$$

since by hypothesis $f(x_0, y_0) = 0$. We now make the change of variables (24)

$$x_1 = x - x_0 = \Delta x, \qquad y_1 = y - y_0 = \Delta y,$$

and use (26) to approximate f and g by their linearizations at (x_0, y_0). The result is that in the neighborhood of the critical point (x_0, y_0), the linearization of the system (22) is

$$(27)\qquad \begin{aligned} x_1' &= \left(\frac{\partial f}{\partial x}\right)_0 x_1 + \left(\frac{\partial f}{\partial y}\right)_0 y_1, \\ y_1' &= \left(\frac{\partial g}{\partial x}\right)_0 x_1 + \left(\frac{\partial g}{\partial y}\right)_0 y_1. \end{aligned}$$

In matrix notation, the linearization is therefore

$$(28)\qquad \mathbf{x}_1' = A\,\mathbf{x}_1, \qquad \text{where} \quad \mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0,y_0)};$$

the matrix A is the Jacobian matrix, evaluated at the critical point (x_0, y_0).

General procedure for sketching the trajectories of non-linear systems.

We can now outline how to sketch in a qualitative way the solution curves of a 2×2 non-linear autonomous system,

$$(29)\qquad \begin{aligned} x' &= f(x, y) \\ y' &= g(x, y). \end{aligned}$$

1. Find all the critical points (i.e., the constant solutions), by solving the system of simultaneous equations

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0\ . \end{aligned}$$

2. For each critical point (x_0, y_0), find the matrix A of the linearized system at that point, by evaluating the Jacobian matrix at (x_0, y_0):

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0,y_0)}.$$

(Alternatively, make the change of variables $x_1 = x - x_0, \quad y_1 = y - y_0$, and drop all terms having order higher than one; then A is the matrix of coefficients for the linear terms.)

3. Find the geometric type and stability of the linearized system at the critical point point (x_0, y_0), by carrying out the analysis in sections 4 and 5.

> *The subsequent steps require that the eigenvalues be* **non-zero, real, and distinct**, *or* **complex, with a non-zero real part**. *The remaining cases: eigenvalues which are* **zero, repeated, or pure imaginary** *are classified as borderline, and the subsequent steps don't apply, or have limited application. See the next section.*

4. According to the above, the acceptable geometric types are a saddle, node (not a star or a defective node, however), and a spiral. Assuming that this is what you have, for each critical point determine enough additional information (eigenvectors, direction of motion) to allow a sketch of the trajectories near the critical point.

5. In the xy-plane, mark the critical points. Around each, sketch the trajectories in its immediate neighborhood, as determined in the previous step, including the direction of motion.

6. Finally, sketch in some other trajectories to fill out the picture, making them compatible with the behavior of the trajectories you have already sketched near the critical points. Mark with an arrowhead the direction of motion on each trajectory.

> If you have made a mistake in analyzing any of the critical points, it will often show up here — it will turn out to be impossible to draw in any plausible trajectories that complete the picture.

Remarks about the steps.

1. In the homework problems, the simultaneous equations whose solutions are the critical points will be reasonably easy to solve. In the real world, they may not be; a simultaneous-equation solver will have to be used (the standard programs — MatLab, Maple, Mathematica, Macsyma — all have them, but they are not always effective.)

2. If there are several critical points, one almost always uses the Jacobian matrix; if there is only one, use your judgment.

3. This method of analyzing non-linear systems rests on the assumption that in the neighborhood of a critical point, the non-linear system will look like its linearization at that point. For the borderline cases this may not be so — that is why they are rejected. The next two sections explain this more fully.

If one or more of the critical points turn out to be borderline cases, one usually resorts to numerical computation on the non-linear system. Occasionally one can use the reduction (section 2) to a first-order equation:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

to get information about the system.

Example 6.3 Sketch some trajectories of the system $\begin{cases} x' = -x + xy \\ y' = -2y + xy \end{cases}$.

Solution. We first find the critical points, by solving $\begin{cases} -x + xy = x(-1+y) = 0 \\ -2y + xy = y(-2+x) = 0 \end{cases}$.

From the first equation, either $x = 0$ or $y = 1$. From the second equation,

$$x = 0 \;\Rightarrow\; y = 0; \qquad y = 1 \;\Rightarrow\; x = 2; \qquad \text{critical points}: \;\; (0,0),\; (2,1).$$

To linearize at the critical points, we compute the Jacobian matrices

$$J = \begin{pmatrix} -1+y & x \\ y & -2+x \end{pmatrix}; \qquad J_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad J_{(2,1)} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Analyzing the geometric type and stability of each critical point:

$(0,0)$: eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = -2$ sink node

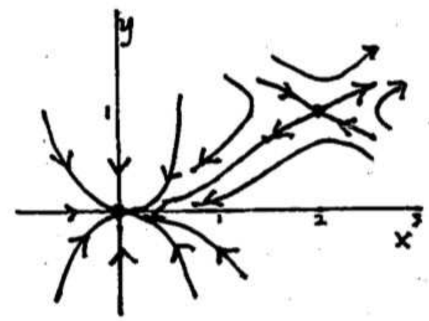
eigenvectors: $\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \vec{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

By the node-sketching principle, trajectories follow $\vec{\alpha}_1$ near the origin, are parallel to $\vec{\alpha}_2$ away from the origin.

$(2,1)$: eigenvalues: $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$ unstable saddle

eigenvectors: $\vec{\alpha}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$; $\vec{\alpha}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

Draw in these eigenvectors at the respective points $(0,0)$ and $(2,1)$, indicating direction of motion (into the critical point if $\lambda < 0$, away from $\lambda > 0$.) Draw in some nearby trajectories.



Then guess at some other trajectories compatible with these. See the figure for one attempt at this. Further information could be gotten by considering the associated first-order ODE in x and y.

7. Structural Stability

In the previous two sections, we described how to get a rough picture of the trajectories of a non-linear system by linearizing at each of its critical points. The basic assumption of the method is that the linearized system will be a good approximation to the original non-linear system if you stay near the critical point.

The method only works however if the linearized system turns out to be a node, saddle, or spiral. What is it about these geometric types that allows the method to work, and why won't it work if the linearized system turns out to be one of the other possibilities (dismissed as "borderline types" in the previous section)?

Briefly, the answer is that nodes, saddles, and spirals are *structurally stable*, while the other possibilities are not. We call a system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

structurally stable if small changes in its parameters (i.e., the constants that enter into the functions on the right hand side) do not change the geometric type and stability of its critical points (or its limit cycles, if there are any — see the next section, and don't worry about them for now.)

Theorem. *The* 2×2 *autonomous linear system*

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{2}$$

is structurally stable if it is a spiral, saddle, or node (but not a degenerate or star node).

Proof. The characteristic equation is

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0,$$

and its roots (the eigenvalues) are

$$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}. \tag{3}$$

Let's look at the cases one-by-one; assume first that the roots λ_1 and λ_2 are real and distinct. The possibilities in the theorem are given by the following (note that since the roots are distinct, the node will not be degenerate or a star node):

$$\begin{aligned} &\lambda_1 > 0,\ \lambda_2 > 0 \ \textit{unstable node} \\ &\lambda_1 < 0,\ \lambda_2 < 0 \ \textit{asymptotically stable node} \\ &\lambda_1 > 0,\ \lambda_2 < 0 \ \textit{unstable saddle.} \end{aligned}$$

The quadratic formula (3) shows that the roots depend continuously on the coefficients a, b, c, d. Thus if the coefficients are changed a little, the roots λ_1 and λ_2 will also be changed a little to λ_1' and λ_2' respectively; the new roots will still be real, and will have the same sign if the change is small enough. Thus the changed system will still have the same geometric type and stability. □

If the roots of the characteristic equation are complex, the reasoning is similar. Let us denote the complex roots by $r \pm si$; we use the root $\lambda = r + si,\ s > 0$; then the possibilities to be considered for structural stability are

$$
\begin{array}{ll}
r > 0,\ s > 0 & \textit{unstable spiral} \\
r < 0,\ s > 0 & \textit{asymptotically stable spiral.}
\end{array}
$$

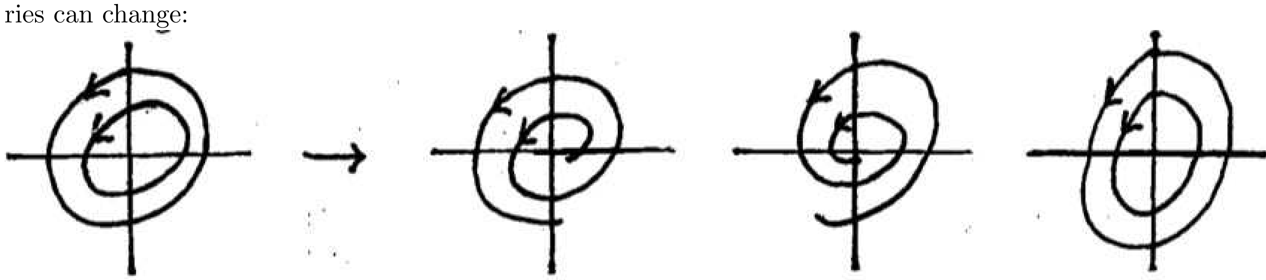
If a, b, c, d are changed a little, the root is changed to $\lambda' = r' + s'i$, where r' and s' are close to r and s respectively, since the quadratic formula (3) shows r and s depend continuously on the coefficients. If the change is small enough, r' will have the same sign as r and s' will still be positive, so the geometric type of the changed system will still be a spiral, with the same stability type. □□

8. The borderline geometric types All the other possibilities for the linear system (2) we call borderline types. We will show now that none of them is structurally stable; we begin with the center.

Eigenvalues pure imaginary. Once again we use the eigenvalue with the positive imaginary part: $\lambda = 0 + si,\ s > 0$. It corresponds to a *center*: the trajectories are a family of concentric ellipses, centered at the origin. If the coefficients a, b, c, d are changed a little, the eigenvalue $0 + si$ changes a little to $r' + s'i$, where $r' \approx 0, s' \approx s$, and there are three possibilities for the new eigenvalue:

$$
\begin{array}{llll}
0 + si \ \to \ r' + s'i: & r' > 0 & r' < 0 & r' = 0 \\
s > 0 & s' > 0 & s' > 0 & s' > 0 \\
\textit{center} & \textit{source spiral} & \textit{sink spiral} & \textit{center}
\end{array}
$$

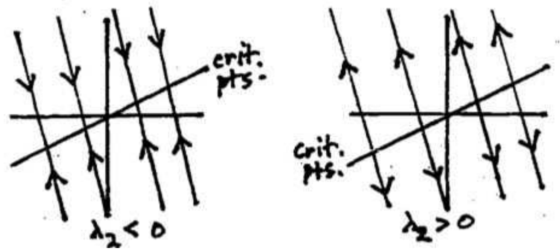
Correspondingly, there are three possibilities for how the geometric picture of the trajectories can change:



Eigenvalues real; one eigenvalue zero. Here $\lambda_1 = 0$, and $\lambda_2 > 0$ or $\lambda_2 < 0$. The general solution to the system has the form (α_1, α_2 are the eigenvectors)

$$\mathbf{x} \ = \ c_1\alpha_1 + c_2\alpha_2 e^{\lambda_2 t}.$$

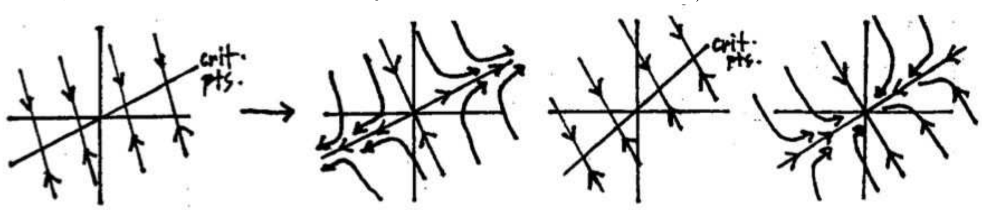
If $\lambda_2 < 0$, the geometric picture of its trajectories shows a line of critical points (constant solutions, corresponding to $c_2 = 0$), with all other trajectories being parallel lines ending up (for $t = \infty$) at one of the critical points, as shown below.



We continue to assume $\lambda_2 < 0$. As the coefficients of the system change a little, the two eigenvalues change a little also; there are three possibilities, since the 0 eigenvalue can become positive, negative, or stay zero:

$\lambda_1 = 0 \ \rightarrow \ \lambda_1'$:	$\lambda_1' > 0$	$\lambda_1' = 0$	$\lambda_1 < 0$
$\lambda_2 < 0 \ \rightarrow \ \lambda_2'$:	$\lambda_2' < 0$	$\lambda_2' < 0$	$\lambda_2' < 0$
critical line	*unstable saddle*	*critical line*	*sink node*

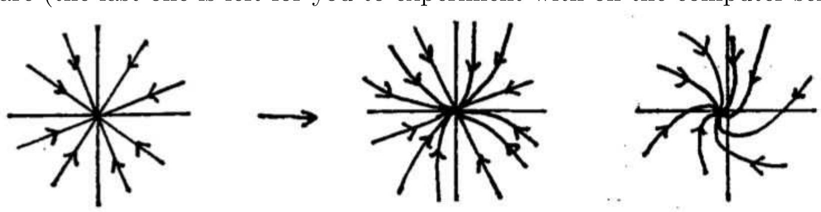
Here are the corresponding pictures. (The pictures would look the same if we assumed $\lambda_2 > 0$, but the arrows on the trajectories would be reversed.)



One repeated real eigenvalue. Finally, we consider the case where $\lambda_1 = \lambda_2$. Here there are a number of possibilities, depending on whether λ_1 is positive or negative, and whether the repeated eigenvalue is complete (i.e., has two independent eigenvectors), or defective (i.e., incomplete: only one eigenvector). Let us assume that $\lambda_1 < 0$. We vary the coefficients of the system a little. By the same reasoning as before, the eigenvalues change a little, and by the same reasoning as before, we get as the main possibilities (omitting this time the one where the changed eigenvalue is still repeated):

$\lambda_1 < 0$	$\rightarrow$	$\lambda_1' < 0$	$r + si$
$\lambda_2 < 0$	$\rightarrow$	$\lambda_2' < 0$	$r - si$
$\lambda_1 = \lambda_2$		$\lambda_1' \neq \lambda_2'$	$r \approx \lambda_1, s \approx 0,$
sink node		*sink node*	*sink spiral*

Typical corresponding pictures for the complete case and the defective (incomplete) case are (the last one is left for you to experiment with on the computer screen)



complete: star node *incomplete: defective node*

Remarks. Each of these three cases—one eigenvalue zero, pure imaginary eigenvalues, repeated real eigenvalue—has to be looked on as a borderline linear system: altering the coefficients slightly can give it an entirely different geometric type, and in the first two cases, possibly alter its stability as well.

Application to non-linear systems.

All the preceding analysis discussed the structural stability of a linear system. How does it apply to non-linear systems?

Suppose our non-linear system has a critical point at P, and we want to study its trajectories near P by linearizing the system at P.

This linearization is only an approximation to the original system, so if it turns out to be a borderline case, i.e., one sensitive to the exact value of the coefficients, *the trajectories near P of the original system can look like any of the types obtainable by slightly changing the coefficients of the linearization.*

It could also look like a combination of types. For instance, if the linearized system had a critical line (i.e., one eigenvalue zero), the original system could have a sink node on one half of the critical line, and an unstable saddle on the other half. (This actually occurs.)

In other words, the method of linearization we used in Sections 6 and 7 to analyze a non-linear system near a critical point doesn't fail entirely, but we don't end up with a definite picture of the non-linear system near P; we only get a list of possibilities. In general one has to rely on computation or more powerful analytic tools to get a clearer answer. The first thing to try is a computer picture of the non-linear system, which often will give the answer.

LC. Limit Cycles

1. Introduction.

In analyzing non-linear systems in the xy-plane, we have so far concentrated on finding the critical points and analysing how the trajectories of the system look in the neighborhood of each critical point. This gives some feeling for how the other trajectories can behave, at least those which pass near anough to critical points.

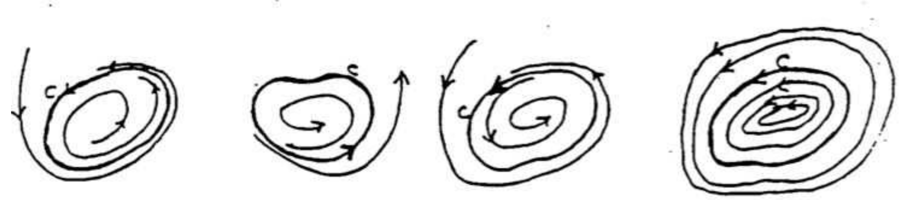
Another important possibility which can influence how the trajectories look is if one of the trajectories traces out a closed curve C. If this happens, the associated solution $\mathbf{x}(t)$ will be geometrically realized by a point which goes round and round the curve C with a certain period T. That is, the solution vector

$$\mathbf{x}(t) = (x(t), y(t))$$

will be a pair of periodic functions with period T :

$$x(t+T) = x(t), \qquad y(t+T) = y(t) \qquad \text{for all} \quad t.$$

If there is such a closed curve, the nearby trajectories must behave something like C. The possibilities are illustrated below. The nearby trajectories can either spiral in toward C, they can spiral away from C, or they can themselves be closed curves. If the latter case does not hold — in other words, if C is an *isolated* closed curve — then C is called a *limit cycle*: stable, unstable, or semi-stable according to whether the nearby curves spiral towards C, away from C, or both.



STABLE LIMIT CYCLE | UNSTABLE LIMIT CYCLE | SEMI-STABLE LIMIT CYCLE | NEUTRALLY-STABLE CENTER

The most important kind of limit cycle is the stable limit cycle, where nearby curves spiral towards C on both sides. Periodic processes in nature can often be represented as stable limit cycles, so that great interest is attached to finding such trajectories if they exist. Unfortunately, surprisingly little is known about how to do this, or how to show that a system has no limit cycles. There is active research in this subject today. We will present a few of the things that are known.

2. Showing limit cycles exist.

The main tool which historically has been used to show that the system

$$x' = f(x, y)$$
$$y' = g(x, y) \tag{1}$$

has a stable limit cycle is the

Poincare-Bendixson Theorem *Suppose R is the finite region of the plane lying between two simple closed curves D_1 and D_2, and* $\mathbf{F}$*is the velocity vector field for the system (1). If*

(i) at each point of D_1 and D_2, the field $\mathbf{F}$*points toward the interior of R, and*

(ii) R contains no critical points,

then the system (1) has a closed trajectory lying inside R.

The hypotheses of the theorem are illustrated by fig. 1. We will not give the proof of the theorem, which requires a background in Mathematical Analysis. Fortunately, the theorem strongly appeals to intuition. If we start on one of the boundary curves, the solution will enter R, since the velocity vector points into the interior of R. As time goes on, the solution can never leave R, since as it approaches a boundary curve, trying to escape from R, the velocity vectors are always pointing inwards, forcing it to stay inside R. Since the solution can never leave R, the only thing it can do as $t \to \infty$ is either approach a critical point — but there are none, by hypothesis — or spiral in towards a closed trajectory. Thus there is a closed trajectory inside R. (It cannot be an unstable limit cycle—it must be one of the other three cases shown above.)

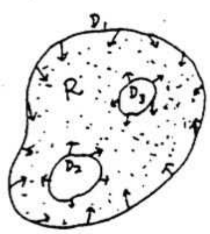


fig. 1.

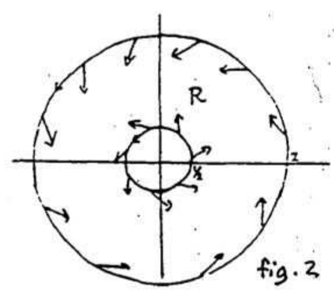


fig. 2

To use the Poincare-Bendixson theorem, one has to search the vector field for closed curves D along which the velocity vectors all point towards the same side. Here is an example where they can be found.

Example 1. Consider the system

$$x' = -y + x(1 - x^2 - y^2)$$
$$y' = x + y(1 - x^2 - y^2) \tag{2}$$

Figure 2 shows how the associated velocity vector field looks on two circles. On a circle of radius 2 centered at the origin, the vector field points inwards, while on a circle of radius 1/2, the vector field points outwards. To prove this, we write the vector field along a circle of radius r as

$$\mathbf{x}' = (-y\,\mathbf{i} + x\,\mathbf{j}) + (1 - r^2)(x\,\mathbf{i} + y\,\mathbf{j})\ . \tag{3}$$

The first vector on the right side of (3) is tangent to the circle; the second vector points radially *in* for the big circle ($r = 2$), and radially *out* for the small circle ($r = 1/2$). Thus the sum of the two vectors given in (3) points inwards along the big circle and outwards along the small one.

We would like to conclude that the Poincare-Bendixson theorem applies to the ring-shaped region between the two circles. However, for this we must verify that R contains no critical points of the system. We leave you to show as an exercise that $(0, 0)$ is the only critical point of the system; this shows that the ring-shaped region contains no critical points.

The above argument shows that the Poincare-Bendixson theorem can be applied to R, and we conclude that R contains a closed trajectory. In fact, it is easy to verify that $x = \cos t,\ y = \sin t$ solves the system, so the unit circle is the locus of a closed trajectory. We leave as another exercise to show that it is actually a stable limit cycle for the system, and the only closed trajectory.

3. Non-existence of limit cycles

We turn our attention now to the negative side of the problem of showing limit cycles exist. Here are two theorems which can sometimes be used to show that a limit cycle does *not* exist.

Bendixson's Criterion *If f_x and g_y are continuous in a region R which is simply-connected (i.e., without holes), and*

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \text{at any point of } R,$$

then the system

$$x' = f(x, y) \qquad (4)$$
$$y' = g(x, y)$$

has no closed trajectories inside R.

Proof. Assume there is a closed trajectory C inside R. We shall derive a contradiction, by applying Green's theorem, in its normal (flux) form. This theorem says

$$(5) \qquad \oint_C (f\,\mathbf{i} + g\,\mathbf{j}) \cdot \mathbf{n}\,ds \quad \equiv \quad \oint_C f\,dy - g\,dx \quad = \quad \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\,dy\ .$$

where D is the region inside the simple closed curve C.

This however is a contradiction. Namely, by hypothesis, the integrand on the right-hand side is continuous and never 0 in R; thus it is either always positive or always negative, and the right-hand side of (5) is therefore either positive or negative.

On the other hand, the left-hand side must be zero. For since C is a closed trajectory, C is always tangent to the velocity field $f\,\mathbf{i} + g\,\mathbf{j}$ defined by the system. This means the normal vector $\mathbf{n}$ to C is always perpendicular to the velocity field $f\,\mathbf{i} + g\,\mathbf{j}$, so that the integrand $f(f\,\mathbf{i} + g\,\mathbf{j}) \cdot \mathbf{n}$ on the left is identically zero.

This contradiction means that our assumption that R contained a closed trajectory of (4) was false, and Bendixson's Criterion is proved. $\square$

Critical-point Criterion *A closed trajectory has a critical point in its interior.*

If we turn this statement around, we see that it is really a criterion for *non-existence*: it says that *if a region R is simply-connected (i.e., without holes) and has no critical points, then it cannot contain any limit cycles.* For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in R since R has no holes.

(Note carefully the distinction between this theorem, which says that limit cycles enclose regions which *do* contain critical points, and the Poincare-Bendixson theorem, which seems to imply that limit cycles tend to lie in regions which *don't* contain critical points. The difference is that these latter regions always contain a hole; the critical points are in the hole. Example 1 illustrated this.

Example 2. For what a and d does $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$ have closed trajectories?

Solution. By Bendixson's criterion, $a + d \neq 0 \quad \Rightarrow$ no closed trajectories.

What if $a + d = 0$? Bendixson's criterion says nothing. We go back to our analysis of the linear system in Notes LS. The characteristic equation of the system is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \ .$$

Assume $a + d = 0$. Then the characteristic roots have opposite sign if $ad - bc < 0$ and the system is a saddle; the roots are pure imaginary if $ad - bc > 0$ and the system is a center, which has closed trajectories. Thus

$$\textit{the system has closed trajectories} \quad \Leftrightarrow \quad a + d = 0, \quad ad - bc > 0.$$

4. The Van der Pol equation.

An important kind of second-order non-linear autonomous equation has the form

$$x'' + u(x)\, x' + v(x) = 0 \qquad \textit{(Liénard equation)}\ . \tag{6}$$

One might think of this as a model for a spring-mass system where the damping force $u(x)$ depends on position (for example, the mass might be moving through a viscous medium of varying density), and the spring constant $v(x)$ depends on how much the spring is stretched—this last is true of all springs, to some extent. We also allow for the possibility that $u(x) < 0$ (i.e., that there is "negative damping").

The system equivalent to (6) is

$$\begin{aligned} x' &= y \\ y' &= -v(x) - u(x)\, y \end{aligned} \tag{7}$$

Under certain conditions, the system (7) has a unique stable limit cycle, or what is the same thing, the equation (6) has a unique periodic solution; and all nearby solutions tend towards this periodic solution as $t \to \infty$. The conditions which guarantee this were given by Liénard, and generalized in the following theorem.

Levinson-Smith Theorem *Suppose the following conditions are satisfied.*

(a) *$u(x)$ is even and continuous,*

(b) *$v(x)$ is odd, $v(x) > 0$ if $x > 0$, and $v(x)$ is continuous for all x,*

(c) *$V(x) \to \infty$ as $x \to \infty$, where $V(x) = \int_0^x v(t)\,dt$,*

(d) *for some $k > 0$, we have*

$$\left.\begin{array}{ll} U(x) < 0, & \text{for } 0 < x < k, \\ U(x) > 0 \text{ and increasing}, & \text{for } x > k, \\ U(x) \to \infty, & \text{as } x \to \infty, \end{array}\right\} \qquad \text{where} \quad U(x) = \int_0^x u(t)\,dt.$$

Then, the system (7) has

i) a unique critical point at the origin;

ii) a unique non-zero closed trajectory C, which is a stable limit cycle around the origin;

iii) all other non-zero trajectories spiralling towards C as $t \to \infty$.

We omit the proof, as too difficult. A classic application is to the equation

$$x'' - a(1 - x^2)\,x' + x = 0 \qquad \textit{(van der Pol equation)} \tag{8}$$

which describes the current $x(t)$ in a certain type of vacuum tube. (The constant a is a positive parameter depending on the tube constants.) The equation has a unique non-zero periodic solution. Intuitively, think of it as modeling a non-linear spring-mass system. When $|x|$ is large, the restoring and damping forces are large, so that $|x|$ should decrease with time. But when $|x|$ gets small, the damping becomes negative, which should make $|x|$ tend to increase with time. Thus it is plausible that the solutions should oscillate; that it has exactly one periodic solution is a more subtle fact.

There is a lot of interest in limit cycles, because of their appearance in systems which model processes exhibiting periodicity. Not a great deal is known about them.

For instance, it is not known how many limit cycles the system (1) can have when $f(x, y)$ and $g(x, y)$ are quadratic polynomials. In the mid-20th century, two well-known Russian mathematicians published a hundred-page proof that the maximum number was three, but a gap was discovered in their difficult argument, leaving the result in doubt; twenty years later the Chinese mathematician Mingsu Wang constructed a system with four limit cycles. The two quadratic polynomials she used contain both very large and very small coefficients; this makes numerical computation difficult, so there is no computer drawing of the trajectories.

There the matter currently rests. Some mathematicians conjecture the maximum number of limit cycles is four, others six, others conjecture that there is no maximum. For autonomous systems where the right side has polynomials of degree higher than two, even less is known. There is however a generally accepted proof that for any particular system for which $f(x, y)$ and $g(x, y)$ are polynomials, the number of limit cycles is finite.

Exercises: Section 5D

Frequency Response

1. Introduction

We will examine the response of a second order linear constant coefficient system to a sinusoidal input. We will pay special attention to the way the output changes as the frequency of the input changes. This is what we mean by the *frequency response* of the system. In particular, we will look at the *amplitude response* and the *phase response*; that is, the amplitude and phase lag of the system's output considered as functions of the input frequency.

In O.4 the Exponential Input Theorem was used to find a particular solution in the case of exponential or sinusoidal input. Here we will work out in detail the formulas for a second order system. We will then interpret these formulas as the frequency response of a mechanical system. In particular, we will look at damped-spring-mass systems. We will study carefully two cases: first, when the mass is driven by pushing on the spring and second, when the mass is driven by pushing on the dashpot.

Both these systems have the same form

$$p(D)x = q(t),$$

but their amplitude responses are very different. This is because, as we will see, it can make physical sense to designate something other than $q(t)$ as the input. For example, in the system

$$mx' + bx' + kx = by'$$

we will consider y to be the input. (Of course, y is related to the expression on the right-hand-side of the equation, but it is not exactly the same.)

2. Sinusoidally Driven Systems: Second Order Constant Coefficient DE's

We start with the second order linear constant coefficient (CC) DE, which as we've seen can be interpreted as modeling a **damped forced harmonic oscillator**. If we further specify the oscillator to be a mechanical system with mass m, damping coefficient b, spring constant k, and with a *sinusoidal* driving force $B\cos\omega t$ (with B constant), then the DE is

$$mx'' + bx' + kx = B\cos\omega t. \tag{1}$$

For many applications it is of interest to be able to predict the periodic response of the system to various values of ω. From this point of view we can picture having a *knob* you can turn to set the input frequency ω, and a screen where we can see how the shape of the system response changes as we turn the ω-knob.

The Exponential Input Theorem (O.4 (4), and see O.4 example 2) tells us how to find a particular solution to (1):

Characteristic polynomial: $p(r) = mr^2 + br + k$.

Complex replacement: $m\tilde{x}'' + b\tilde{x}' + k\tilde{x} = Be^{i\omega t}$, $x = \mathrm{Re}(\tilde{x})$.

Exponential Input Theorem: $\tilde{x}_p = \dfrac{Be^{i\omega t}}{p(i\omega)} = \dfrac{Be^{i\omega t}}{k - m\omega^2 + ib\omega}$

thus,

$$x_p \;=\; \mathrm{Re}(\tilde{x}_p) \;=\; \frac{B}{|p(i\omega)|}\cos(\omega t-\phi) \;=\; \frac{B}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}\cos(\omega t-\phi), \qquad (2)$$

where $\phi = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\dfrac{b\omega}{k-m\omega^2}\right)$. (In this case ϕ must be between 0 and π. We say ϕ is in the first or second quadrants.)

Letting $A = \dfrac{B}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}$, we can write the periodic response x_p as

$$x_p = A\cos(\omega t-\phi).$$

The *complex gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *complexified* equation, is

$$\tilde{g}(\omega) = \frac{1}{p(i\omega)} \;=\; \frac{1}{k-m\omega^2+ib\omega}.$$

The *gain*, which is defined as the ratio of the amplitude of the output to the amplitude of the input in the *real* equation, is

$$g = g(\omega) = \frac{1}{\mid p(i\omega)\mid} \;=\; \frac{1}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}. \qquad (3)$$

The *phase lag* is

$$\phi = \phi(\omega) = \mathrm{Arg}(p(i\omega) = \tan^{-1}(\frac{b\omega}{k-m\omega^2}) \qquad (4)$$

and we also have the *time lag* = ϕ/ω.

Terminology of Frequency Response

We call the gain $g(\omega)$ the **amplitude response** of the system. The phase lag $\phi(\omega)$ is called the **phase response** of the system. We refer to them collectively as the **frequency response** of the system.

Notes:

1. Observe that the whole DE scales by the input amplitude B.

2. All that is needed about the input for these formulas to be valid is that it is of the form (*constant*) $\times$ (a *sinusoidal* function). Here we have used the notation $B\cos\omega t$ but the amplitude factor in front of the cosine function can take any form, including having the constants depend on the system parameters and/or on ω. (And of course one could equally-well use $\sin\omega t$, or any other shift of cosine, for the sinusoid.) This point is very important in the physical applications of this DE and we will return to it again.

3. Along the same lines as the preceding: we always define the gain as the *the amplitude of the periodic output divided by the amplitude of the periodic input*. Later we will see examples where the gain is *not* just equal to $\frac{1}{p(i\omega)}$ (for complex gain) or $\frac{1}{|p(i\omega)|}$ (for real gain) – stay tuned!

3. Frequency Response and Practical Resonance

The gain or amplitude response to the system (1) is a function of ω. It tells us the size of the system's response to the given input frequency. If the amplitude has a peak at ω_r we call this the **practical resonance frequency.** If the damping b gets too large then, for the system in equation (1), there is no peak and, hence, no practical resonance. The following figure shows two graphs of $g(\omega)$, one for small b and one for large b.

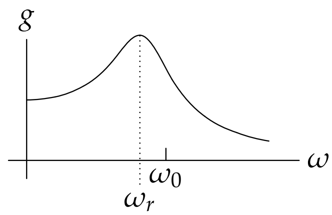


Fig 1a. Small b (has resonance).

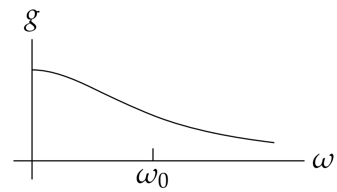


Fig 1b. Large b (no resonance)

In figure (1a) the damping constant b is small and there is practical resonance at the frequency ω_r. In figure (1b) b is large and there is no practical resonant frequency.

Finding the Practical Resonant Frequency.
We now turn our attention to finding a formula for the practical resonant frequency -if it exists- of the system in (1). Practical resonance occurs at the frequency ω_r where $g(w)$ has a maximum. For the system (1) with gain (3) it is clear that the maximum gain occurs when the expression under the radical has a minimum. Accordingly we look for the minimum of

$$f(\omega) = (k - m\omega^2)^2 + b^2\omega^2.$$

Setting $f'(\omega) = 0$ and solving gives

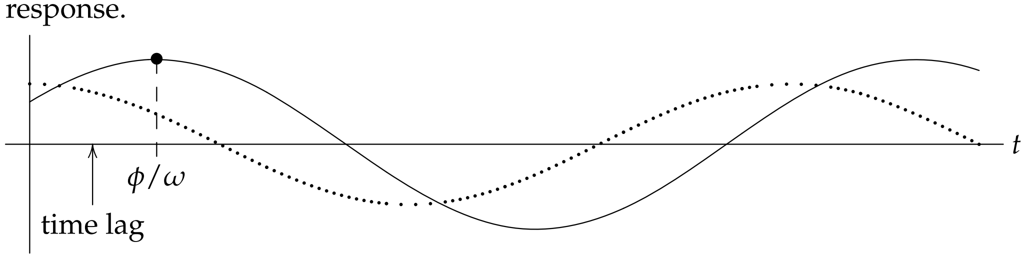
$$\begin{aligned} f'(\omega) &= -4m\omega(k - m\omega^2) + 2b^2\omega = 0 \\ &\Rightarrow \omega = 0 \text{ or } m^2\omega^2 = mk - b^2/2. \end{aligned}$$

We see that if $mk - b^2/2 > 0$ then there is a practical resonant frequency

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}. \tag{5}$$

Phase Lag:
In the picture below the dotted line is the input and the solid line is the response.

The damping causes a lag between when the input reaches its maximum and when the output does. In radians, the angle ϕ is called the *phase lag* and in units of time ϕ/ω is the *time lag*. The lag is important, but in this class we will be more interested in the amplitude response.



4. Mechanical Vibration System: Driving Through the Spring

The figure below shows a spring-mass-dashpot system that is driven through the spring.

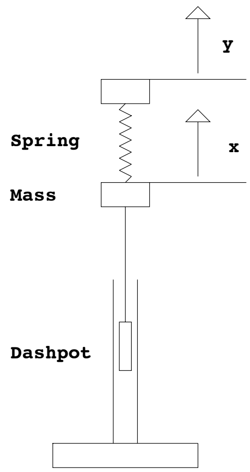


Figure 1. Spring-driven system

Suppose that y denotes the displacement of the plunger at the top of the spring and $x(t)$ denotes the position of the mass, arranged so that $x = y$ when the spring is unstretched and uncompressed. There are two forces acting on the mass: the spring exerts a force given by $k(y - x)$ (where k is the spring constant) and the dashpot exerts a force given by $-bx'$ (against the motion of the mass, with damping coefficient b). Newton's law gives

$$mx'' = k(y - x) - bx'$$

or, putting the system on the left and the driving term on the right,

$$mx'' + bx' + kx = ky\,. \tag{6}$$

In this example it is natural to regard y, rather than the right-hand side $q = ky$, as the input signal and the mass position x as the system response. Suppose that y is sinusoidal, that is,

$$y = B_1 \cos(\omega t).$$

Then we expect a sinusoidal solution of the form

$$x_p = A\cos(\omega t - \phi).$$

By definition the *gain* is the ratio of the amplitude of the system response to that of the input signal. Since B_1 is the amplitude of the input we have $g = A/B_1$.

In equations (3) and (4) we gave the formulas for g and ϕ for the system (1). We can now use them with the following small change. The k on the right-hand-side of equation (6) needs to be included in the gain (since we don't include it as part of the input). We get

$$\begin{aligned} g(\omega) &= \frac{A}{B_1} = \frac{k}{|p(i\omega)|} = \frac{k}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \\ \phi(\omega) &= \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right). \end{aligned}$$

Note that the gain is a function of ω, i.e. $g = g(\omega)$. Similarly, the *phase lag* $\phi = \phi(\omega)$ is a function of ω. The entire story of the steady state system response $x_p = A\cos(\omega t - \phi)$ to sinusoidal input signals is encoded in these two functions of ω, the gain and the phase lag.

We see that choosing the input to be y instead of ky scales the gain by k and does not affect the phase lag.

The factor of k in the gain does not affect the frequency where the gain is greatest, i.e. the practical resonant frequency. From (5) we know this is

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

Note: Another system leading to the same equation is a series RLC circuit. We will favor the mechanical system notation, but it is interesting to note the mathematics is exactly the same for both systems.

5. Mechanical Vibration System: Driving Through the Dashpot

Now suppose instead that we fix the top of the spring and drive the system by moving the bottom of the dashpot instead.

Suppose that the position of the bottom of the dashpot is given by $y(t)$ and the position of the mass is given by $x(t)$, arranged so that $x = 0$ when the spring is relaxed. Then the force on the mass is given by

$$mx'' = -kx + b\,\frac{d}{dt}(y - x)$$

since the force exerted by a dashpot is supposed to be proportional to the speed of the piston moving through it. This can be rewritten as

$$mx'' + bx' + kx = by'\,. \tag{7}$$

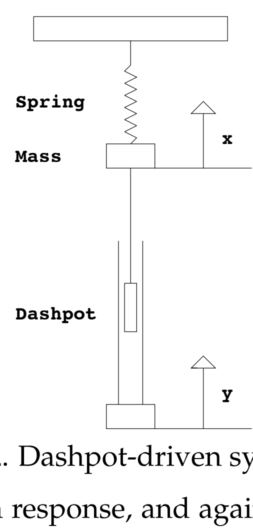


Figure 2. Dashpot-driven system

We will consider x as the system response, and again on physical grounds we specify as the input signal the position y of the back end of the dashpot. Note that the *derivative* of the input signal (multiplied by b) occurs on the right hand side of the equation.

Again we suppose that the input signal is of sinusoidal form

$$y = B_1 \cos(\omega t).$$

We will now work out the frequency response analysis of this problem.

First, $y = B_1 \cos(\omega t) \Rightarrow y' = -\omega B_1 \sin(\omega t)$, so our equation is

$$mx'' + bx' + kx = -b\omega B_1 \sin(\omega t)\,. \tag{8}$$

We know that the periodic system response will be sinusoidal, and as usual we choose the amplitude-phase form with the cosine function

$$x_p = A\cos(\omega t - \phi)\,.$$

Since $y = B_1 \cos(\omega t)$ was chosen as the input, the gain g is given by $g = \frac{A}{B_1}$.

As usual, we compute the gain and phase lag ϕ by making a complex replacement.

One natural choice would be to regard $q(t) = -b\omega B_1 \sin(\omega t)$ as the imaginary part of a complex equation. This would work, but we must keep in mind that the input signal is $B_1 \cos(\omega t)$ and also that we want to express the solution x_p as $x_p = A\cos(\omega t - \phi)$.

Instead we will go back to equation (7) and complexify before taking the derivative of the right-hand-side. Our input $y = B_1 \cos(\omega t)$ becomes $\tilde{y} = B_1 e^{i\omega t}$ and the DE becomes

$$mz'' + bz' + kz = b\tilde{y}' = i\omega b B_1 e^{i\omega t}. \tag{9}$$

Since $y = \mathrm{Re}(\tilde{y})$ we have $x = \mathrm{Re}(z)$; that is, the sinusoidal system response x_p of (8) is the real part of the exponential system response z_p of (9). The Exponential Input Theorem gives

$$z_p = \frac{i\omega b B_1}{p(i\omega)}\, e^{i\omega t}$$

where

$$p(s) = ms^2 + bs + k$$

is the characteristic polynomial.

The complex gain (scale factor that multiplies the input signal to get the output signal) is

$$\tilde{g}(\omega) = \frac{i\omega b}{p(i\omega)}.$$

Thus, $z_p = B_1 \tilde{g}(\omega) e^{i\omega t}$.

We can write $\tilde{g} = |\tilde{g}| e^{-i\phi}$, where $\phi = -\mathrm{Arg}(\tilde{g})$. (We use the minus sign so ϕ will come out as the phase lag.) Substitute this expression into the formula for z_p to get

$$z_p = B_1 |\tilde{g}|\, e^{i(\omega t - \phi)}.$$

Taking the real part we have

$$x_p = B_1 |\tilde{g}|\, \cos(\omega t - \phi).$$

All that's left is to compute the gain $g = |\tilde{g}|$ and the phase lag $\phi = -\text{Arg}(\tilde{g})$. We have

$$p(i\omega) = m(i\omega)^2 + bi\omega + k = (k - m\omega^2) + bi\omega\,,$$

so,

$$\tilde{g} = \frac{i\omega b}{p(i\omega)} = \frac{i\omega b}{(k - m\omega^2) + bi\omega}. \tag{10}$$

This gives

$$g(\omega) = |\tilde{g}| = \frac{\omega b}{|p(i\omega)|} = \frac{\omega b}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

In computing the phase ϕ we have to be careful not to forget the factor of i in the numerator of $\tilde{g}$. After a little algebra we get

$$\phi(\omega) = -\text{Arg}(\tilde{g}) = \tan^{-1}(-(k - m\omega^2)/(b\omega)).$$

As with the system driven through the spring, we try to find the input frequency $\omega = \omega_r$ which gives the largest system response. In this case we can find ω_r without any calculus by using the following shortcut: divide the numerator and denominator in (10) by $bi\omega$ and rearrange to get

$$\tilde{g} = \frac{1}{1 + (k - m\omega^2)/(i\omega b)} = \frac{1}{1 - i(k - m\omega^2)/(\omega b)}\,.$$

Now the gain $g = |\tilde{g}|$ can be written as

$$g = \frac{1}{\sqrt{1 + (k - m\omega^2)^2/(\omega b)^2}}.$$

Because squares are always positive, this is clearly largest when the term $k - m\omega^2 = 0$. At this point $g = 1$ and $\omega_r = \sqrt{k/m} = \omega_0$, i.e. the resonant frequency is the natural frequency.

Since $\tilde{g}(\omega_0) = 1$, we also see that the phase lag $\phi = \text{Arg}(\tilde{g})$ is 0 at ω_r Thus the input and output sinusoids are in phase at resonance.

We have found interesting and rather surprising results for this dashpot-driven mechanical system, namely, that the resonant frequency occurs at the system's natural undamped frequency ω_0; that this resonance is independent of the damping coefficient b; and that the maximum gain which can be obtained is $g = 1$. We can contrast this with the spring-side driven system worked out in the previous note, where the resonant frequency certainly *did* depend on the damping coefficient. In fact, there was no resonance at all if the system is too heavily damped. In addition, the gain could, in principle, be arbitarily large.

Comparing these two mechanical systems side-by-side, we can see the importance of the choice of the specification for the input in terms of understanding the resulting behavior of the physical system. In both cases the right-hand side of the DE is a sinusoidal function of the form $B\cos\omega t$ or $B\sin\omega t$, and the resulting mathematical formulas are essentially the same. The key difference lies in the dependence of the constant B on either the system parameters m, b, k and/or the input frequency ω. It is in fact the dependence of B on ω and b in the dashpot-driven case that results in the radically different result for the resonant input frequency ω_r.

Poles, Amplitude Response, Connection to the Exponential Input Theorem

1. Introduction

For our standard linear constant coefficient system

$$p(D)x = f \tag{1}$$

the transfer function is $W(s) = 1/p(s)$. In this case, what we will call the poles of $W(s)$ are simply the zeros of the characteristic polynomial $p(s)$ (also known as the characteristic roots). We have had lots of experience using these roots in this course and know they give important information about the system. The reason we talk about the poles of the transfer function instead of just sticking with the characteristic roots is that the system (1) is a special case of a **Linear Time Invariant** (LTI) system and all LTI systems have a transfer function, while the characteristic polynomial is defined only for systems described by constant coefficient linear ODE's like (1).

We have seen (Notes S) that the stability of the system (1) is determined by the roots of the characteristic polynomial. We saw as well that the amplitude response of the system to a sinusoidal input of frequency ω is also determined by the characteristic polynomial, namely formula FR (2) says the gain is

$$g(\omega) = \frac{1}{|p(i\omega)|}$$

The Laplace transform gives us another view of a signal by transforming it from a function of t, say $f(t)$, to a function $F(s)$ of the complex frequency s.

A key object from this point of view is the transfer function. For the system (1), if we consider $f(t)$ to be the input and $x(t)$ to be the output, then the transfer function is $W(s) = 1/p(s)$, which is again determined by the characteristic polynomial.

We will now learn about poles and the pole diagram of an LTI system. This ties together the notions of stability, amplitude response and transfer function, all in one diagram in the complex s-plane. The pole diagram gives us a way to visualize systems which makes many of their important properties clear at a glance; in particular, and remarkably, the pole diagram

1. shows whether the system stable;
2. shows whether the unforced system is oscillatory;
3. shows the exponential rate at which the unforced system returns to equilibrium (for stable systems); and
4. gives a rough picture of the amplitude response and practical resonances of the system.

For these reasons the pole diagram is a standard tool used by engineers in understanding and designing systems.

We conclude by reminding you that every LTI system has a transfer function. Everything we learn in this session will apply to such systems, including those not modeled by DE's of the form (1)

2. Definition of Poles

2.1. Rational Functions A **rational function** is a ratio of polynomials $q(s)/p(s)$.

Examples. The following are all rational functions. $(s^2+1)/(s^3+3s+1)$, $1/(ms^2+bs+k)$, $s^2+1+(s^2+1)/1$.

If the numerator $q(s)$ and the denominator $p(s)$ have no roots in common, then the rational function $q(s)/p(s)$ is in **reduced form**

Example. The three functions in the example above are all in reduced form.

Example. $(s-2)/(s^2-4)$ is not in reduced form, because $s=2$ is a root of both numerator and denominator. We can rewrite this in reduced form as

$$\frac{s-2}{s^2-4} = \frac{s-2}{(s-2)(s+2)} = \frac{1}{s+2}.$$

2.2. Poles For a rational function in reduced form the **poles** are the values of s where the denominator is equal to zero; or, in other words, the points where the rational function is not defined. We allow the poles to be complex numbers here.

Examples. a) The function $1/(s^2+8s+7)$ has poles at $s=-1$ and $s=-7$.
b) The function $(s-2)/(s^2-4) = 1/(s+2)$ has only one pole, $s=-2$.
c) The function $1/(s^2+4)$ has poles at $s=\pm 2i$.
d) The function s^2+1 has no poles.
e) The function $1/(s^2+8s+7)(s^2+4)$ has poles at -1, -7, $\pm 2i$. (Notice that this function is the product of the functions in (a) and (c) and that its poles are the union of poles from (a) and (c).)

Remark. For ODE's with system function of the form $1/p(s)$, the poles are just the roots of $p(s)$. These are the familiar characteristic roots, which are important as we have seen.

2.3. Graphs Near Poles

We start by considering the function $F_1(s) = \frac{1}{s}$. This is well defined for every complex s except $s=0$. To visualize $F_1(s)$ we might try to graph it. However it will be simpler, and yet still show everything we need, if we graph $|F_1(s)|$ instead.

To start really simply, let's just graph $|F_1(s)| = \frac{1}{|s|}$ for s real (rather than complex).

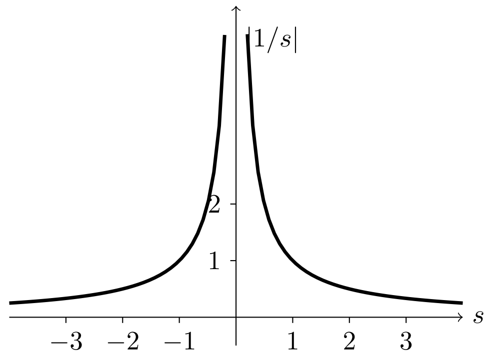


Figure 1: Graph of $\frac{1}{|s|}$ for s real.

Now let's do the same thing for $F_2(s) = 1/(s^2 - 4)$. The roots of the denominator are $s = \pm 2$, so the graph of $|F_2(s)| = \frac{1}{|s^2-4|}$ has vertical asymptotes at $s = \pm 2$.

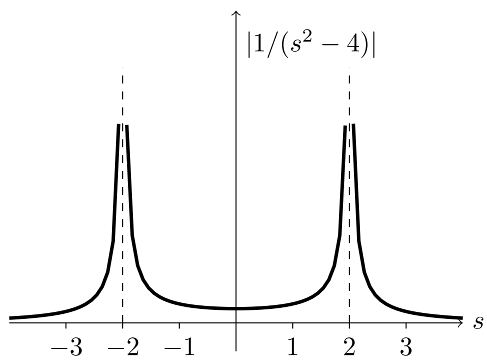


Figure 2: Graph of $\frac{1}{|s^2-4|}$ for s real.

As noted, the vertical asymptotes occur at values of s where the denominator of our function is 0. These are what we defined as the poles.

- $F_1(s) = \frac{1}{s}$ has a single pole at $s = 0$.
- $F_2(s) = \frac{1}{s^2-4}$ has two poles, one each at $s = \pm 2$.

Looking at Figures 1 and 2 you might be reminded of a tent. The poles of the tent are exactly the vertical asympotes which sit at the poles of the function.

Let's now try to graph $|F_1(s)|$ and $|F_2(s)|$ when we allow s to be complex. If $s = a + ib$ then $F_1(s)$ depends on two variables a and b, so the graph requires three dimensions: two for a and b, and one more (the vertical axis) for the value of $|F_1(s)|$. The graphs are shown in Figure 3 below. They are 3D versions of the graphs above in Figures 1 and 2. At each pole there is a conical shape rising to infinity, and far from the poles the function fall off to 0.

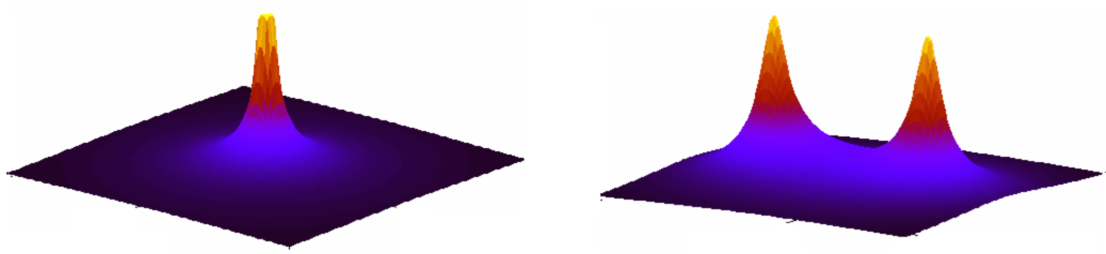


Figure 3: The graphs of $|1/s|$ and $1/|s^2 - 4|$.

Roughly speaking, the poles tell you the shape of the graph of a function $|F(s)|$: it is *large near the poles*. In the typical pole diagams seen in practice, the $|F(s)|$ is also small far away from the poles.

2.4. Poles and Exponential Growth Rate

If $a > 0$, the exponential function $f_1(t) = e^{at}$ grows rapidly to infinity as $t \to \infty$. Likewise the function $f_2(t) = e^{at} \sin bt$ is oscillatory with the amplitude of the oscillations growing exponentially to infinity as $t \to \infty$. In both cases we call a the *exponential growth rate* of the function.

The formal definition is the following

Definition: The **exponential growth rate** of a function $f(t)$ is the smallest value a such that

$$\lim_{t \to \infty} \frac{f(t)}{e^{bt}} = 0 \quad \text{for all } b > a. \tag{2}$$

In words, this says $f(t)$ grows slower than any exponential with growth rate larger than a.

Examples.

1. e^{2t} has exponential growth rate 2.

2. e^{-2t} has exponential growth rate -2. A negative growth rate means that the function is *decaying* exponentially to zero as $t \to \infty$.

3. $f(t) = 1$ has exponential growth rate 0.

4. $\cos t$ has exponential growth rate 0. This follows because $\lim_{t \to \infty} \dfrac{\cos t}{e^{bt}} = 0$ for all positive b.

5. $f(t) = t$ has exponential growth rate 0. This may be surprising because $f(t)$ grows to infinity. But it grows linearly, which is slower than any positive exponential growth rate.

6. $f(t) = e^{t^2}$ does not have an exponential growth rate since it grows faster than any exponential.

Poles and Exponential Growth Rate

We have the following theorem connecting poles and exponential growth rate.

Theorem: The exponential growth rate of the function $f(t)$ is the largest real part of all the poles of its Laplace transform $F(s)$.

Examples. We'll check the theorem in a few cases.

1. $f(t) = e^{3t}$ clearly has exponential growth rate equal to 3. Its Laplace transform is $1/(s-3)$ which has a single pole at $s = 3$,and this agrees with the exponential growth rate of $f(t)$.

2. Let $f(t) = t$, then $F(s) = 1/s^2$. $F(s)$ has one pole at $s = 0$. This matches the exponential growth rate zero found in (5) from the previous set of examples.
3. Consider the function $f(t) = 3e^{2t} + 5e^{t} + 7e^{-8t}$. The Laplace transform is $F(s) = 3/(s-2) + 5/(s-1) + 7/(s+8)$, which has poles at $s = 2,\ 1,\ -8$. The largest of these is 2. (Don't be fooled by the absolute value of -8, since $2 > -8$, the largest pole is 2.) Thus, the exponential growth rate is 2. We can also see this directly from the formula for the function. It is clear that the $3e^{2t}$ term determines the growth rate since it is the dominant term as $t \to \infty$.
4. Consider the function $f(t) = e^{-t}\cos 2t + 3e^{-2t}$ The Laplace transform is $F(s) = \frac{s}{(s+1)^2+4} + \frac{3}{s+2}$. This has poles $s = -1 \pm 2i$, -2. The largest real part among these is -1, so the exponential growth rate is -1.

Note that in item (4) in this set of examples the growth rate is negative because $f(t)$ actually *decays* to 0 as $t \to \infty$. We have the following

Rule:

1. If $f(t)$ has a negative exponential growth rate then $f(t) \to 0$ as $t \to \infty$.

2. If $f(t)$ has a positive exponential growth rate then $f(t) \to \infty$ as $t \to \infty$.

2.5. An Example of What the Poles Don't Tell Us

Consider an arbitrary function $f(t)$ with Laplace transform $F(s)$ and $a > 0$. Shift $f(t)$ to produce $g(t) = u(t-a)f(t-a)$, which has Laplace transform $G(s) = e^{-as}F(s)$. Since e^{-as} does not have any poles, $G(s)$ and $F(s)$ have exactly the same poles. That is, the poles can't detect this type of shift in time.

3. Pole Diagrams

3.1. Definition of the Pole Diagram

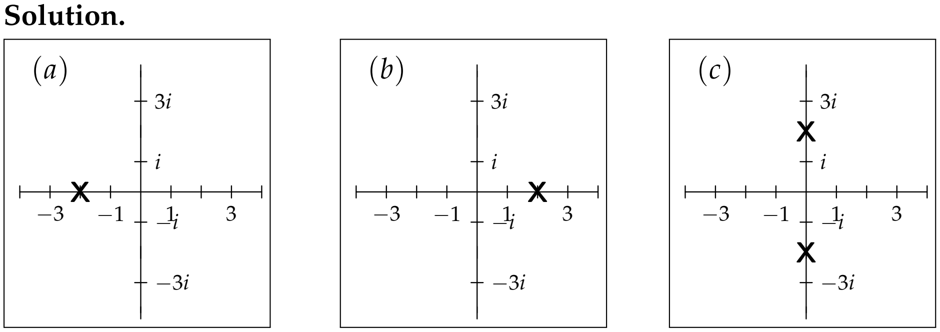
The **pole diagram** of a function $F(s)$ is simply the complex s-plane with an **X** marking the location of each pole of $F(s)$.

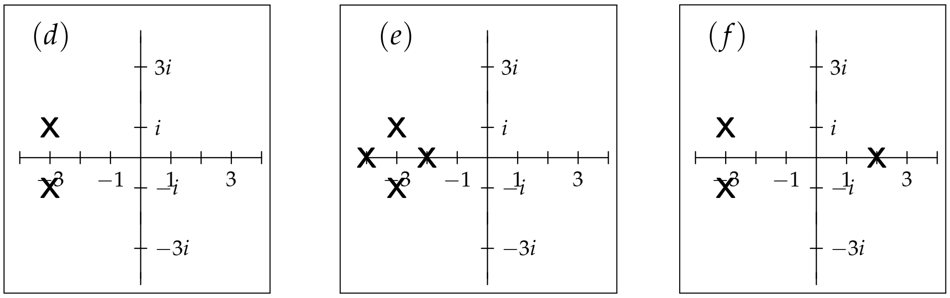
Example 1. Draw the pole diagrams for each of the following functions.

a) $F_1(s) = \frac{1}{s+2}$ b) $F_2(s) = \frac{1}{s-2}$ c) $F_3(s) = \frac{1}{s^2+4}$

d) $F_4(s) = \frac{s}{s^2+6s+10}$ e) $F_5(s) = \frac{1}{((s^2+3)^2+1)(s+2)(s+4)}$ f) $F_6(s) = \frac{1}{((s+3)^2+1)(s-2)}$

Solution.



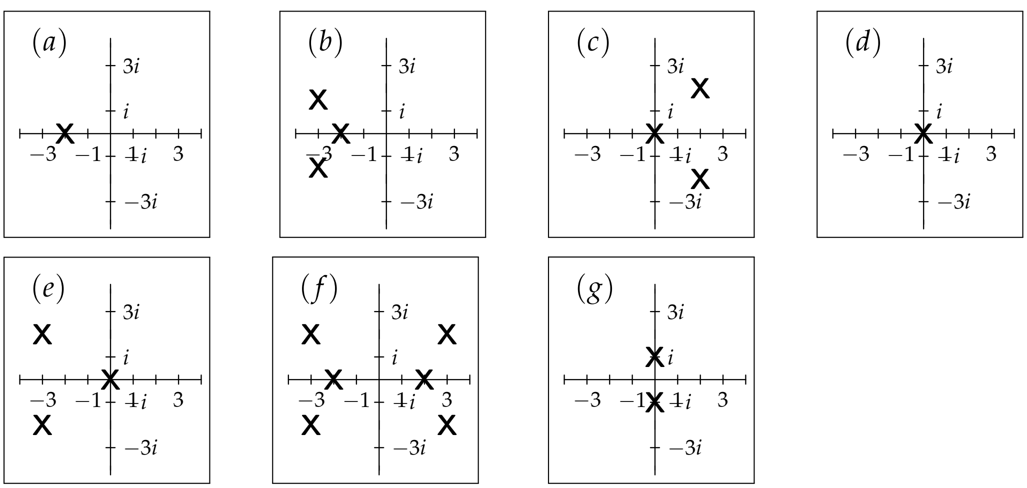


For (d) we found the poles by first completing the square: $s^2 + 6s + 10 = (s+3)^2 + 1$, so the poles are at $s = -3 \pm i$.

Example 2. Use the pole diagram to determine the exponential growth rate of the inverse Laplace transform of each of the functions in example 1.

Solution.
a) The largest pole is at -2, so the exponential growth rate is -2.
b) The largest pole is at 2, so the exponential growth rate is 2.
c) The poles are $\pm 2i$, so the largest real part of a pole is 0. The exponential growth rate is 0.
d) The largest real part of a pole is -3. The exponential growth rate is -3.
e) The largest real part of a pole is -2. The exponential growth rate is -2.
f) The largest real part of a pole is 2. The exponential growth rate is 2.

Example 3. Each of the pole diagrams below is for a function $F(s)$ which is the Laplace transform of a function $f(t)$. Say whether
(i) $f(t) \to 0$ as $t \to \infty$
(ii) $f(t) \to \infty$ as $t \to \infty$
(iii) You don't know the behavior of $f(t)$ as $t \to 0$,



Solution. a) Exponential growth rate is -2, so $f(t) \to 0$.
b) Exponential growth rate is -2, so $f(t) \to 0$.
c) Exponential growth rate is 2, so $f(t) \to \infty$.
d) Exponential growth rate is 0, so we can't tell how $f(t)$ behaves.
Two examples of this: (i) if $F(s) = 1/s$ then $f(t) = 1$, which stays bounded; (ii) if $F(s) = 1/s^2$ then $f(t) = t$, which does go to infinity, but more slowly than any positive exponential.

e) Exponential growth rate is 0, so don't know the behavior of $f(t)$.
f) Exponential growth rate is 3, so $f(t) \to \infty$.
g) Exponential growth rate is 0, so don't know the behavior of $f(t)$. (e.g. both $\cos t$ and $t\cos t$ have poles at $\pm i$.

3.2. The Pole Diagram for an LTI System

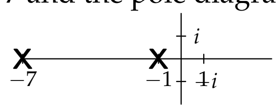
Definition: The pole diagram for an LTI system is defined to be the pole diagram of its transfer function.

Example 4. Give the pole diagram for the system

$$x'' + 8x' + 7x = f(t),$$

where we take $f(t)$ to be the input and $x(t)$ the output.

Solution. The transfer function for this system is $W(s) = \dfrac{1}{s^2+8s+1} = \dfrac{1}{(s+1)(s+7)}$. Therefore, the poles are $s = -1, -7$ and the pole diagram is



Example 5. Give the pole diagram for the system

$$x'' + 4x' + 6x = y',$$

where we consider $y(t)$ to be the input and $x(t)$ to be the output.

Solution. Assuming rest IC's, Laplace transforming this equation gives us $(s^2+4s+6)X = sY$. This implies $X(s) = \dfrac{s}{s^2+4s+6}Y(s)$ and the transfer function is $W(s) = \dfrac{s}{s^2+4s+6}$. This has poles at $s = -2 \pm \sqrt{2}\,i$.

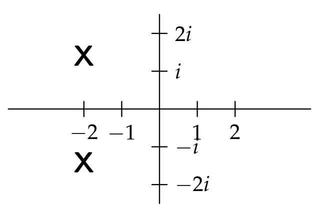


Figure: Pole diagram for the system in example 5.

4. Poles and Stability

Recall that the LTI system

$$p(D)x = f \tag{3}$$

has an associated homogeneous equation

$$p(D)x = 0 \tag{4}$$

In Notes S we saw the following stability criteria.
1. The system is stable if every solution to (4) goes to 0 as $t \to \infty$.
In words, the unforced system always returns to equilibrium.

2. Equivalently, the system is stable if all the roots of the characteristic equation have negative real part.

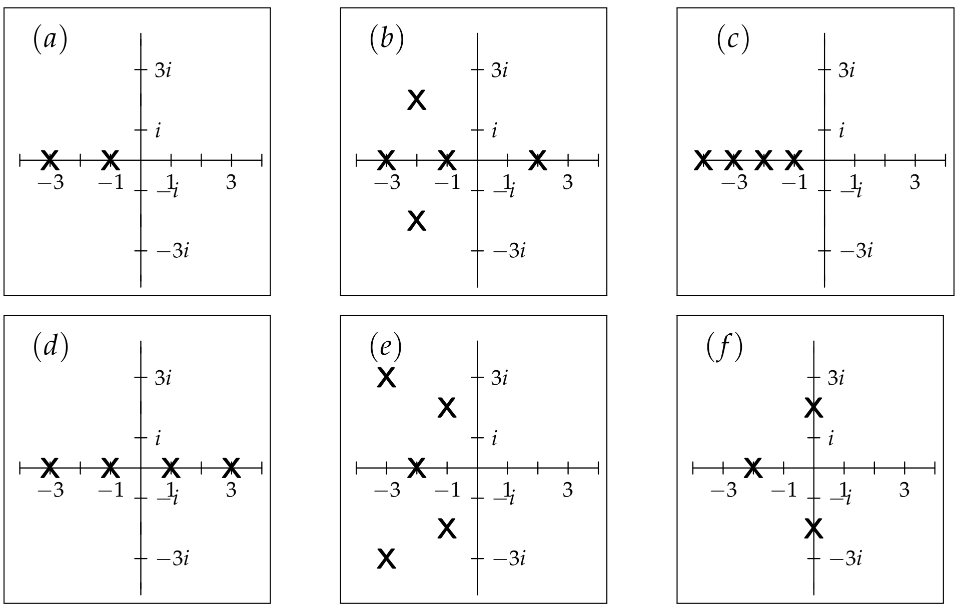
Now, since the transfer function for the system in (3) is $\dfrac{1}{p(s)}$ the poles of the system are just the characteristic roots. Comparing this with the stability criterion 2, gives us another way of expressing the stability criteria.

3. The system is stable if all its poles have negative real part.

4. Equivalently, the system is stable if all its poles lie strictly in the left half of the complex plane $\mathrm{Re}(s) < 0$.

Criterion 4 tells us how to see at a glance if the system is stable, as illustrated in the following example.

Example. Each of the following six graphs is the pole diagram of an LTI system. Say which of the systems are stable.



Solution. (a), (c) and (e) have all their poles in the left half-plane, so they are stable. The others do not, so they are not stable.

5. Poles and Amplitude Response

We started by considering the poles of functions $F(s)$, and saw that, by definition, the graph of $|F(s)|$ went off to infinity at the poles. Since it tells us where $|F(s)|$ is infinite, the pole diagram provides a crude graph of $|F(s)|$: roughly speaking, $|F(s)|$ will be large for values of s near the poles. In this note we show how this basic fact provides a useful graphical tool for spotting resonant or near-resonant frequencies for LTI systems.

Example 1. Figure 1 shows the pole diagram of a function $F(s)$. At which of the points A, B, C on the diagram would you guess $|F(s)|$ is largest?

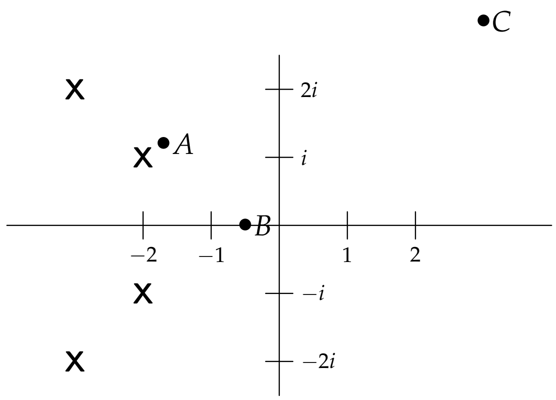


Figure 2: Pole diagram for example 1.

Solution. Point A is close to a pole and B and C are both far from poles so we would guess point $|F(s)|$ is largest at point A.

Example 2. The pole diagram of a function $F(s)$ is shown in Figure 2. At what point s on the positive imaginary axis would you guess that $|F(s)|$ is largest?

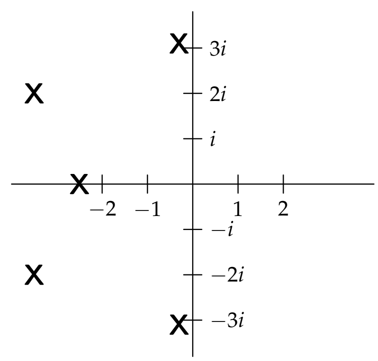


Figure 2: Pole diagram for example 2.

Solution. We would guess that s should be close to $3\,i$, which is near a pole. There is not enough information in the pole diagram to determine the exact location of the maximum, but it is most likely to be near the pole.

5.1. Amplitude Response and the System Function

Consider the system

$$p(D)x = f(t). \tag{5}$$

where we take $f(t)$ to be the input and $x(t)$ to be the output. The transfer function of this system is

$$W(s) = \frac{1}{p(s)}. \tag{6}$$

If $f(t) = B\cos(\omega t)$ then equation FR (2) gives the following periodic solution to (5)

$$x_p(t) = \frac{B\cos(\omega t - \phi)}{|p(i\omega)|}, \quad \text{where } \phi = \text{Arg}(p(i\omega)). \tag{7}$$

If the system is stable, then all solutions are asymptotic to the periodic solution in (7). In this case, we saw (FR (3)) that the amplitude response of the system as a function of ω is

$$g(\omega) = \frac{1}{|p(i\omega)|.} \tag{8}$$

Comparing (6) and (8), we see that for a stable system the amplitude response is related to the transfer function by

$$g(\omega) = |W(i\omega)|. \tag{9}$$

Note: The relation (9) holds for all stable LTI systems.

Using equation (9) and the language of amplitude response we will now re-do example 2 to illustrate how to use the pole diagram to estimate the practical resonant frequencies of a stable system.

Example 3. Figure 3 shows the pole diagram of a stable LTI system. At approximately what frequency will the system have the biggest response?

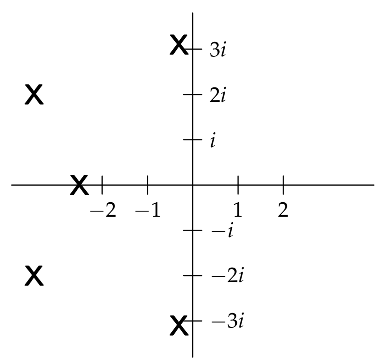


Figure 3: Pole diagram for example 3 (same as Figure 2).

Solution. Let the transfer function be $W(s)$. Equation (9) says the amplitude response $g(\omega) = |W(i\omega)|$. Since $i\omega$ is on the positive imaginary axis, the amplitude response $g(\omega)$ will be largest at the point $i\omega$ on the imaginary axis where $|W(i\omega)|$ is largest. This is exactly the point found in example 2. Thus, we choose $i\omega \approx 3i$, i.e. the practical resonant frequency is approximately $\omega = 3$.

Note: Rephrasing this in graphical terms: we can graph the magnitude of the system function $|W(s)|$ as a surface over the s-plane. The amplitude response of the system $g(\omega) = |W(i\omega)|$ is given by the part of the system function graph that lies above the imaginary axis. This is all illustrated beautifully by the applet `Amplitude: Pole Diagram`.

18.03 LA.1: Phase plane and linear systems

[**1**] Energy conservation
[**2**] Energy loss
[**3**] Companion system
[**4**] Heat flow
[**5**] Elimination

[1] Energy conservation

Let's think again about the harmonic oscillator, $m\ddot{x} + kx = 0$. The solutions are sinusoids of angular frequency $\omega = \sqrt{k/m}$. The mass bounces back and forth without damping. Let's check conservation of energy.

From physics we know that energy is the sum of kinetic plus potential. Kinetic energy is given by

$$\mathrm{KE} = \frac{m\dot{x}^2}{2}$$

Potential energy can be determined by computing work done. If we declare the relaxed spring to have PE $= 0$, then using Hooke's law when the mass is at position x

$$\mathrm{PE} = -\int_0^x -k\overline{x}\,d\overline{x} = \frac{kx^2}{2}$$

(When you're fighting the force, you increase potential energy, so the potential energy is given by $-\int F\,dx$. But the direction of force in Hooke's law opposes the displacement, so the force is $-kx$.)

So the total energy of the spring system is given by

$$E = \frac{kx^2}{2} + \frac{m\dot{x}^2}{2} \tag{1}$$

Let's see how it changes with time:

$$\dot{E} = kx\dot{x} + m\dot{x}\ddot{x} = \dot{x}(kx + m\ddot{x}) = 0$$

Conservation of energy is in force!

Energy is a function of two variables, x and $\dot{x}$. The plane with these coordinates is called the *phase plane*. It is traditional to draw x horizontally and $\dot{x}$ vertically. Contours of constant energy are curves on this plane, namely ellipses. Rearranging (1),

$$\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1$$

so the maximal value (i.e. the amplitude) of x is $\sqrt{2E/k}$ and the amplitude of $\dot{x}$ is $\sqrt{2E/m}$. These are the semi-axes of the ellipse. These formulas make sense: For given energy, small spring constant means big swing; small mass means large velocity.

As time increases, the point $(x(t), \dot{x}(t))$ traces out this ellipse. Which ellipse depends on the energy. You get a whole family of nested non-intersecting curves. This is called the *phase diagram* of this system.

Question 10.1. In which direction is the ellipse traversed?

1. Clockwise
2. Counterclockwise
3. Depends
4. I don't know

Well, above the axis $\dot{x} > 0$, which means that x is increasing. So the answer is 1, clockwise.

In a phase diagram, trajectories move to the right above the horizontal axis, and to the left below it.

How about when the trajectory crosses the horizontal? Well there $\dot{x} = 0$: the tangent is vertical. It crosses at right angles.

As the point moves around the ellipse, energy in the system sloshes back and forth between potential (when $|x|$ is large and $|\dot{x}|$ is small) and kinetic (when $|\dot{x}|$ is large and $|x|$ is small).

[2] Energy loss

What happens when we introduce some damping? So now

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{2}$$

Our equation for the energy is unchanged, but now

$$\dot{E} = \dot{x}(kx + m\ddot{x}) = -b\dot{x}^2$$

Energy is lost to friction. The dashpot heats up. The loss is largest when $|\dot{x}|$ is largest, and zero when $\dot{x} = 0$.

The effect of the friction is that the trajectory crosses through the equal-energy ellipses; it spirals in towards the origin, clockwise.

The x and $\dot{x}$ graphs are damped sinusoids, out of phase with each other.

Wait, that's just the underdamped case. The other option is that the trajectory just curves in to zero without spiralling. These are clearly shown in the Mathlet "Linear Phase Portraits: Matrix Entry."

[3] The companion system

In introducing $\dot{x}$ as an independent variable, we've done something important. We've rewritten the original equation as a first order equation. To be clear about this, let's write y for the new variable. It's related to x: $y = \dot{x}$. What's $\dot{y}$? To keep the notation clean, let me replace b by $\frac{b}{m}$ and k by $\frac{k}{m}$.

$$\dot{y} = \ddot{x} = -kx - b\dot{x} = -kx - by$$

So we have a pair of linked differential equations, a *system* of ordinary differential equations. (Sorry about the overuse of the word "system.")

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx - by \end{cases}$$

This is called the *companion system* of the equation (2).

We might as well use vector notation:

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{so} \quad \dot{\mathbf{u}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

so we can write our equation using matrix multiplication:

$$\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \mathbf{u}$$

The matrix here is the *companion matrix* of the differential operator $D^2 + bD + kI$.

This is pretty amazing! We have replaced a second order equation with a first order (but now vector-valued) equation!

You can even incorporate a forcing term:

$$\ddot{x} + b\dot{x} + kx = q(t)$$

translates as

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -kx - by + q(t) \end{cases}$$

or

$$\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0 \\ q(t) \end{bmatrix}$$

Note also that the *linear* second order equation translated into a *linear* first order vector valued equation, and that homogeneity is preserved.

You can do this with higher order equations too: For a third order equation $\dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$, say $y = \dot{x}$, $z = \dot{y}$, so $\dot{z} = -a_2 z - a_1 y - a_0 x$ and

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Side note: Numerical schemes work with first order equations only. To solve a higher order equation numerically, one replaces it by its companion first order system. We'll discuss this in December.

[4] Heat flow

Suppose I have a thermally insulated rod, and I'm interested in how heat flows through it.

I'll install a thermometer every foot, and just think about heat transfers between those points. Suppose for a start that the rod is three feet long. So there are four temperatures, x_0, x_1, x_2, and x_3, to consider.

Let's suppose that the temperatures at the end are fixed, constant, but x_1 and x_2 are functions of time.

Well, the linear or Newtonian approach goes as follows: if $x_0 > x_1$, there will be an upward pressure on x_1 that is proportional to the difference $x_0 - x_1$. Independently of this, if $x_2 > x_1$, there will be an upward pressure on x_1 that is proportional to the difference $x_2 - x_1$.

Putting them together,

$$\dot{x}_1 = k((x_0 - x_1) + (x_2 - x_1)) = k(x_0 - 2x_1 + x_2)$$

Similarly,

$$\dot{x}_2 = k(x_1 - 2x_2 + x_3)$$

This is a linked system of equations, which we can express in matrix form, using

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + k\begin{bmatrix} x_0 \\ x_3 \end{bmatrix} \tag{3}$$

A four foot section would lead to

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + k\begin{bmatrix} x_0 \\ 0 \\ x_4 \end{bmatrix}$$

and you can imagine from this what longer sections would look like.

The general picture is this: We have a system whose state is determined not by a single number but rather by a vector $\mathbf{u}$ of n numbers. (Maybe the vector is $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, but maybe not.) The rate of change of $\mathbf{u}$ at time t is determined by the value of $\mathbf{u}(t)$ (and perhaps the value of t), and we will mainly assume that this determination is *linear*, which means that the equation is

$$\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}(t)$$

where A is an $n \times n$ matrix A. The matrix A represents the physical system. $\mathbf{q}(t)$ is a "driving term" (perhaps the end point temperatures of the rod). To get a particular solution one has to specify an initial condition, $\mathbf{u}(0)$, consisting of n numbers.

There is a slight change of convention in force here: We are now isolating the $\dot{\mathbf{u}}$ on the left, and putting everything else—representation of the system and input signal alike—on the right. There's a sign reversal that happens; this is reflected in the signs in the formula for the companion matrix.

First order systems of linear equations incorporate higher order linear differential equations, while giving natural representations of other types of physical phenomena. To gain the deepest possible insight into their behavior, it's best to develop some of the theory of matrices, especially the square matrices that show up in the model above.

[5] Elimination

Do you think we can rewrite the equation (3) as a second order equation? Let's try. Maybe I'll take $x_0 = x_3 = 0$, the homogeneous case. If we succeed:

Question 10.2 Do you expect this to be underdamped or overdamped?

1. Underdamped
2. Overdamped

Well, do you really expect oscillations in the heat distribution? Let's see.

$$\begin{cases} \dot{x}_1 &= & -2kx_1 & + & kx_2 \\ \dot{x}_2 &= & kx_1 & - & 2kx_2 \end{cases}$$

Let's try to eliminate x_2. I can use the first equation to solve for it in terms of x_1: $kx_2 = \dot{x}_1 + 2kx_1$. If I substitute this into k times the second equation I get

$$\ddot{x}_1 + 2k\dot{x}_1 = k^2 x_1 - 2k(\dot{x}_1 + 2kx_1)$$

or

$$\ddot{x}_1 + 4k\dot{x}_1 + 3k^2 x_1 = 0$$

OK, the characteristic polynomial is

$$p(s) = s^2 + 4ks + 3k^2 = (s+k)(s+k)$$

so the roots are -1 and -3. Overdamped. General solution:

$$x_1 = ae^{-kt} + be^{-3kt}$$

You can always do this; but it's not very systematic or insightful, and it breaks the symmetry of the variables. Linear algebra offers a better way.

18.03 LA.2: Matrix multiplication, rank, solving linear systems

[1]Matrix Vector Multiplication $A\mathbf{x}$

Example 1:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

I like to think of this multiplication as a linear combination of the columns:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}.$$

Many people think about taking the dot product of the rows. That is also a perfectly valid way to multiply. But this column picture is very nice because it gets right to the heart of the two fundamental operation that we can do with vectors.

We can multiply them by scalar numbers, such as x_1 and x_2, and we can add vectors together. This is *linearity*.

[2] When is $A\mathbf{x} = \mathbf{b}$ Solvable?

There are two main components to linear algebra:

1. Solving an equation $A\mathbf{x} = \mathbf{b}$

2. Eigenvalues and Eigenvectors

This week we will focus on this first part. Next week we will focus on the second part.

Given an equation $A\mathbf{x} = \mathbf{b}$, the first questions we ask are :

Question Is there a solution?

Question If there is a solution, how many solutions are there?

Question What is the solution?

Example 2: Let's start with the first question. Is there a solution to this equation?

$$\underset{3 \times \mathbf{2}}{\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}} \quad \underset{\mathbf{2} \times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \quad = \underset{3 \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Notice the dimensions or shapes. The number of columns of A must be equal to the number of rows of $\mathbf{x}$ to do the multiplication, and the vector we get has the dimension with the same number of rows as A and the same number of columns as $\mathbf{x}$.

Solving this equation is equivalent to finding x_1 and x_1 such that the linear combination of columns of A gives the vector $\mathbf{b}$.

Example 3:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

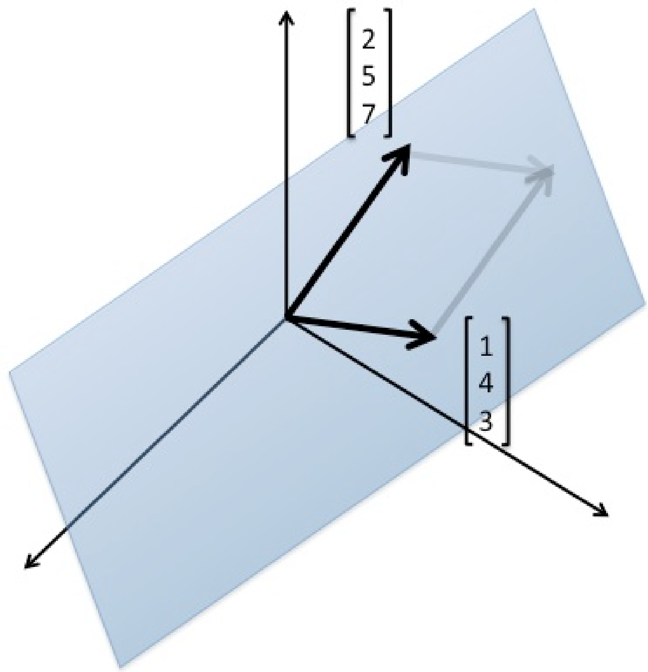
The vector $\mathbf{b}$ is the same as the 2nd column of A, so we can find this solution by inspection, the answer is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

But notice that this more general linear equation from Example 2:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is a system with 3 equations and 2 unknowns. This most likely doesn't have a solution! This would be like trying to fit 3 points of data with a line. Maybe you can, but most likely you can't!

To understand when this system has a solution, let's draw a picture.



All linear combinations of columns of A lie on a plane.

What are *all* linear combinations of these two columns vectors? How do you describe it?

It's a plane! And it is a plane that goes through the origin, because

$$0\begin{bmatrix}1\\4\\3\end{bmatrix}+0\begin{bmatrix}2\\5\\7\end{bmatrix}=\mathbf{0}.$$

It is helpful to see the geometric picture in these small cases, because the goal of linear algebra is to deal with very large matrices, like 1000 by 100. (That's actually a pretty small matrix.)

What would the picture of a 1000 by 100 matrix be?

It lives in 1000-dimensional space. What would the 100 column vectors span? Or, what is the space of all possible solutions?

Just a hyperplane, a flat, thin, at most 100-dimensional space inside of 1000-dimensions. But linear algebra gets it right! That is the power of linear algebra, and we can use our intuition in lower dimensions to do math on much larger data sets.

The picture exactly tells us what are the possible right hand sides to the equation. $A\mathbf{x} = \mathbf{b}/$ If $\mathbf{b}$ is in the plane, then there is a solution to the equation!

$$\text{All possible } \mathbf{b} \Longleftrightarrow \text{all possible combinations } A\mathbf{x}.$$

In our case, this was a plane. We will call this plane, this subspace, the *column space* of the matrix A.

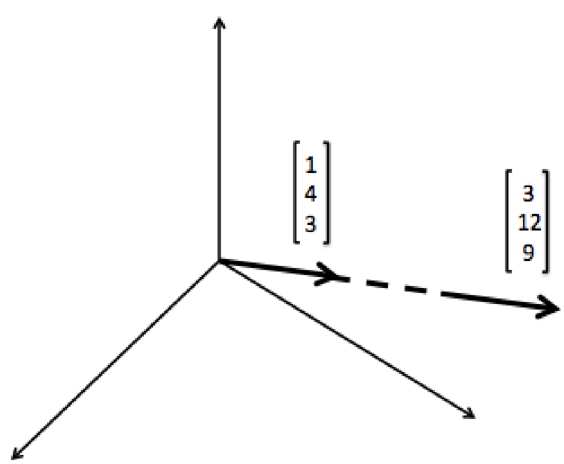
If $\mathbf{b}$ is not on that plane, not in that column space, then there is no solution.

Example 4: What do you notice about the columns of the matrix in this equation?

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The 2nd column is a multiple of the 1st column. We might say these columns are *dependent*.

The picture of the column space in this case is a *line*.



All linear combinations of columns of A lie along a line.

[3] Reduced Row Echelon Form and Rank

You learned about reduced row echelon form in recitation. Matlab and other learn algebra systems do these operation.

Example 5: Find the reduced row echelon form of our matrix:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}.$$

It is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is very obvious that the reduced row echelon form of the matrix has 2 columns that are independent.

Let's introduce a new term the **rank** of a matrix.

$$\textit{Rank} \text{ of } A \quad = \text{the number of independent columns of } A.$$

Example 6: Find the row echelon form of

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix}.$$

But what do you notice about the rows of this matrix? We made this matrix by making the columns dependent. So the rank is 1. We didn't touch the rows, this just happened.

How many independent rows are there?

1!

The reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}?$$

This suggests that we can define rank equally well as the number of independent rows of A.

$$\begin{aligned} \textit{Rank} \text{ of } A \quad &= \text{the number of independent columns of } A \\ &\quad \text{the number of independent rows of } A \end{aligned}$$

The process of row reduction provides the algebra, the mechanical steps that make it obvious that the matrix in example 5 has rank 2! The steps of row reduction don't change the rank, because they don't change the number of independent rows!

Example 7: What if I create a 7x7 matrix with random entries. How many linearly independent columns does it have?

7 if it is random!

What is the row echelon form? It's the identity.

To create a random 7x7 matrix with entries random numbers between 0 and 1, we write

```
rand(7).
```

The command for reduced row echelon form is rref. So what you will see is that

```
rref(rand(7))=eye(7).
```

The command for the identity matrix is

```
eye(n).
```

That's a little Matlab joke.

[4] Matrix Multiplication

We thought of $A\mathbf{x} = \mathbf{b}$ as a combination of columns. We can do the same thing with matrix multiplication.

$$AB = A\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \\ | & | & & | \end{bmatrix}$$

This perspective leads to a rule $A(BC) = (AB)C$. This seemingly simple rule takes some messing around to see. But this observation is key to many ideas in linear algebra. Note that in general $AB \neq BA$.

[5] Key Connection to Differential Equations

We want to find solutions to equations $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ is an unknown.

You've seen how to solve differential equations like

$$\frac{dx}{dt} - 3t^5 x = b(t).$$

The key property of this differential equation is that it is *linear* in x. To find a complete solution we need to find one particular solution and add to it all the homogeneous solution. This complete solution describes all solutions to the differential equation.

The same is true in linear algebra! Solutions behave the same way. This is a consequence of **linearity**.

Example: 8 Let's solve

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 6 \end{bmatrix}.$$

This matrix has rank 1, it has dependent columns. We chose $\mathbf{b}$ so that this equation is solvable. What is one particular solution?

$$\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is one solution. There are many solutions. For example

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -22 \\ 8 \end{bmatrix}$$

But we only need to choose one particular solution. And we'll choose [2;0].

What's the homogeneous solution? We need to solve the equation

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One solution is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and all solutions are given by scalar multiples of this one solution, so all homogenous solutions are described by

$$\mathbf{x}_h = c \begin{bmatrix} -3 \\ 1 \end{bmatrix} \qquad c \text{ a real number.}$$

The complete solution is

$$\mathbf{x}_{complete} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Here is how linearity comes into play!

$$A\mathbf{x}_{complete} = A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

18.03 LA.3: Complete Solutions, Nullspace, Space, Dimension, Basis

[1] Particular solutions

Matrix Example

Consider the matrix equation

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

The complete solution to this equation is the line $x_1 + x_2 = 8$. The homogeneous solution, or the *nullspace* is the set of solutions $x_1 + x_2 = 0$. This is all of the points on the line through the origin. The homogeneous and complete solutions are picture in the figure below.

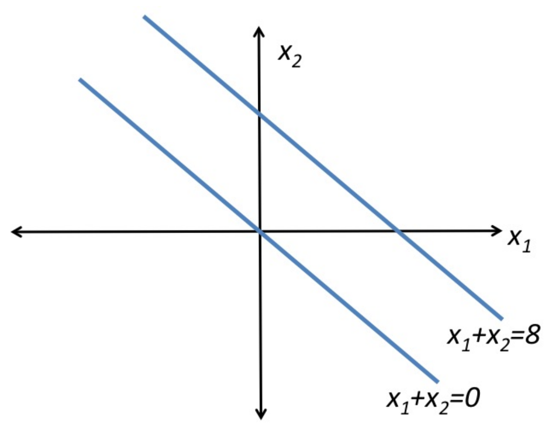


Figure 1: The homogeneous and complete solutions

To describe a complete solution it suffices to choose one particular solution, and add to it, any homogeneous solution. For our particular solution, we might choose

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

If we add any homogeneous solution to this particular solution, you move along the line $x_1 + x_2 = 8$. All this equation does is take the equation for the homogeneous line, and move the origin of that line to the particular solution!

How do solve this equation in Matlab? We type

```
x = [1 1] \ [8]
```

In general we write

```
x = A \ b
```

Differential Equations Example

Let's consider the linear differential equation with initial condition given:

$$\frac{dy}{dt} + y = 1$$
$$y(0)$$

To solve this equation, we can find one particular solution and add to it any homogeneous solution. The homogeneous solution that satisfies the initial condition is $x_h = y(0)e^{-t}$. So then a particular solution must satisfy $y_p(0) = 0$ so that $x_p(0) + x_h(0) = y(0)$, and such a particular solution is $y_p = 1 - e^{-t}$. The complete solution is then:

complete solution		**particular solution**		**homogeneous solution**
y	$=$	$1 - e^{-t}$	$+$	$y(0)e^{-t}$

However, maybe you prefer to take the steady state solution. The steady state solution is when the derivative term vanishes, $\frac{dy}{dt} = 0$. So instead we can choose the particular solution $y_p = 1$. That's an excellent solution to choose. Then in order to add to this an homogeneous solution, we add some multiple of e^{-t} so that at $t = 0$ the complete solution is equal to $y(0)$ and we find

$$\begin{array}{ccccc} \textbf{complete solution} & & \textbf{particular solution} & & \textbf{homogeneous solution} \\ & & & & \\ y & = & 1 & + & (y(0)-1)e^{-t} \\ & & \uparrow & & \uparrow \\ & & \textit{steady state solution} & & \textit{transient solution} \end{array}$$

The solution 1 is an important solution, because all solutions, no matter what initial condition, will approach the steady state solution $y = 1$.

There is not only 1 particular solution. There are many, but we have to choose 1 and live with it. But any particular solution will do.

[2] Complete Solutions

Matrix Example

Let's solve the system:

$$\begin{array}{cccc} x_1 & & +cx_3 & = b_1 \\ & x_2 & +dx_3 & = b_2 \end{array}$$

What is the matrix for this system of equations?

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

Notice that A is already in row echelon form! But we could start with any system

$$\begin{array}{cccc} x_1 & +3x_2 & +5x_3 & = b_1 \\ 4x_1 & +7x_2 & +19x_3 & = b_2 \end{array}$$

and first do a sequence of row operations to obtain a row echelon matrix. (Don't forget to do the same operations to b_1 and b_2:

$$\begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 4 & 7 & 17 & \vdots & b_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3/2 & 5/2 & \vdots & b_1/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -11 & \vdots & 5b_1/2 - 3b_2/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

Let's find the complete solution to $Ax = b$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}.$$

Geometrically, what are we talking about?

The solution to each equation is a plane, and the planes intersect in a line. That line is the complete solution. It doesn't go through 0! Only solutions to the equation $A\mathbf{x} = \mathbf{0}$ will go through 0!

So let's find 1 particular solution, and all homogeneous solutions.

Recommended particular solution: Set the free variable $\mathbf{x}_3 = 0$. Then

$$\mathbf{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}.$$

We could let the free variable be any value, but 0 is a nice choice because with a reduced echelon matrix, it is easy to read off the solution.

So what about the homogenous, or null solution. I will write $\mathbf{x}_n$ instead of $\mathbf{x}_h$ for the null solution of a linear system, but this is the same as the homogeneous solution. So now we are solving $A\mathbf{x} = \mathbf{0}$. The only bad choice is $x_3 = 0$, since that is the zero solution, which we already know. So instead we choose $\mathbf{x}_3 = 1$. We get

$$\mathbf{x}_n = C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$$

The complete solution is

$$x_{complete} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}.$$

This is the power of the row reduced echelon form. Once in this form, you can read everything off!

Differential Equations Example

Let's consider the differential equation $y'' + y = 1$. We can choose the steady state solution for the particular solution $y_p = 1$.

Let's focus on solving $y'' + y = 0$. What is the nullspace of this equation?

We can't say vectors here. We have to say functions. But that's OK. We can add functions and we can multiply them by constants. That's all we could do with vectors too. Linear combinations are the key.

So what are the homogeneous solutions to this equation? Give me just enough, but not too many.

One answer is $y_h = c_1 \cos(t) + c_2 \sin(t)$. Using linear algebra terminology, I would say there is a *2-dimensional* nullspace. There are two independent solutions $\cos(t)$ and $\sin(t)$, and linear combinations of these two solutions gives all solutions!

$\sin(t)$ and $\cos(t)$ are a *basis* for the nullspace.

A **basis** means each element of the basis is a solution to $A\mathbf{x} = \mathbf{0}$. Can multiply by a constant and we still get a solution. And we can add together and still get a solution. Together we get all solutions, but the $\sin(t)$ and $\cos(t)$ are different or *indepenedent* solutions.

What's another description of the nullspace?

$$C_1 e^{it} + C_2 e^{-it}$$

This description is just as good. Better in some ways (fulfills the pattern better), not as good in others (involves complex numers). The basis in this case is e^{it} and e^{-it}. They are independent solutions, but linear combinations give all null solutions.

If you wanted to mess with your TA, you could choose $y_h = Ce^{it} + D\cos(t)$. This is just as good.

We've introduced some important words. The *basis for the nullspace.* In this example, the beauty is that the nullspace will always have 2 functions in it. 2 is a very important number.

- The degree of the ODE is 2
- There are 2 constants
- 2 initial conditions are needed
- The dimension of the nullspace is 2.

[3] The nullspace

Suppose we have the equation $R\mathbf{x} = 0$. The collection of $\mathbf{x}$ that solve this equation form the *nullspace.* The nullspace always goes through the origin.

Example

Suppose we have a 5 by 5 matrix. Does it have an inverse or doesn't it? Look at the nullspace! If only solution in the nullspace is $\mathbf{0}$, then yes, it is invertible. However, if there is some nonzero solution, then the matrix is not invertible.

The other importan work we used is space.

Matrix Example

Let NR denote the nullspace of R:

$$R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

What's a basis for the nullspace? A basis could be $\begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$. Or we could take $\begin{bmatrix} -2c \\ -2d \\ 2 \end{bmatrix}$. The dimension is $3 - 2 = 1$. So there is only one element in the basis.

Why can't we take 2 vectors in the basis?

Because they won't be independent elements!

Differential Equations Example

For example, $Ce^{it} + D\cos(t) + E\sin(t)$ does not form a basis because they are not independent! Euler's formula tells us that $e^{it} = \cos(t) + i\sin(t)$, so e^{it} *depends* on $\cos(t)$ and $\sin(t)$.

[4] **Space, Basis, Dimension** There are a lot of important words that have been introduced.

- Space
- Basis for a Space
- Dimension of a Space

We have been looking at small sized examples, but these ideas are not small, they are very central to what we are studying.

First let's consider the word space. We have two main examples. The column space and the nullspace.

A	**Column Space**	**Nullspace**
Definition	All linear cominations of the columns of A	All solutions to $A\mathbf{x} = \mathbf{0}$
50×70 matrix	Column space lives in $\mathbb{R}^{50}$	Nullspace lives in $\mathbb{R}^{70}$
$m \times n$ matrix	Column space lives in $\mathbb{R}^m$	Nullspace lives in $\mathbb{R}^n$

Definition V is a space (or *vector* space) when: if x and y are in the space, then for any constant c, cx is in the space, and $x + y$ is also in the space. That's superposition!

Let's make sense of these terms for a larger matrix that is in row echelon form.

Larger Matrix Example

$$R = \begin{bmatrix} = 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first and third columns are the *pivot* columns. The second and forth are *free* columns.

What is the column space, $C(R)$?

All linear combinations of the columns. Is $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ in the column space? No it's not. The column space is the xy-plane, all vectors $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. The dimension is 2, and a basis for the column space can be taken to be the pivot columns.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Note, if your original matrix wasn't in rref form, you must take the original form of the pivot columns as your basis, not the row reduced form of them!

What is a basis for the nullspace, $N(R)$?

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

The reduced echelon form makes explicit the linear relations between the columns.

The relationships between the columns of A are the same as the linear relationships between the columns of any row-equivalent matrix, such as the reduced echelon form R. So a pivot indicates that this column is independent of the previous columns; and, for example, the 2 in the second column in this reduced form is a record of the fact that the second column is 2 times the first. This is why the reduced row echelon form is so useful to us. It allows us to immediately read off a basis for both the independent columns, and the nullspace.

Note that this line of thought is how you see that the reduced echelon form is well-defined, independent of the sequence of row operations used to obtain it.

18.03 LA.4: Inverses and Determinants

[**1**] Transposes
[**2**] Inverses
[**3**] Determinants

[1] Transposes

The transpose of a matrix A is denoted A^T, or in Matlab, A'.

The transpose of a matrix exchanges the rows and columns. The ith column becomes the ith row. Or the a_{ij} entry becomes the a_{ji} entry.

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 7 \end{bmatrix}$$

Symmetric Matrices are square matrices that satisfy $A = A^T$.

Example:

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 3 & 2 & 5 \\ 9 & 5 & 8 \end{bmatrix}$$

We'll see that the eigenvalues of symmetric matrices are great. The eigenvectors are even better! And symmetric matrices come up all of the time.

Property of transposes:

$$(AB)^T = B^T A^T$$

[2] Inverses

Important questions:

- Is a matrix A invertible?
- How do you compute the inverse?

Let A be a square matrix. Suppose it has an inverse. We denote the inverse by A^{-1}, and it has the property that

$$AA^{-1} = I \qquad A^{-1}A = I.$$

The fact that the inverse is simultaneously a right and left inverse is not immediately obvious. See if you can use the associative property $(AB)C = A(BC)$ to see why this must be the case when A is square.

If the inverse of A and B both exists, and both matrices have the same shape, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Corny Example:

If B represents taking off your jacket, and A represents taking off your sweater, then it makes sense that you first take of your jacket, and then take off your sweater. To find the inverse, which is to reverse this process, it makes sense that we have to reverse the order. First you put the sweater back on, and then you put your jacket on.

So let's start to answer our question: when is a matrix invertible? To answer this question, we'll look at when it is NOT invertible first.

A is NOT invertible when:

- The determinant of A is zero.
- There exists a nonzero vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{0}$.

Examples of matrices that are easy to invert:

2 by 2 matrices are easy to invert:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

Check that the product of this matrix with its inverse is the identity.

The quantity in the denominator $ad - bc$ is the determinant of that matrix. That's why it can't be zero. This is true for 5 by 5 matrices, 10 by 10 matrices, the inverse will always involve dividing by the determinant.

Diagonal matrices are easy to invert:

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & 1/d_3 & \\ & & & 1/d_4 \end{bmatrix}$$

Example of non-invertible matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ -4 & 7 & -3 \\ -3 & -2 & 5 \end{bmatrix}$$

We notices that the sum of each row is zero. So

$$\begin{bmatrix} 3 & -1 & -2 \\ -4 & 7 & -3 \\ -3 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

The vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ is in the *nullspace* of A.

Let's see why if there is a nonzero vector in the nullspace, then A can't be invertible.

Proof: Suppose $\mathbf{x} \neq 0$ and $A\mathbf{x} = \mathbf{0}$. If A^{-1} existed, then

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{0} \\ I\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0} \end{aligned}$$

This contradiction forces us to accept that an inverse must not exist!
QED

Conditions for Invertibility:

- $\det A \neq 0$.
- Nullspace of A is $\mathbf{0}$.
- Columns are independent.
- A has full rank (rank of $A = n$ if A is an n by n matrix).
- Rows are independent.
- The row echelon form of A has a full set of nonzero pivots.
- rref A is the identify.
- $A\mathbf{x} = \mathbf{b}$ has a solution for every b.

This last condition is the question we've been interested in, when can we solve this equation. The idea is that when it is always solvable, then $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

QED

Triangular Matrices

If A is upper triangular,

$$A = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

we can tell whether or not A is invertible immediately by looking at the diagonal, or pivot entries. If all the diagonal entries are nonzero, the matrix is invertible.

For example, the matrix

$$\begin{bmatrix} 3 & \pi & \mathrm{e} & \rho \\ 0 & 2 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is invertible, and its determinant is the product of these pivots, which is 42.

Computing Inverses

We've seen a bunch of connected ideas about when a matrix is invertible. And it is important to understand that you basically never want to compute the inverse. In Matlab, you could solve the equation $A\mathbf{x} = \mathbf{b}$ by typing

```
inv(A)*b
```

This is correct, but not smart. Instead, use

```
A \ b
```

which cleverly checks for certain forms of a matrix, and ultimately will row reduce to find the solution.

However, if we want to compute A^{-1} there is a complicated formula involving determinants. But what you want to do is really solve the equation

$AA^{-1} = I$. Let $\mathbf{a}'_1, \mathbf{a}'_2, \ldots, a'_n$ denote the columns of A^{-1}. Then solving for A^{-1} is equivalent to solving the n equations:

$$A\mathbf{a}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\mathbf{a}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \cdots \qquad A\mathbf{a}'_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Recall that to solve $A\mathbf{x} = \mathbf{b}$, we augment the matrix A with the column $\mathbf{b}$ and do row operations until we end up with a row reduced matrix where we can read off the solutions:

$$[A|\mathbf{b}] \longrightarrow \text{ Apply row operations } \longrightarrow [R|\mathbf{d}]$$

It turns out we can solve all n of our equations simulaneously by augmenting the matrix A with the matrix I, $[A|I]$ and then performing row operations. Because A is invertible, its reduced row echelon form is I, and what you end up with is I on the left, augmented by the solutions to $A\mathbf{x} = I$ on the right. But that solution is exactly A^{-1}.

Computing the inverse

- Augment A with the identity matrix: $[A|I]$
- Apply row operations until A reaches row reduced echelon form (rref)
- What you are left with on the augmented side is the collection of columns of A^{-1}: $[I|A^{-1}]$

[3] Determinants

Example 2 by 2

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In general, the determinant of a *square* matrix is a single number. This entry depends on all of the entries of the matrix.

Properties of the determinant:

- $\det I = 1$
- If you subtract m times row i and subtract that from row j, the determinants is unchanged! Example, subtract 4 times row 1 from row 2:
$$\det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 7 & 2 & 1 \end{bmatrix}$$
- If you exchange rows, the determinant changes sign. Example:
$$\det I = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
- If you multiply a row by a number c, the determinant is multiplied by c.
$$\det \begin{bmatrix} c & c & c \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 7 & 2 & 1 \end{bmatrix}$$
- $\det(AB) = \det A \det B$

Example

Suppose I have a matrix A such that two of its rows are equal. Then if I exchange those rows, the matrix is unchanged. But by the third property, this implies that $\det A = -\det A$, which can only be true if $\det A$ is zero. This tells us that any matrix whose rows and not independent has determinant equal to zero.

Example 3 by 3

The 3 by 3 determinant has $6 = 3!$ terms. Each term is a multiple of a three entries, one from each row and column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = 1\cdot 3\cdot 5 + 2\cdot 4\cdot 3 + 3\cdot 2\cdot 4 - 2\cdot 2\cdot 5 - 1\cdot 4\cdot 4 - 3\cdot 3\cdot 3$$

This matrix satisfies the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so the determinant must be zero.

In general, the determinant of an n by n matrix is a sum of $n!$ terms all combined into one number. A 4 by 4 matrix already has 24 terms! That is a lot of terms. The key idea here is that if a matrix is not invertible, its determinant is zero.

18.03 LA.5: Eigenvalues and Eigenvectors

[1] Eigenvectors and Eigenvalues

Example from Differential Equations

Consider the system of first order, linear ODEs.

$$\frac{dy_1}{dt} = 5y_1 + 2y_2$$
$$\frac{dy_2}{dt} = 2y_1 + 5y_2$$

We can write this using the companion matrix form:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Note that this matrix is symmetric. Using notation from linear algebra, we can write this even more succinctly as

$$\mathbf{y}' = A\mathbf{y}.$$

This is a coupled equation, and we want to uncouple it.

Method of Optimism

We've seen that solutions to linear ODEs have the form e^{rt}. So we will look for solutions

$$y_1 = e^{\lambda t} a$$
$$y_2 = e^{\lambda t} b$$

Writing in vector notation:

$$\mathbf{y} = e^{\lambda t} \begin{bmatrix} a \\ b \end{bmatrix} = e^{\lambda t} \mathbf{x}$$

Here λ is the eigenvalue and $\mathbf{x}$ is the eigenvector.

To find a solution of this form, we simply plug in this solution into the equation $\mathbf{y}' = A\mathbf{y}$:

$$\frac{d}{dt} e^{\lambda t} \mathbf{x} = \lambda e^{\lambda t} \mathbf{x}$$
$$A e^{\lambda t} \mathbf{x} = e^{\lambda t} A \mathbf{x}$$

If there is a solution of this form, it satisfies this equation

$$\lambda e^{\lambda t} \mathbf{x} = e^{\lambda t} A \mathbf{x}.$$

Note that because $e^{\lambda t}$ is never zero, we can cancel it from both sides of this equation, and we end up with the central equation for eigenvalues and eigenvectors:

$$\boxed{\lambda \mathbf{x} = A\mathbf{x}}$$

Definitions

- A nonzero vector $\mathbf{x}$ is an *eigenvector* if there is a number λ such that $A\mathbf{x} = \lambda \mathbf{x}$.

- The scalar value λ is called the *eigenvalue*.

Note that it is always true that $A\mathbf{0} = \lambda \cdot \mathbf{0}$ for any λ. This is why we make the distinction than an eigenvector must be a nonzero vector, and an eigenvalue must correspond to a nonzero vector. However, the scalar value λ can be any real or complex number, including 0.

This is a subtle equation. Both λ and $\mathbf{x}$ are unknown. This isn't exactly a linear problem. There are more unknowns.

What is this equation saying? It says that we are looking for a vector $\mathbf{x}$ such that $\mathbf{x}$ and $A\mathbf{x}$ point in the same direction. But the length can change, the length is scaled by λ.

Note that this isn't true for most vectors. Typically $A\mathbf{x}$ does not point in the same direction as $\mathbf{x}$.

Example

If $\lambda = 0$, our central equation becomes $A\mathbf{x} = 0\mathbf{x} = 0$. The eigenvector $\mathbf{x}$ corresponding to the eigenvalue 0 is a vector in the nullspace!

Example

Let's find the eigenvalues and eigenvectors of our matrix from our system of ODEs. That is, we want to find $\mathbf{x}$ and λ such that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \lambda \begin{bmatrix} ? \\ ? \end{bmatrix}$$

By inspection, we can see that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We have found the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda_1 = 7$.

So a solution to a differential equation looks like

$$\mathbf{y} = e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Check that this is a solution by pluging

$$\begin{aligned} y_1 &= e^{7t} \qquad \text{and} \\ y_2 &= e^{7t} \end{aligned}$$

into the system of differential equations.

We can find another eigenvalue and eigenvector by noticing that

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We've found the nonzero eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with corresponding eigenvalue $\lambda_2 = 3$.

Check that this also gives a solution by plugging

$$\begin{aligned} y_1 &= e^{3t} \qquad \text{and} \\ y_2 &= -e^{3t} \end{aligned}$$

back into the differential equations.

Notice that we've found two independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. More is true, you can see that $\mathbf{x}_1$ is actually perpendicular to $\mathbf{x}_2$. This is because the matrix was symmetric. Symmetric matrices always have perpendicular eigenvectors.

[2] Observations about Eigenvalues

We can't expect to be able to eyeball eigenvalues and eigenvectors everytime. Let's make some useful observations.

We have

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

and eigenvalues

$$\begin{aligned} \lambda_1 &= 7 \\ \lambda_2 &= 3 \end{aligned}$$

- The sum of the eigenvalues $\lambda_1 + \lambda_2 = 7 + 3 = 10$ is equal to the sum of the diagonal entries of the matrix A is $5 + 5 = 10$.

The sum of the diagonal entries of a matrix A is called the *trace* and is denoted $\mathrm{tr}\,(A)$.

It is always true that

$$\lambda_1 + \lambda_2 = \mathrm{tr}\,(A).$$

If A is an n by n matrix with n eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\boxed{\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{tr}\,(A)}$$

- The product of the eigenvalues $\lambda_1 \lambda_2 = 7 \cdot 3 = 21$ is equal to $\det A = 25 - 4 = 21$.

In fact, it is always true that

$$\boxed{\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A}.$$

For a 2 by 2 matrix, these two pieces of information are enough to compute the eigenvalues. For a 3 by 3 matrix, we need a 3rd fact which is a bit more complicated, and we won't be using it.

[3] Complete Solution to system of ODEs

Returning to our system of ODEs:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

We see that we've found 2 solutions to this homogeneous system.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \textit{and} \qquad e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution is obtained by taking linear combinations of these two solutions, and we obtain the general solution of the form:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The complete solution for any system of two first order ODEs has the form:

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2,$$

where c_1 and c_2 are constant parameters that can be determined from the initial conditions $y_1(0)$ and $y_2(0)$. It makes sense to multiply by this parameter because when we have an eigenvector, we actually have an entire line of eigenvectors. And this line of eigenvectors gives us a line of solutions. This is what we're looking for.

Note that this is the general solution to the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. We will also be interested in finding particular solutions $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. But this isn't where we start. We'll get there eventually.

Keep in mind that we know that all linear ODEs have solutions of the form e^{rt} where r can be complex, so this method has actually allowed us to find *all* solutions. There can be no more and no less than 2 independent solutions of this form to this system of ODEs.

In this example, our matrix was *symmetric*.

- Symmetric matrices have real eigenvalues.
- Symmetric matrices have perpendicular eigenvectors.

[4] Computing Eigenvectors

Let's return to the equation $A\mathbf{x} = \lambda\mathbf{x}$.

Let's look at another example.

Example

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

This is a 2 by 2 matrix, so we know that

$$\lambda_1 + \lambda_2 = \operatorname{tr}(A) = 5$$
$$\lambda_1 \cdot \lambda_2 = \det(A) = 6$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. In fact, because this matrix was upper triangular, the eigenvalues are on the diagonal!

But we need a method to compute eigenvectors. So lets' solve

$$A\mathbf{x} = 2\mathbf{x}.$$

This is back to last week, solving a system of linear equations. The key idea here is to rewrite this equation in the following way:

$$(A - 2I)\mathbf{x} = \mathbf{0}$$

How do I find $\mathbf{x}$? I am looking for $\mathbf{x}$ in the nullspace of $A - 2I$! And we already know how to do this.

We've reduced the problem of finding eigenvectors to a problem that we already know how to solve. Assuming that we can find the eigenvalues λ_i, finding $\mathbf{x}_i$ has been reduced to finding the nullspace $N(A - \lambda_i I)$.

And we know that $A - \lambda I$ is singular. So let's compute the eigenvector $\mathbf{x}_1$ corresponding to eigenvalue 2.

$$A - 2I = \begin{bmatrix} 0 & 4 \\ 0 & 1 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By looking at the first row, we see that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a solution. We check that this works by looking at the second row.

Thus we've found the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to eigenvalue $\lambda_1 = 2$.

Let's find the eigenvector $\mathbf{x}_2$ corresponding to eigenvalue $\lambda_2 = 3$. We do this by finding the nullspace $N(A - 3I)$, we wee see is

$$A - 3I = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second eigenvector is $\mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ corresponding to eigenvalue $\lambda_2 = 3$.

Important observation: this matrix is NOT symmetric, and the eigenvectors are NOT perpendicular!

[5] Method for finding Eigenvalues

Now we need a general method to find eigenvalues. The problem is to find λ in the equation $A\mathbf{x} = \lambda\mathbf{x}$.

The approach is the same:

$$(A - \lambda I)\mathbf{x} = 0.$$

Now I know that $(A - \lambda I)$ is singular, and singular matrices have determinant 0! This is a key point in LA.4. To find λ, I want to solve $\det(A - \lambda I) = 0$. The beauty of this equation is that $\mathbf{x}$ is completely out of the picture!

Consider a general 2 by 2 matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}.$$

The determinant is a polynomial in λ:

$$\det(A - \lambda I) = \lambda^2 - \underset{\substack{\uparrow \\ \mathrm{tr}\,(A)}}{(a+d)\lambda} + \underset{\substack{\uparrow \\ \det(A)}}{(ad - bc)} = 0$$

This polynomial is called the *characteristic polynomial*. This polynomial is important because it encodes a lot of important information.

The determinant is a polynomial in λ of degree 2. If A was a 3 by 3 matrix, we would see a polynomial of degree 3 in λ. In general, an n by n matrix would have a corresponding nth degree polynomial.

Definition

The *characteristic polynomial* of an n by n matrix A is the nth degree polynomial $\det(A - \lambda I)$.

- The roots of this polynomial are the eigenvalues of A.
- The constant term (the coefficient of λ^0) is the determinant of A.
- The coefficient of λ^{n-1} term is the trace of A.
- The other coefficients of this polynomial are more complicated invariants of the matrix A.

Note that it is not fun to try to solve polynomial equations by hand if the degree is larger than 2! I suggest enlisting some computer help.

But the fundamental theorem of arithmetic tells us that this polynomial always has n roots. These roots can be real or complex.

Example of imaginary eigenvalues and eigenvectors

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Take $\theta = \pi/2$ and we get the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

What does this matrix do to vectors?

To get a sense for how this matrix acts on vectors, check out the Matrix Vector Mathlet `http://mathlets.org/daimp/MatrixVector.html`

Set $a = d = 0$, $b = -1$ and $c = 1$. You see the input vector $\mathbf{v}$ in yellow, and the output vector $A\mathbf{v}$ in blue.

What happens when you change the radius? How is the magnitude of the output vector related to the magnitude of the input vector?

Leave the radius fixed, and look at what happens when you vary the angle of the input vector. What is the relationship between the direction of the input vector and the direction of the output vector?

This matrix rotates vectors by 90 degrees! For this reason, there can be no *real* nonzero vector that points in the same direction after being multiplied by the matrix A. Let's look at the characteristic polynomial and find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

Let's do a quick check:

- $\lambda_1 + \lambda_2 = i - i = \operatorname{tr}(A)$
- $\lambda_1 \cdot \lambda_2 = (i)(-i) = -1 = \det(A)$

Let's find the eigenvector corresponding to eigenvalue i:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}$$

Solving for the nullspace we must find the solution to the equation:

$$\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To solve this equation, I look at the first row, and checking against the second row we find that the solution is

$$\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

What ODE does this correspond to?

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This is the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 \end{aligned}$$

Using the method of elimination we get that:

$$y_1'' = -y_2' = -y_1$$

We are very familiar with this differential equation, it is the harmonic oscillator $y'' + y = 0$. This linear, 2nd order equation parameterized motion around a circle! It is a big example and physics, and we know that the solution space has a basis spanned by e^{it} and e^{-it}. Notice that the i and $-i$ are the eigenvalues!

Properties of Eigenvalues

Suppose A has eigenvalue λ and nonzero eigenvector $\mathbf{x}$.

- The the eigenvalues of A^2 are λ^2.

Why?

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

We see that the vector $\mathbf{x}$ will also be an eigenvector corresponding to λ. However, be careful!!! In the example above, $\lambda_1 = i$ and $\lambda_2 = -1$, we get repeated eigenvalues $\lambda_1 = \lambda_2 = -1$. And in fact

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Since $-I\mathbf{x} = -\mathbf{x}$ for all nonzero vectors $\mathbf{x}$, in fact every vector in the plane is an eigenvector with eigenvalue -1!

We know that the exponential function is important.

- The eigenvalues of e^A are e^λ, with eigenvector $\mathbf{x}$.

If $e^A\mathbf{x}$ had meaning,

$$e^A\mathbf{x} = e^\lambda\mathbf{x}$$

where $\mathbf{x}$ is an eigenvector of A, and λ is the corresponding eigenvalue.

- The eigenvalues of e^{-1} are λ^{-1}, with eigenvector $\mathbf{x}$.

Let's look at the example $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$, which had eigenvalues 7 and 3. Check that A^{-1} has eigenvalues $1/7$ and $1/3$. We know that $\det(A) * \det(A^{-1}) = 1$, and $\det(A) = 21$ and $\det(A^{-1}) = 1/21$, which is good.

- The eigenvalues of $A + 12I$ are $\lambda + 12$, with eigenvector $\mathbf{x}$.

Check this with our favorite symmetric matrix A above.

Nonexamples

Let A and B be n by n matrices.

- The eigenvalues of $A + B$ are generally NOT the eigenvalues of A plus eigenvalues of B.
- The eigenvalues of AB are generally NOT the eigenvalues of A times the eigenvalues of B.

Question: What would be necessary for the eigenvalues of $A + B$ to be the sum of the eigenvalues of A and B? Similarly for AB.

Keep in mind that $A\mathbf{x} = \lambda\mathbf{x}$ is NOT an easy equation.

In matlab, the command is

```
eig(A)
```

18.03 LA.6: Diagonalization and Orthogonal Matrices

[1] Diagonal factorization

Recall: if $A\mathbf{x} = \lambda\mathbf{x}$, then the system $\dot{\mathbf{y}} = A\mathbf{y}$ has a general solution of the form

$$y = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2,$$

where the λ_i are eigenvalues with corresponding eigenvectors $\mathbf{x}_i$.

I'm never going to see eigenvectors without putting them into a matrix. And I'm never going to see eigenvalues without putting them into a matrix. Let's look at an example from last class.

$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$. We found that this had eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

I'm going to form a matrix out of these eigenvectors called the *eigenvector matrix* S:

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then lets look at what happens when we multiply AS, and see that we can factor this into S and a diagonal matrix Λ:

$$\underset{A}{\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 7 & 3 \\ 7 & -3 \end{bmatrix} = \underset{S}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}}$$

We call matrix Λ with eigenvalues λ on the diagonal the *eigenvalue matrix*.

So we see that $AS = S\Lambda$, but we can multiply both sides on the right by S^{-1} and we get a factorization $A = S\Lambda S^{-1}$. We've factored A into 3 pieces.

Properties of Diagonalization

- $A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$
- $A^{-1} = (S\Lambda S^{-1})^{-1} = (S^{-1})^{-1}\Lambda^{-1}S^{-1} = S\Lambda^{-1}S^{-1}$

Diagonal matrices are easy to square and invert because you simply square or invert the elements along the diagonal!

[2] Solving systems of first order differential equations

The entire reason we are finding eigenvectors is to solve differential equations. Let's express our solution to the differential equation in terms of S and Λ:

$$y = \underbrace{\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathbf{c}}$$

What determines $\mathbf{c}$? Suppose we have an initial condition $\mathbf{y}(0)$. Plugging this into our vector equation above we can solve for $\mathbf{c}$:

$$\begin{aligned} \mathbf{y}(0) &= SI\mathbf{c} \\ S^{-1}\mathbf{y}(0) &= \mathbf{c} \end{aligned}$$

The first line simply expresses our initial condition as a linear combination of the eigenvectors, $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. The second equation just multiplies the first by S^{-1} on both sides to solve for $\mathbf{c}$ in terms of $\mathbf{y}(0)$ and S^{-1}, which we know, or can compute from what we know.

Steps for solving a differential equation

Step 0. Find λ_i and $\mathbf{x}_i$.

Step 1. Use the initial condition to compute the parameters:

$$\mathbf{c} = S^{-1}\mathbf{y}(0)$$

Step 2. Multiply $\mathbf{c}$ by $e^{\Lambda t}$ and S:

$$\mathbf{y} = Se^{\Lambda t}S^{-1}\mathbf{y}(0).$$

[3] Symmetric and Orthonormal Matrices

In our example, we saw that A was symmetric ($A = A^T$) implied that the eigenvectors were perpendicular, or orthogonal. Perpendicular and orthogonal are two words that mean the same thing.

Now, the eigenvectors we chose

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

had length $\sqrt{2}$. If we make them unit length, we can choose eigenvectors that are both orthogonal and unit length. This is called *orthonormal*.

Question: Are the unit length vectors also eigenvectors?

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Yes! If $A\mathbf{x} = \lambda\mathbf{x}$, then

$$A\frac{\mathbf{x}}{||\mathbf{x}||} = \lambda\frac{\mathbf{x}}{||\mathbf{x}||}.$$

It turns out that finding the inverse of a matrix whose columns are orthonormal is extremely easy! All you have to do is take the transpose!

Claim

If S has orthonormal columns, then $S^{-1} = S^T$.

Example

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \qquad\qquad\qquad S^T = I$$

If the inverse exists, it is unique, so S^T must be the inverse!

If we set $\theta = \pi/4$ we get

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

but what we found was

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Fortunately we can multiply the second column by negative 1, and it is still and eigenvector. So in the 2 by 2 case, we can always choose the eigenvectors of a symmetric matrix so that the eigenvector matrix is not only orthonormal, but also so that it is a rotation matrix!

In general, a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is said to be orthonormal if the dot product of any vector with itself is 1:

$$\mathbf{x}_i \cdot \mathbf{x}_i = \mathbf{x}_i^T \mathbf{x}_i = 1,$$

and the dot product of any two vectors that are not equal is zero:

$$\mathbf{x}_i \cdot \mathbf{x}_j = \mathbf{x}_i^T \mathbf{x}_j = 0,$$

when $i \neq j$.

This tells us that the matrix product:

$$\begin{bmatrix} - & \mathbf{x}_1^T & - \\ - & \mathbf{x}_2^T & - \\ - & \mathbf{x}_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2 & \mathbf{x}_1^T\mathbf{x}_3 \\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \mathbf{x}_2^T\mathbf{x}_3 \\ \mathbf{x}_3^T\mathbf{x}_1 & \mathbf{x}_3^T\mathbf{x}_1 & \mathbf{x}_3^T\mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S^T \qquad\qquad S \qquad\qquad\qquad\qquad\qquad = I$$

Example

We've seen that 2 by 2 orthonormal eigenvector matrices can be chosen to be rotation matrices.

Let's look at a 3 by 3 rotation matrix:

$$S = \frac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

As an exercise, test that all vector dot products are zero if the vectors are not equal, and are one if it is a dot product with itself. This is a particularly nice matrix because there are no square roots! And this is also a rotation matrix! But it is a rotation is 3 dimensions.

Find a symmetric matrix A whose eigenvector matrix is S.

All we have to do is choose any Λ with real entries along the diagonal, and then $A = S\Lambda S^T$ is symmetric!

Recall that $(AB)^T = B^T A^T$. We can use this to check that this A is in fact symmetric:

$$\begin{aligned} A^T &= (S\Lambda S^T)^T \\ &= S^{TT}\Lambda^T S^T \\ &= S\Lambda S^T \end{aligned}$$

This works because transposing a matrix twice returns the original matrix, and transposing a diagonal matrix does nothing!

In physics and engineering this is called the principal axis theorem. In math, this is the spectral theorem.

Why is it called the principal axis theorem?

An ellipsoid whose principal axis are along the standard x, y, and z axes can be written as the equation $ax^2 + by^2 + cz^2 = 1$, which in matrix form is

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

However, what you consider a general ellipsoid, the 3 principal direction can be pointing in any direction. They are orthogonal direction though! And this means that we can get back to the standard basis elements by applying a rotation matrix S whose columns are orthonormal. Thus our equation for a general ellipsoid is:

$$\left(S \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \left(S \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = 1$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \left(S^T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} S \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

18.03 LA.7: Two dimensional dynamics

[**1**] Rabbits
[**2**] Springs

[1] Rabbits

Farmer Jones and Farmer McGregor have adjacent farms, both afflicted with rabbits. Let's model this. Write $x(t)$ for the number of rabbits in Jones's farm, and $y(t)$ for the number in McGregor's.

Rabbits breed fast: growth rate of 5 per year: $\dot{x} = 5x$, $\dot{y} = 5y$.

But wait, these two systems are coupled. The rabbits can jump over the hedge between the farms. McGregor's grass is greener, so it happens twice as often into his than out if his, per unit population. So we have

$$\begin{cases} \dot{x} &= 3x + y \\ \dot{y} &= 2x + 4y \end{cases}$$

The equation is *homogeneous*, at least till McGregor gets his gun. In matrices, with $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$\dot{\mathbf{u}} = A\mathbf{u} \quad , \quad A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

We could eliminate, but now we know better: we look for solutions of the form

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{v} \quad , \mathbf{v} \neq \mathbf{0}$$

That is, you separate the time dependence from the high dimensionality. You look for *ray solutions*. He pointed out what happens when you substitute this into the equation:

$$\dot{\mathbf{u}} = \lambda e^{\lambda t}\mathbf{v}$$

while

$$Ax = Ae^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$$

and the only way these can be equal is if

$$A\mathbf{v} = \lambda\mathbf{v}$$

That is, λ is an eigenvalue of A, and $\mathbf{v}$ is a nonzero eigenvector.

Let's try it: The characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$$

and the roots of this polynomial are $\lambda_1 = 2$, $\lambda_2 = 5$.

So we have two "normal mode" solutions, one growing like e^{2t} and the other much faster, like e^{5t}. (They both get large as t grows, but when $e^{2t} = 100$, $e^{5t} = 100,000$.)

Then find nonzero eigenvectors by finding nonzero vectors killed by $A-\lambda I$. With $\lambda = 2$,

$$A - (2I) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad : \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or any nonzero multiple. In general one has the row reduction algorithm, but for 2×2 cases you can just eyeball it. I like to look at one of the rows, reverse the order and change one sign. Then check your work using the other row. Remember, $A - \lambda I$ is supposed to be a *singular* matrix, zero determinant, so the rows should say the same things.

The other eigenvalue gives

$$A - (5I) = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \quad : \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The two *normal mode* solutions are thus

$$e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad , \quad e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the general solution is a linear combination of these two.

This way of solving is much more perspicacious than the elimination we did back in September: the variables are equally important and are put on equal footing.

Remember the phase diagram: Plot the trajectory of $\mathbf{x}(t)$. There are two sets of ray solutions, along the two eigenvectors. All solutions except the constant one at $\mathbf{0}$ go off exponentially to infinity. Other solutions are linear combinations of these two. As $t \to -\infty$, both exponentials get small, but e^{5t} gets smaller much faster, so the solutions become asymptotic to the other eigenline.

The picture is this shown well on the Mathlet "Linear Phase Portraits: Matrix Entry." This phase portrait is called a "Node."

[2] **Springs again.** Another source of systems is the companion matrix: The companion matrix of

$$\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x$$

for example is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

In the harmonic oscillator $\ddot{x} + \omega^2 x = 0$ for example the companion matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

We know the solutions of the harmonic oscillator, but let's solve using eigenvectors.

The characteristic polynomial is $p_A(\lambda) = \lambda^2 + \omega^2$. This is a general fact, true in any dimension:

The characteristic polynomial of an LTI operator is the same as that of its companion matrix.

The eigenvalues here are $\pm\omega i$. Plunge on and find corresponding eigenvectors: For $\lambda_1 = i\omega$,

$$A - \lambda I = \begin{bmatrix} -i\omega & 1 \\ -\omega^2 & -i\omega \end{bmatrix} \quad : \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$$

(Check the second row!) Complex eigenvalues give rise to complex eigenvectors. The other eigenvalue is $-i\omega = \overline{i\omega}$, and the corresponding eigenvector is the complex conjugate of $\mathbf{v_1}$.

These are the normal modes: $e^{\pm i\omega t}\begin{bmatrix} 1 \\ \pm i\omega \end{bmatrix}$. We can extract real solutions in the usual way, by taking real and imaginary parts:

$$\mathbf{x_1} = \begin{bmatrix} \cos(\omega t) \\ -\omega\sin(\omega t) \end{bmatrix} \quad , \quad \mathbf{x_2} = \begin{bmatrix} \sin(\omega t) \\ \omega\cos(\omega t) \end{bmatrix}$$

Now the trajectories are ellipses. This phase portrait is called a *center*.

18.03 LA.8: Stability

[**1**] Tr-Det plane
[**2**] More on the Tr-Det plane
[**3**] Stability and the Tr-Det plane

[1] The Trace-determinant plane

The system will oscillate if there are non-real eigenvalues. This is true in any number of dimensions. In two dimenions we can decide whether eigenvalues are real or not by completing the square:

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = \left(\lambda - \frac{\operatorname{tr} A}{2}\right)^2 - \left(\frac{(\operatorname{tr} A)^2}{4} - \det A\right)$$

so

$$\lambda_{1,2} = \frac{\operatorname{tr} A}{2} \pm \sqrt{\frac{(\operatorname{tr} A)^2}{4} - \det A}$$

has

$$\text{real roots if} \quad \det A \leq \frac{1}{4}(\operatorname{tr} A)^2$$

$$\text{non-real roots if} \quad \det A > \frac{1}{4}(\operatorname{tr} A)^2$$

The trace and determinant determine the eigenvalues, and conversely:

$$\operatorname{tr} A = \lambda_1 + \lambda_2 \quad , \quad \det A = \lambda_1 \lambda_2$$

Let's draw a plane with the trace horizontally and the determinant vertically. There's a big division of behavior depending on whether you're above or below the *critical parabola*

$$\det A = \frac{(\operatorname{tr} A)^2}{4}$$

If you're above the critical parabola, you get spirals (or a special type of spiral, a center, if you are at $\operatorname{tr} A = 0$ since that's where $\operatorname{Re} \lambda = 0$).

If you're below the parabola, the roots are real. You get two differing types of behavior depending on whether the eigenvalues are of the same sign or of opposite signs. Since the determinant is the product, $\det A > 0$ if the eigenvalues are of the same sign, $\det A < 0$ if they are of opposite sign.

The big categories of behavior:

Spirals if $\operatorname{Im}\lambda \neq 0$; of angular frequency $\operatorname{Im}\lambda = \omega_d$.

Nodes if the eigenvalues are real and of the same sign: as in the rabbits example.

Saddles if the eigenvalues are real and of opposite sign.

Here's a saddle example: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This has already been "decoupled": it's already diagonal. The eigenvalues are $+1$, -1, with nonzero eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The normal mode solutions are $e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. There are two pairs of ray trajectories, and everything else moves along hyperbolas.

You can see how this works out in general using the Mathlet "Linear Phase Portraits."

[2] More about the trace-determinant plane.

Let's look in more detail at these dynamical systems. Start with this unstable spiral and decrease the trace. Remember: the trace is the sum of the eigenvalues, so in this complex case it's twice the real part. You are making the real part smaller, so the rate of expansion gets smaller and the spiral gets tighter.

Question 17.1. If I increase the determinant,

1. The spirals will get tighter
2. The spirals will get looser
3. Neither (but the spirals will change in some other way)
4. Don't know

Well, the determinant is the product of the eigenvalues. In this complex case, the eigenvalues are complex conjugates of each other, so their product is the square of their common magnitude. With fixed real part, the only way that can increase is for the imaginary part to increase. When that happens, you make more loops for a given amount of expansion. So I expect the spirals to get tighter. Let's see. Yes.

As I decrease the determinant, the spirals get looser but also flatter, and if we push all the way to the critical parabola the long direction becomes an eigendirection. This marginal case is called a "defective node." There's a repeated eigenvalue but only a one dimensional space of eigenvectors. Any non-diagonal 2×2 matrix with a repeated eigenvalue has this property. You can read more about these marginal cases in the notes.

If I now move on into node territory, you see the single eigenline splitting into two; there are now two eigenvalues of the same sign.

Maybe now is a good time to talk about this box at top right. Fixing the trace and determinant give you two equations. But the space of 2×2 matrices is 4 dimensional, so there are two degrees of freedom within the set of matrices with given trace and determinant. They are recorded in this box.

Side comment: These alterations are accomplished by replacing A by SAS^{-1}, where S is invertible. Notice that this doesn't change the characteristic polynomial: To compute the characteristic polynomial of SAS^{-1}, we notice that $SAS^{-1} - \lambda I = SAS^{-1} - S\lambda IS^{-1} = S(A - \lambda I)S^{-1}$. So, since $\det(AB) = (\det A)(\det B)$, we have $p_{SAS^{-1}}(\lambda) = p_A(\lambda)$. So SAS^{-1} and A have the same eigenvalues; and the same trace and determinant.

One thing I can do is rotate the whole picture. (This uses a rotation matrix for S. These are discussed in LA.6)

The other thing I can do is change the angle between the two eigenlines.

If I look back at the degenerate node, this angle parameter shifts the picture like this. In the middle, you find a "star node": repeated eigenvalue, but this time you do get two independent eigenvectors. In fact the matrix is diagonal, a multiple of the identity matrix, and every vector is an eigenvector!

When we move down to the $\det = 0$ axis, we are forcing one of the eigenvalues to be 0. That indicates a nonzero constant solution. The other eigenline is here. This phase portrait is called a "comb." A comb is intermediate between a node and a saddle.

The saddles behave like the nodes.

If I go back up to the spirals, now in the end, you see that the box splits into two parts. As I push up towards the break, I get more symmetric spirals. The upper part is clockwise motion, the lower counterclockwise. You can't tell which you have from the trace and determinant alone.

[3] Stability

Here's the trace-determinant plane again, visible using "Linear Phase Portraits: Cursor Entry." This plane descibes the various behaviors exhibited by a 2D homogeneous linear system, $\dot{\mathbf{u}} = A\mathbf{u}$. Above the critical parabola $\det A = (\operatorname{tr} A)^2/4$ the eigenvalues are non-real and the solution trajectories are spirals. The entire phase portrait is actually called a spiral as well. To the left and right, the eigenvalues are real and of the same sign; the phase portrait is a "node." Below the axis, the eigenvalues are real and of opposite sign; the phase portraits are "saddles." (I think this comes from the similarity of this picture with the picture of level sets of $x^2 - y^2$, from 18.02.)

There's an even more basic dichotomy:

Mostly, solutions either blow up or decay to zero.

How can we tell which happens? Well the solutions will always involve exponentials with exponents given by the eigenvalues of the matrix. The exponent might be real, it might be complex. But in every case, its growth as t increases is determined by *the sign of the real part*.

Here's the summary, valid for $n \times n$ systems.

Unstable: Most solutions blow up as $t \to \infty$: the real part of *some* root is positive.

Stable: All solutions decay to zero as $t \to \infty$: the real parts of *all* roots are negative.

In the 2×2 case, we can work this out.

If $\det A < 0$, the roots must be real and of opposite sign: unstable.

If $\det A > 0$ and the roots are real, they must be either both positive ($\operatorname{tr} A > 0$: unstable) or both negative ($\operatorname{tr} A < 0$: stable).

If $\det A > 0$ and the roots are not real, then then $\operatorname{Re} \lambda = \frac{\operatorname{tr} A}{2}$: so again $\operatorname{tr} A > 0$: unstable; $\operatorname{tr} A < 0$: stable.

So in the trace-det plane the stable region is the northwest quadrant only.

18.03 LA.9: Decoupling

[1] Multiple springs.

The "Coupled Oscillators" Mathlet shows a system with three springs connecting two masses. The ends of the springs are fixed, and the whole thing is set up so that there is a position in which all three springs are relaxed. Let's see some of the behaviors that are possible.

Wow, this is pretty complicated. Imagine what happens with five springs, or a hundred ....

Let's analyze this. The spring constants are k_1, k_2, k_3; the masses are m_1, m_2. The displacement from relaxed positions are x_1, x_2.

Let's look at the special case when $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$. You can put the subscripts back in on your own.

The forces on the objects are given by

$$\begin{cases} m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 = kx_1 - 2kx_2 \end{cases}$$

Let's divide through by m, and agree to write $k/m = \omega^2$.

There's a matrix here, $B = \omega^2 \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$. (This is the same as the heat conduction matrix!) We have

$$\ddot{\mathbf{x}} = B\mathbf{x}$$

What do do about the second derivative? Let's do the companion trick! Set $\mathbf{y} = \dot{\mathbf{x}}$, so

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{y} \\ \dot{\mathbf{y}} = B\mathbf{x} \end{cases}$$

Breaking this down even further, $y_1 = \dot{x}_1$, $y_2 = \dot{x}_2$; so we have four equations in four unknown functions:

$$\begin{aligned}
\dot{x}_1 &= & & & y_1 & \\
\dot{x}_2 &= & & & & y_2 \\
\dot{y}_1 &= -2\omega^2 x_1 & + \omega^2 x_2 & & & \\
\dot{y}_2 &= \omega^2 x_1 & - 2\omega^2 x_2 & & &
\end{aligned}$$

We might be quite uncomfortable about the prospect of computing eigenvalues of 4×4 matrices without something like Matlab. But we have a block matrix here:

$$A = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$$

and

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

Let's think about the eigenvector equation: It's

$$\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \lambda\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

This breaks down to two simpler equations:

$$\begin{cases} \mathbf{y} = \lambda\mathbf{x} \\ B\mathbf{x} = \lambda\mathbf{y} \end{cases}$$

Plugging the first equation into the second gives

$$B\mathbf{x} = \lambda^2\mathbf{x}$$

This says that the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector for B associated to the eigenvalue λ^2.

We have learned that the four eigenvalues of A are the square roots of the two eigenvalues of B. And the eigenvectors are gotten by putting $\lambda\mathbf{x}$ below the $\mathbf{x}$.

Well let's see. The characteristic polynomial of B is

$$p_B(\lambda) = \lambda^2 + 4\omega^2\lambda + 3\omega^4 = (\lambda + \omega^2)(\lambda + 3\omega^2)$$

so its eigenvalues are $-\omega^2$ and $-3\omega^2$.

That says the eigenvalues of A are $\pm i\omega$ and $\pm\sqrt{3}i\omega$.

We're almost there. The eigenvectors for B: For $-\omega^2$ we want to find a nonzero vector killed by $B - (-\omega^2 I) = \omega^2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do. The matrix is symmetric so eigenvectors for different eigenvalues are orthogonal; an eigenvector for value $-3\omega^2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So the eigenvectors for A are given by

$$\lambda = \pm i\omega \, : \begin{bmatrix} 1 \\ 1 \\ \pm i\omega \\ \pm i\omega \end{bmatrix} \quad , \quad \lambda = \pm\sqrt{3}i\omega \, : \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{3}i\omega \\ \mp\sqrt{3}i\omega \end{bmatrix}$$

This gives us exponential solutions!

$$e^{i\omega t} \begin{bmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{bmatrix} \quad , \quad e^{i\sqrt{3}\omega t} \begin{bmatrix} 1 \\ -1 \\ \sqrt{3}i\omega \\ -\sqrt{3}i\omega \end{bmatrix}$$

and their complex conjugates. We can get real solutions by taking real and imaginary parts. Let's just write down the top halves; the bottom halves are just the derivatives.

$$\begin{bmatrix} \cos(\omega t) \\ \cos(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\omega t) \\ \sin(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \cos(\sqrt{3}\omega t) \\ -\cos(\sqrt{3}\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\sqrt{3}\omega t) \\ -\sin(\sqrt{3}\omega t) \end{bmatrix}$$

The first two combine to give the general sinusoid of angular frequency ω for x_1 *and* $x_2 = x_1$. In this mode the masses are moving together; the spring between them is relaxed.

The second two combine to give a general sinusoid of angular frequency $\sqrt{3}\omega$ for x_1, *and* $x_2 = -x_1$. In this mode the masses are moving back and forth relative to each other.

These are "normal modes." I have used this term as a synonym for "exponential solutions" earlier in the course, but now we have a better definition. From Wikipedia:

A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation.

We can see them here on the Mathlet, if I adjust the initial conditions.

Behind the chaotic movement there are two very regular, sinuoidal motions. They happen at different frequencies, and that makes the linear combinations look chaotic. In fact the two frequencies never match up, because $\sqrt{3}$ is an irrational number. Except for the normal mode solutions, no solutions are perioidic.

The physics department has graciously provided some video footage of this in real action': `http://www.youtube.com/watch?v=zlzns5PjmJ4`Coupled Air Carts You can't see the springs here, and at the outset the masses have brakes on. When they turn the switch, the carts become elevated and move.

Check out the action at 5:51 as well: five carts!

I didn't really finish what could be gleaned from the "Coupled Oscillators" applet. All three springs have the same strength k, and the masses all have the same value m. What do you think? Are you seeing periodic motion here?

[The class was split on this question.]

Let's see. We calculated that there at least two families of periodic, even sinusoidal, solutions. They come from the eigenvalues $\pm\omega i$ and $\pm\sqrt{3}\omega i$, where $\omega = \sqrt{k/m}$. I have the applet set to $m = 1$ and $k = 4$. These special "normal mode" solutions are:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} \cos(\omega t - \phi) \\ \cos(\omega t - \phi) \end{bmatrix} \quad , \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} \cos(\sqrt{3}\omega t - \phi) \\ -\cos(\sqrt{3}\omega t - \phi) \end{bmatrix}$$

The general solution is a linear combination of these two. I claim that if both normal modes occur in the linear combination, then you will definitely NOT have a periodic solutions. This comes from the number-theoretic fact that $\sqrt{3}$ is irrational!

[2] Heat Phase Portrait.

I want to discuss how initial conditions can be used to specify which linear combination you get, in a situation like this—we are talking about $\dot{\mathbf{x}} = A\mathbf{x}$. For an example I'd like to go back to the insulated rod from LA.1.

For 3 thermometers, placed at 1, 2, and 3 feet, the details are a little msesy. The same ideas are already visible with two thermometers, so let's focus on that. The temperatures at 0, 1, 2, and 3 feet are x_0, x_1, x_2, x_3. The equation controlling this is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_0 \\ x_3 \end{bmatrix}$$

We're interested in the homogeneous case, so we'll take $x_0 = x3 = 0$ (in degrees centegrade, not Kelvin!), and I'm taking the conduction constant to be $k = 1$.

You can easily find eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. This tells us right off that some solutions decay like e^{-t}, some others decay like e^{-3t}, and in general you get a mix of the two. In paricular: no oscillation, and stable. In fact we can answer:

Question 18.1 This is a

1. Stable Spiral
2. Stable Saddle
3. Stable Node
4. Unstable Spiral
5. Unstable Saddle
6. Unstable Node
7. Don't know.

[There was uniform agreement that **3** is correct. Of course there is no such thing as a "stable saddle."

To get more detail we need to know the eigenvectors. Get them by subtracting the eigenvalue from the diagonal entry and finding a vector killed by the result. For $\lambda = -1$ we get $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, which kills $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and all multiples. We could do the same for the other eigenvalue, but as pointed out in LA.6 symmetric matrices have orthogonal eigenvectors (for distict eigenvalues), so a nonzero eigenvector for $\lambda = -3$ is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So we get two basic exponential solutions: $e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We can now draw the two eigenlines. Ray solutions converge exponentially to zero along them. These solutions are characterized by the fact that the ratio between x_1 and x_2 is constant in time. These are the **normal modes** for the heat model. The Wikipedia definition was applicable only to oscillating motion. You know, the word "normal mode" is like the word "happiness." It's hard to define, you know it when you see it, and it's really important, the basis of everything.

Other solutions are a mix of these two normal modes. IMPORTANT: The smaller the (real part of) the eigenvalue, the quicker the decay. $e^{-3t} = (e^{-t})^3$, so when $e^{-t} = 0.1$, $e^{-3t} = 0.001$. So the component of $\mathbf{v_2}$ decays much more rapidly than the component of $\mathbf{v_1}$, so these other trajectories approach tangency with the eigenline for value -1.

[3] Initial conditions.

Suppose I have a different initial condition, maybe $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$: so the temperatures are $0, 1, 2, 0$. What happens? What are c_1 and c_2 in

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

How can we find c_1 and c_2?

More generally, when we are studying $\dot{\mathbf{x}} = A\mathbf{x}$, and we've found eigenvalues $\lambda_1, \ldots$ and nonzero eigenvectors $\mathbf{v_1}, \ldots$, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2} + \cdots$$

We're interested in the cofficients c_i giving specified initial condition $\mathbf{x}(0) = \mathbf{v}$. Since $e^{\lambda\cdot 0} = 1$ alwasy, our equation is

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$$

As with many computations, the first step is to rewrite this as a matrix equation:

$$\mathbf{v} = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We will write S for the matrix whose columns are the eigenvectors, so $\mathbf{v} =$

$S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Now it's clear: we find $c_1, \ldots$ by inverting S:

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}\mathbf{v}$$

This is an important principle!

The entries in $S^{-1}\mathbf{v}$ are the coefficients in

$$\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$$

In our example, $S^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

sou our particular solution is

$$\mathbf{x}(t) = \frac{3}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The second term decays much faster than the first.

[4] Coordinates.

The big idea: a system can seem complicated just because we are using the wrong coordinates to describe it.

I want to let $\mathbf{x}$ be any vector; maybe it varies with time, as a solution of $\dot{\mathbf{x}} = A\mathbf{x}$. We can write it as a linear combination of $\mathbf{v_1}, \ldots$, but now the coefficients can vary too. I'll write $y_1, y_2, \ldots$ for them to make them look more variable:

$$\mathbf{x} = y_1\mathbf{v_1} + y_2\mathbf{v_2} + \cdots$$

or

$$\mathbf{x} = S\mathbf{y}$$

For example,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$$

This is a change of cooridinates. The y_1 axis (where $y_2 = 0$) is the λ_1 eigenline; the y_2 axis (where $y_1 = 0$) is the λ_2 eigenline.

We can change back using S^{-1}:

$$\mathbf{y} = S^{-1}\mathbf{x}$$

so for us

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i.e.

$$y_1 = \frac{x_1 + x_2}{2} \quad , \quad y_2 = \frac{x_1 - x_2}{2}$$

[5] Diagonalization.

OK, but these vectors $\mathbf{v_i}$ have something to do with the matrix A: they are eigenvectors for it. What does this mean about the relationship between S and A? Well, the eigenvector equation is

$$A\mathbf{v} = \lambda\mathbf{v}$$

that is,

$$A\mathbf{v_1} = \lambda_1\mathbf{v_1} \quad , \quad \cdots \quad , \quad A\mathbf{v_n} = \lambda_n\mathbf{v_n}$$

Line these up as the columns of a matrix product:

$$AS = A\begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v_1} & \cdots & \lambda_n\mathbf{v_n} \end{bmatrix}$$

Now comes a clever trick: the right hand side is a matrix product as well:

$$\cdots = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = S\Lambda$$

where Λ (that's a big "lambda") is the "eigenvalue matrix," with little λ's down the diagonal. That is:

$$AS = S\Lambda$$

or

$$A = S\Lambda S^{-1}$$

This is a *diagonalization* of A. It exhibits the simlicity hidden inside of A. There are only n eigenvalues, but n^2 entries in A. They don't completely determine A, of course, but they say a lot about it.

[6] Decoupling.

Now let's apply this to the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. I'm going to plug

$$\mathbf{x} = S\mathbf{y} \quad \text{and} \quad A = S\Lambda S^{-1}$$

into this equation.

$$\dot{\mathbf{x}} = \frac{d}{dt}S\mathbf{y} = S\dot{\mathbf{y}}$$

and

$$A\mathbf{x} = S\Lambda S^{-1}\mathbf{x} = S\Lambda\mathbf{y}$$

Put it together and cancel the S:

$$\dot{\mathbf{y}} = \Lambda\mathbf{y}$$

Spelling this out:

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2 \\ &\cdots \\ \dot{y}_n &= \lambda_n y_n \end{aligned}$$

Each variable keeps to itself; its derivatives don't depend on the other variables. They are **decoupled**.

In our example,

$$\begin{array}{lll} y_1 = \frac{x_1+x_2}{2} & \text{satisfies} & \dot{y}_1 = -y_1 \\ y_2 = \frac{x_1-x_2}{2} & \text{satisfies} & \dot{y}_3 = -3y_2 \end{array}$$

So the smart variables to use to record the state of our insulated bar are not x_1 and x_2, but rather the *average* and the *difference* (or half the difference). The average decays exponentially like e^{-t}. The difference decays much faster. So quite quickly, the two temperatures become very close, and then the both of them die off exponentially to zero.

18.03 LA.10: The Matrix Exponential

[1] Exponentials
[2] Exponential matrix
[3] Fundamental matrices
[4] Diagonalization
[5] Exponential law

[1] Exponentials

What is e^x?

Very bad definition: e^x is the xth power of the number $e \sim 2.718281828459045\ldots$

Two problems with this: (1) What is e? (2) What does it mean to raise a number to the power of, say, $\sqrt{2}$, or π?

Much better definition: $y(x) = e^x$ is the solution to the differential equation $\dfrac{dy}{dx} = y$ with initial condition $y(0) = 1$.

Now there's no need to know about e in advance; e is *defined* to be e^1. And e^x is just a function, which can be evaluated at $\sqrt{2}$ or at π just as easily as at an integer.

Note the sublety: you can't use this definition to describe e^x for any *single* x (except $x = 0$); you need to define the entire function at once, and then evaluate that function at the value of x you may want.

As you know, this gives us solutions to other equations: I claim that $y = e^{rt}$ satisfies $\dfrac{dy}{dt} = ry$. This comes from the chain rule, with $x = rt$:

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = ry$$

A further advantage of this definition is that it can be extended to other contexts in a "brain-free" way.

A first example is Euler's definition

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We *defined* $x(t) = e^{(a+bi)t}$ to be the solution to $\dot{x} = (a+bi)x$, and then *calculated* that

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$$

In all these cases, you get the solution for any initial condition: $e^{rt}x(0)$ is the solution to $\dot{x} = rx$ with initial condition $x(0)$.

[2] Matrix exponential

We're ready for the next step: We have been studying the equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where A is a square (constant) matrix.

Definition. e^{At} is the matrix of functions such that the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, in terms of its initial condition, is $e^{At}\mathbf{x}(0)$.

How convenient is that!

If we take $\mathbf{x}(0)$ to be the vector with 1 at the top and 0 below, the product $e^{At}\mathbf{x}(0)$ is the first column of e^{At}. Similarly for the other columns. So:

Each column of e^{At} is a solution of $\dot{\mathbf{x}} = A\mathbf{x}$. We could write this:

$$\frac{d}{dt}e^{At} = Ae^{At}$$

e^{At} is a matrix-valued solution! It satisfies a simple initial condition:

$$e^{A0} = I$$

Not everything about 1×1 matrices extends to the general $n \times n$ matrix. But everything about 1×1 matrices *does* generalize to *diagonal* $n\times n$ matrices.

If $A = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, the given coordinates are already decoupled: the equation $\dot{\mathbf{x}} = A\mathbf{x}$ is just $\dot{x}_1 = \lambda_1 x_1$ and $\dot{x}_2 = \lambda_2 x_2$. Plug in initial condition $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: the first column of $e^{\Lambda t}$ is $\begin{bmatrix} e^{\lambda_1 t} \\ 0 \end{bmatrix}$. Plug in initial condition $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: the second column is $\begin{bmatrix} 0 \\ e^{\lambda_2 t} \end{bmatrix}$. So

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Same works for $n \times n$, of course.

[3] Fundamental matrices

Here's how to compute e^{At}. Suppose we've found the right number (n) independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$: say $\mathbf{u_1(t)}, \ldots, \mathbf{u_n(t)}$. Line them up in a row: this is a "fundamental matrix" for A:

$$\Phi(t) = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \cdots & \mathbf{u_n} \end{bmatrix}$$

The general solution is

$$\mathbf{x}(t) = \Phi(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\Phi(t)$ may not be quite e^{At}, but it's close. Note that $\mathbf{x}(0) = \Phi(0) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, or $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi(0)^{-1}\mathbf{x}(0)$. Thus

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$$

So

$$e^{At} = \Phi(t)\Phi(0)^{-1}$$

for any fundamental matrix $\Phi(t)$.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Characteristic polynomial $p_A(\lambda) = \lambda^2 + 1$, so the eigenvalues are $\pm i$. The phase portrait is a "center." Eigenvectors for $\lambda = i$ are killed by $A - iI = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$; for example $\begin{bmatrix} 1 \\ i \end{bmatrix}$. So the exponential solutions are given by

$$e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and its complex conjugate. To find real solutions, take just the right linear combinations of these to get the real and imaginary parts:

$$\mathbf{u_1}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad , \quad \mathbf{u_2}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

These both parametrize the unit circle, just starting at different places. The corresponding fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

We luck out, here: $\Phi(0) = I$, so

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

[4] Diagonalization

Suppose that A is diagonalizable: $A = S\Lambda S^{-1}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. You can find the eigenvalues as roots of the characteristic polynomial, but you might as well remember that the eigenvalues of an upper (or lower) triangular matrix are the diagonal entries: here 1 and 3. Also an eigenvalue for 1 is easy: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For the other, subtract 3 from the diagonal entries: $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ kills $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Suppose $A = S\Lambda S^{-1}$. Then we have exponential solutions corresponding to the eigenvalues:

$$\mathbf{u_1}(t) e^{\lambda_1 t}\mathbf{v_1}\,, \ldots$$

These give a fine fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v_1} & \ldots & e^{\lambda_n t}\mathbf{v_n} \end{bmatrix}$$

$$= Se^{\Lambda t} \quad , \quad S = \begin{bmatrix} \mathbf{u_1} & \ldots & \mathbf{u_n} \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Then $\Phi(0) = S$, so

$$e^{At} = Se^{\Lambda t}S^{-1}$$

In our example,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

You could multiply this out, but, actually, the exponential matrix is often a pain in the neck to compute, and is often more useful as a symbolic device. Just like e^x, in fact!

[5] The exponential law

I claim that

$$e^{A(t+s)} = e^{At}e^{As}$$

This is a consequence of "time invariance." We have to see that both sides are equal after multiplying by an arbitrary vector $\mathbf{v}$. Let $\mathbf{x}(t)$ be the solution of $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condtion $\mathbf{x}(0) = \mathbf{v}$: so $\mathbf{x}(t) = e^{At}\mathbf{v}$. Now fix s and let

$$\mathbf{y}(t) = \mathbf{x}(t+s) = e^{A(t+s)}\mathbf{v}$$

Calculate using the chain rule:

$$\frac{d}{dt}\mathbf{y}(t) = \frac{d}{dt}\mathbf{x}(t+s) = \dot{\mathbf{x}}(t+s) = A\mathbf{x}(t+s) = A\mathbf{y}(t)$$

So $\mathbf{y}$ is the solution to $\dot{\mathbf{y}} = A\mathbf{y}$ with $\mathbf{y}(0) = \mathbf{x}(s) = e^{As}\mathbf{v}$. That means that $\mathbf{y}(t) = e^{At}e^{As}\mathbf{v}$. QED

This is the proof of the exponential law even in the 1×1 case; and you will recall that as such it contains the trigonometric addition laws. Powerful stuff!

18.03 LA.11: Inhomogeneous Systems

[**1**] Planetarium
[**2**] Romance
[**3**] Family
[**4**] Variations
[**5**] Exponential Input/Response

[1] Planetarium

So now we have a symbol expressing a solution of

$$\dot{\mathbf{x}} = A\mathbf{x}$$

in terms of initial conditions:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0)$$

We saw for example that for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

[I used the opposite signs on Monday; the calculation is the same.] I want you to have the following vision: The differential equation causes a flow around the origin. As time increases, the whole plane rotates. The differential equation says: the velocity vector is always perpendicular to the position vector (and points off to the left). After a time t has elapsed, whatever starting vector you had has rotated by t radians, counterclockwise.

So let's see: If I start at $\mathbf{v}$ and let s seconds elapse, I'm at $e^{As}\mathbf{v}$. Then I let another t seconds elapse. The effect is that I rotate this new vector by t radians:

$$e^{At}(e^{As})\mathbf{v}$$

On the other hand, all that has happened is that $t + s$ seconds has elapsed since time 0, so

$$e^{At}e^{As}\mathbf{v} = e^{A(t+s)}\mathbf{v}$$

This is the exponential law. I had you thinking of the rotation matrix, but the same reasoning works in complete generality. (Note that I used associativity.)

It shows for example that

$$(e^{At})^{-1} = e^{-At}$$

I think you need another example.

[2] The Romance of matrices

The MIT Humanities people have analyzed the plot of Shakespeare's *Romeo and Juliet*, and determined that it is well modeled by the following system of equations. The two parameters are

$$R = \text{Romeo's affection for Juliet}$$

$$J = \text{Juliet's affection for Romeo}$$

These two young people are totally unaffected by any outside influences; this will be a *homogeneous* system. Romeo's affection for Juliet changes over time. The rate of change is based on how he currently feels and also on how she currently feels. Similarly for Juliet. The Humanities people have discovered that

$$\dot{R} = R - J \quad , \quad \dot{J} = R + J$$

that is

$$\frac{d}{dt}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

So Juliet is a straightforward girl: if she likes him, that causes her to start to like him better. If he likes her, same thing.

Romeo is more complicated. If he likes her, that causes him to start to like her even more. That's normal. But if he sees that she is starting to like him, well, that exerts a negative influence on his feelings towards her.

Question 20.1. This relationship

1. is stable and will settle down to calm old age along a spiral
2. is stable and will settle down along a node
3. is unstable and will blow up along a node
4. is unstable and will probably blow up along a saddle

5. is unstable and will spiral out of control
6. Who's Shakespeare?

Let's see how this plot develops. As the play opens, Romeo has noticed Juliet at a dance, and immediately $R = 1$. Juliet is otherwise occupied, though, and $J = 0$. But the deriviative is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: a promising sign.

Soon she does notice him and her affection rises. That very fact slows his growth of passion, though, so the plot trajectory curves down. Notice that $\dot{R} = 0$ when $R = J$: he peaks out when their affections towards each other are equal.

This continues. The more she likes him, the stronger a drag that is on his growth of affection for her, till he starts to actively dislike the girl. This continues till Juliet starts to lose heart. $\dot{J} = 0$ when $J = -R$. Then she starts to like him less; but she still does love him, so that causes Romeo to like her less, with predictable effects on her feelings towards him.

Presently her feelings become negative, and his continue to crater, but as she like him less he gets more interested. His feelings bottom out over here, but she continues to stay away. She starts to flirt with his friends. Things are very bad, but his complicated nature brings things around. Soon he starts to like her again; that decreases her rate of decline, and soon she bottoms out. As he gets more passionate, she does too, and after a while we're back to $J = 0$.

But what's R now? Well, you can work it out. For the present, let's just notice that $\operatorname{tr} A = 2$ and $\det A = 2$. This puts us in the unstable part of the spiral segment, so we could have predicted all this quite simply.

Shall we solve? $p_A(\lambda) = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1$ has roots $1 \pm i$. Positive real part, nonzero imaginary part: so unstable spiral. Eigenvectors for $1 + i$ are killed by $A - (1 + i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$; for example $\begin{bmatrix} 1 \\ i \end{bmatrix}$. So exponential solution $e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$, with real part $\mathbf{u_1}(t) = e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and imaginary part $\mathbf{u_2}(t) = e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. The general solution is a linear combination. Luckily the starting point we had above is here: our story line was $\mathbf{u_1}$.

We're in luck here again: these two solutions are normalized and we've computed that

$$e^{At} = e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

The second factor is the rotation matrix we had above. So as time increases, the whole romance spins around while expanding exponentially. After one rotation, 2π time has elapsed, so now $R = e^{2\pi} \sim 535.49$! So my picture wasn't very accurate.

[3] Inhomogeneous

Of course this is somewhat oversimplified. The fact is that Romeo and Juliet were influenced by their families, the Montagues and Capulets. So this is not a homogeneous situation after all. A better model might be:

$$\frac{d}{dt}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

The two families in Verona agree on one thing: this romance is a terrible idea. Will they succeed in damping it down?

What's the general form of the answer going to be?

We've learned to look for

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$$

We know about $\mathbf{x_h}$. What might we take for a particular solution, with this constant forcing term (parental pressure)?

How about trying for a constant solution? It would have to be so that the left hand side is zero, so

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

This we can do; we have to compute

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

so

$$\mathbf{x_p} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

So there is a stable point, an equilibrium, where $R = 10$ and $J = 0$, when everything is in balance. It must be quite uncomfortable for poor Romeo. But as soon as someone exhales, you add in a nonzero homogeneous equation and the couple quickly spirals out of control. This is an unstable equilibrium! The attempt at parental control works only if Romeo and Juliet happen to be in this state; and it's very unlikely to work for very long.

[4] **Variations** But what if the input signal here isn't constant? I want to solve

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{q}(t) \quad \text{or} \quad \dot{\mathbf{x}} - A\mathbf{x} = \mathbf{q}$$

Again it suffices to find a single "particular solution."

Let's try what we did before: "variation of parameters." We know that the general solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is of the form

$$c_1\mathbf{u_1(t)} + \cdots + c_n\mathbf{u_n(t)}$$

where $\mathbf{u_1}, \ldots, \mathbf{u_n}$ are independent solutions, and we now have a slick way to write this, using the corresponding "fundamental matrix"

$$\Phi(t) = \begin{bmatrix} \mathbf{u_1}(t) & \mathbf{u_2}(t) & \cdots & \mathbf{u_n}(t) \end{bmatrix}$$

The fact that the columns are solutions is recorded by the matrix equation

$$\frac{d}{dt}\Phi(t) = A\Phi(t)$$

We can write the general solution of the homogeneous equation as $\Phi(t)\mathbf{c}$. The coefficient vector $\mathbf{c}$ is a vector of "parameters." Let's let it vary: try for

$$\mathbf{x}(t) = \Phi(t)\mathbf{u}(t)$$

Plug into the equation:

$$\begin{array}{rcrcl} \dot{\mathbf{x}}(t) & = & A\Phi(t)\mathbf{u}(t) & + & \Phi(t)\dot{\mathbf{u}}(t) \\ -A\mathbf{x}(t) & = & -A\Phi(t)\mathbf{u}(t) & & \\ \hline \mathbf{q}(t) & = & & & \Phi(t)\dot{\mathbf{u}}(t) \end{array}$$

Solve for $\dot{\mathbf{u}}$:

$$\dot{\mathbf{u}}(t) = \Phi(t)^{-t}\mathbf{q}(t)$$

Then integrate and multiply by $\Phi(t)^{-1}$:

$$\mathbf{x}(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{q}(t)\, dt$$

Notice that the indefinite integral is only well defined up to adding a constant vector, which then gets multiplied by $\Phi(t)$: so this is the general solution in a nutshell.

Since we don't care about initial conditions at this point, there's no need to use the exponential matrix and it's generally more convenient not to. If you do, though, you can use the fact that $(e^{At})^{-1} = e^{-At}$ to write

$$\mathbf{x} = e^{At} \int e^{-At} \mathbf{q}(t)\, dt$$

[5] Exponential Input/Response

In this section we consider exponential input signals. Recall two things about ordinary constant coefficient equations. First, with exponential input we can use the method of optimism which leads to algebraic methods. Second, using complex replacement sinusoidal input signals can be handled in the same way as exponential signals.

Let A be a constant matrix, a a constant and $\mathbf{K}$ a constant column vector. We consider here the system

$$\dot{\mathbf{x}} = A\mathbf{x} + e^{at}\mathbf{K}$$

To solve this equation we use the method of optimism and try a particular solution of the form $\mathbf{x_p} = e^{at}\mathbf{v}$, where $\mathbf{v}$ is an unknown constant vector. Substituting $\mathbf{x_p}$ into the equation and doing some simple algebra we find

$$\begin{array}{rrcl} & \mathbf{x_p} & = & A\mathbf{x} + e^{at}\mathbf{K} \\ \Leftrightarrow & ae^{at}\mathbf{v} & = & e^{at}A\mathbf{v} + e^{at}\mathbf{K} \\ \Leftrightarrow & (aI - A)\mathbf{v} & = & \mathbf{K} \\ \Leftrightarrow & \mathbf{v} & = & (aI - A)^{-1}\mathbf{K} \\ \Leftrightarrow & \mathbf{x_p} & = & e^{at}\mathbf{v} \\ & & = & e^{at}(aI - A)^{1}\mathbf{K} \end{array}$$

Notes: 1. In the third equation we replaced $a\mathbf{v}$ by $aI\mathbf{v}$. The identity matrix changes nothing, but allows us to do the algebra of matrix subtraction.

2. This is only valid if $(aI - A)^{-1}$ exists, that is if $\det(aI - A) \neq 0$. (Note, this is equivalent to saying a is *not* an eigenvalue of A.)

Following our usage for ordinary differential equations we call the formula $\mathbf{x} = e^{at}(aI - A)^{-}\mathbf{K}$ the exponential response formula or the exponential input theorem.

Examples

1. Solve

$$\begin{aligned} \dot{x} &= 3x - y + e^{2t} \\ \dot{y} &= 4x - y - e^{2t} \end{aligned}$$

Solution. Here $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $a = 2$, $\mathbf{K} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$(2I - A)^{-1} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \Rightarrow \boxed{\mathbf{x_p} = e^{2t} \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.}$

2. (Same matrix as in example 1.) Solve

$$\begin{aligned} \dot{x} &= 3x - y + 3 \\ \dot{y} &= 4x - y + 2 \end{aligned}$$

Solution: $a = 0,\ \mathbf{K} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad -A^{-1} = -\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$.

$$\boxed{\mathbf{x_p} = -A^{-1} \cdot \mathbf{K} = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.}$$

Note: Here the input is a constant so our method of optimism is equivalent to guessing a constant solution.

3. Solve

$$\begin{aligned} \dot{x} &= x + 2y + \cos t \\ \dot{y} &= 2x + y \end{aligned}$$

Solution: We use complex replacement:

$$\dot{\mathbf{z}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{z} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{x} = \mathrm{Re}\,(\mathbf{z}).$$

The exponential response formula gives

$$\mathbf{z_p}(t) = e^{it}(iI - A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have

$$iI - A = \begin{bmatrix} i-1 & -2 \\ -2 & i-1 \end{bmatrix} \Rightarrow (iI-A)^{-1} = \frac{1}{-4-2i}\begin{bmatrix} i-1 & 2 \\ 2 & i-1 \end{bmatrix}$$

So (not showing all of the complex arithmetic),

$$\begin{aligned} \mathbf{z_p}(t) &= \tfrac{1}{-4-2i} e^{it} \begin{bmatrix} i-1 & 2 \\ 2 & i-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (\cos t + i \sin t) \tfrac{1}{10} \begin{bmatrix} 1-3i \\ -4+2i \end{bmatrix} \\ &= \tfrac{1}{10} \left(\begin{bmatrix} \cos t + 3\sin t \\ -4\cos t - 2\sin t \end{bmatrix} + i \begin{bmatrix} -3\cos t + \sin t \\ 2\cos t - 4\sin t \end{bmatrix} \right) \end{aligned}$$

$$\boxed{\mathbf{x_p}(t) = \mathrm{Re}\,(\mathbf{z_p})(t) = \tfrac{1}{10}\begin{bmatrix} \cos t + 3\sin t \\ -4\cos t - 2\sin t \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \end{bmatrix}}$$

Superposition

For linear constant coefficient equations the principle of superposition allows us to use the exponential input method for input functions like $\mathbf{f} = \begin{bmatrix} 3e^{2t} \\ -e^t \end{bmatrix}$. That is we can split $\mathbf{f}$ into a sum:

$$\mathbf{f} = \begin{bmatrix} 3e^{2t} \\ -e^t \end{bmatrix} = e^{2t}\begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

and solve with each piece separately and then sum the two solutions.

18.03 PDE.1: Fourier's Theory of Heat

1. Temperature Profile.

2. The Heat Equation.

3. Separation of Variables (the birth of Fourier series)

4. Superposition.

In this note we meet our first partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

This is the equation satisfied by the temperature $u(x,t)$ at position x and time t of a bar depicted as a segment,

$$0 \le x \le L, \quad t \ge 0$$

The constant k is the conductivity of the material the bar is made out of.

We will focus on one physical experiment. Suppose that the initial temperature is 1, and then the ends of the bar are put in ice. We write this as

$$\boxed{u(x,0) = 1, \quad 0 \le x \le L} \qquad \boxed{u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0}.$$

The value(s) of $u = 1$ at $t = 0$ are called *initial conditions*. The values at the ends are called *endpoint or boundary conditions*. We think of the initial and endpoint values of u as the input, and the temperature $u(x,t)$ for $t > 0$, $0 < x < L$ as the response. (For simplicity, we assume that only the ends are exposed to the lower temperature. The rest of the bar is insulated, not subject to any external change in temperature. Fourier's techniques also yield answers even when there is heat input over time at other points along the bar.)

As time passes, the temperature decreases as cooling from the ends spreads toward the middle. At the midpoint, $L/2$, one finds Newton's law of cooling,

$$u(L/2,t) \approx ce^{-t/\tau}, \quad t > \tau$$

The so-called characteristic time τ is inversely proportional to the conductivity of the material. If we choose units so that $\tau = 1$ for copper, then according to Wikipedia,

$$\tau \sim 7 \quad \text{(cast iron)}; \quad \tau \sim 7000 \quad \text{(dry snow)}$$

The constant c, on the other hand, is **universal**:

$$c \approx 1.3$$

It depends only on the fact that the shape is a bar (modeled as a line segment).

Fourier figured out not only how to explain c using differential equations, but the whole

$$\textbf{temperature profile:} \quad u(x,t) \approx e^{-t/\tau}h(x); \quad h(x) = \frac{4}{\pi}\sin\left(\frac{\pi}{L}x\right), \quad t > \tau.$$

The shape of h reflects how much faster the temperature drops near the ends than in the middle. It's natural that h should be some kind of hump, symmetric around $L/2$.

We looked at the heat equation applet to see this profile emerge as t increases. It's remarkable that a sine function emerges out of the input $u(x,0) = 1$. There is no evident mechanism creating a sine function, no spring, no circle, no periodic input. The sine function and the number $4/\pi$ arise naturally out of differential equations alone.

Deriving the heat equation. To explain the heat equation, we start with a thought experiment. If we fix the temperature at the ends, $u(0,t) = 0$ and $u(L,t) = T$, what will happen in the long term as $t \to \infty$? The answer is that

$$u(x,t) \to U_{\text{steady}}(x), \quad t \to \infty$$

where U_{steady} is the steady, or equilbrium, temperature, and

$$U_{\text{steady}}(x) = \frac{T}{L}x \quad \text{(linear)}$$

The temperature $u(L/2,t)$ at the midpoint $L/2$ tends to the average of 0 and T, namely $T/2$. At the point $L/4$, half way between 0 and $L/2$, the temperature tends to the average of the temperature at 0 and $T/2$, and so forth.

At a very small scale, this same mechanism, the tendency of the temperature profile toward a straight line equilibrium means that if u is concave down then the temperature in the middle should decrease (so the profile becomes closer to being straight). If u is concave up, then the temperature in the middle should increase (so that, once again, the profile becomes closer to being straight). We write this as

$$\frac{\partial^2 u}{\partial x^2} < 0 \implies \frac{\partial u}{\partial t} < 0$$
$$\frac{\partial^2 u}{\partial x^2} > 0 \implies \frac{\partial u}{\partial t} > 0$$

The simplest relationship that reflects this is a linear (proportional) relationship,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad k > 0$$

Fourier's reasoning. Fourier introduced the heat equation, solved it, and confirmed in many cases that it predicts correctly the behavior of temperature in experiments like the one with the metal bar.

Actually, Fourier crushed the problem, figuring out the whole formula for $u(x,t)$ and not just when the initial value is $u(x,0) = 1$, but also when the initial temperature varies with x. His formula even predicts accurately what happens when $0 < t < \tau$.

Separation of Variables. For simplicity, take $L = \pi$ and $k = 1$. The idea is not to try to solve for what looks like the simplest initial condition namely $u(x,0) = 1$, but instead to look for solutions of the form

$$u(x,t) = v(x)w(t)$$

Plugging into the equation, we find

$$\frac{\partial u}{\partial t} = v(x)\dot{w}(t), \quad \frac{\partial^2 u}{\partial^2 x} = v''(x)w(t)$$

Therefore, since $k = 1$,

$$v(x)\dot{w}(t) = v''(x)w(t) \implies \frac{\dot{w}(t)}{w(t)} = \frac{v''(x)}{v(x)} = c \quad \text{(constant).}$$

This is the first key step. We divided by $v(x)$ and $w(t)$ to "separate" the variables. But the function $\dot{w}(t)/w(t)$ is independent of x, whereas $v''(x)/v(x)$ is indendent of t. And since these are equal, this function depends neither on x nor on t, and must be a constant. Notice also that the constant, which we are calling c for the time being, is the same constant in two separate ordinary differential equations:

$$\dot{w}(t) = cw(t), \quad v''(x) = cv(x).$$

The best way to proceed is to remember the endpoint conditions

$$u(0,t) = u(\pi, 0) = 0 \implies v(0) = v(\pi) = 0.$$

We know what the solutions to $v''(x) = cv(x)$, $v(0) = v(\pi) = 0$ look like. They are

$$v_n(x) = \sin nx, \quad n = 1,\ 2,\ 3,\ \ldots$$

Moreover, $v_n''(x) = -n^2 \sin nx = -n^2 v_n(x)$, so that $c = -n^2$. We now turn to the equation for w, which becomes

$$\dot{w}_n(t) = -n^2 w(t) \implies w_n(t) = e^{-n^2 t}.$$

(We may as well take $w(0) = 1$. We will be taking multiples later.) In summary, we have found a large collection of solutions to the equation, namely,

$$u_n(x,t) = v_n(x) w_n(t) = e^{-n^2 t} \sin nx$$

For these solutions, the endpoint condition $u_n(0,t) = u_n(\pi, t) = 0$ is satisfied, but the initial condition is

$$u_n(x,0) = v_n(x) = \sin nx\,.$$

This is where Fourier made an inspired step. What if we try to write the function $u(x,0) = 1$ as a linear combination of $v_n(x) = \sin nx$?

On the face of it, expressing 1 as a sum of terms like $\sin nx$ makes no sense. We know that $\sin nx$ is zero at the ends $x = 0$ and $x = \pi$. But something tricky should be happening at the ends because the boundary conditions are discontinuous in time. At $t = 0$ we had temperature 1 at the ends, then suddenly when we plunged the ends in ice, we had temperature 0. So it's not crazy that the endpoints should behave in a peculiar way.

If there is any chance to write

$$u(x,0) = 1 = \sum b_n \sin nx, \quad 0 < x < \pi,$$

then it must be that the function is odd. In other words, we need to look at

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Moreover, the function has to be periodic of period 2π. This is none other than the square wave $f(x) = Sq(x)$, the very first Fourier series we computed.

$$1 = Sq(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \right), \quad 0 < x < \pi.$$

Now since initial conditions $v_n(x)$ yield the solution $u_n(x,t)$, we can apply the

Principle of Superposition $\quad u(x,0) = \sum b_n \sin nx \implies u(x,t) = \sum b_n e^{-n^2 t} \sin nx$

In other words, if $u(x, 0) = 1$, $0 < x < \pi$, then

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + \frac{1}{3}e^{-3^2 t}\sin 3x + \frac{1}{5}e^{-5^2 t}\sin 5x + \cdots\right) \quad 0 \le x \le \pi, \quad t > 0.$$

The exact formula for the solution u to the heat equation is this series; it **cannot be expressed in any simpler form**. But often one or two terms already give a good approximation. Fourier series work as well, both numerically and conceptually, as any finite sum of terms involving functions like e^{-t} and $\sin x$. Look at the Heat Equation applet to see the first term (main hump) emerge, while the next term $b_3 e^{-9t}\sin 3x$ tends to zero much more quickly. (The other terms are negligible after an even shorter time.)

For this example, the characteristic time is $\tau = 1$, $e^{-t/\tau} = e^{-t}$, and

$$u(x,t) = \frac{4}{\pi}e^{-t}\sin x + \text{smaller terms as } t \to \infty.$$

To get an idea how small the smaller terms are, take an example.

Example. Fix $t_1 = \ln 2$, then $e^{-t_1} = 1/2$, and

$$u(x,t_1) = \frac{4}{\pi}\left(\frac{1}{2}\sin x + \frac{1}{3\cdot 2^9}\sin 3x + \cdots\right) = \frac{2}{\pi}\sin x \pm 10^{-3}$$

18.03 PDE.2: Decoupling; Insulated ends

1. Normal Modes: $e^{\lambda_k t} v_k$

2. Superposition

3. Decoupling; dot product

4. Insulated ends

In this note we will review the method of separation of variables and relate it to linear algebra. There is a direct relationship between Fourier's method and the one we used to solve systems of equations.

We compare a system of ODE $\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)$ where A is a matrix and $\boldsymbol{u}(t)$ is a vector-valued function of t to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u,\ 0 < x < \pi,\ t > 0; \quad u(0,t) = u(\pi,t) = 0$$

with zero temperature ends. To establish the parallel, we write

$$\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t) \qquad ----- \qquad \dot{u} = \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u \quad (A = (\partial/\partial x)^2)$$

To solve the equations we look for normal modes:

$$\text{Try } \boldsymbol{u}(t) = w(t)\boldsymbol{v}. \qquad ----- \qquad \text{Try } u(x,t) = w(t)v(x).$$

This leads to equations for eigenvalues and eigenvectors:

$$\begin{cases} A\boldsymbol{v} = \lambda \boldsymbol{v} \\ \dot{w} = \lambda w \end{cases} \qquad ------ \qquad \begin{cases} Av = v''(x) = \lambda v(x) \text{ [and } v(0) = v(\pi) = 0] \\ \dot{w}(t) = \lambda w(t) \end{cases}$$

There is one new feature: in addition to the differential equation for $v(x)$, there are endpoint conditions. The response to the system $\dot{u} = A\boldsymbol{u}$ is determined by the initial condition $\boldsymbol{u}(0)$, but the heat equation response is only uniquely identified if we know the endpoint conditions as well as $u(x,0)$.

Eigenfunction Equation. The solutions to

$$v''(x) = \lambda v(x) \text{ and } v(0) = v(\pi) = 0,$$

are known as *eigenfunctions.* They are

$$v_k(x) = \sin kx, \quad k = 1,\, 2,\, \ldots$$

and the eigenvalues $\lambda_k = -k^2$ lead to $w_k(t) = e^{-k^2 t}$.

$$\text{normal modes} : e^{\lambda_k t}\boldsymbol{v}_k \qquad ------ \qquad e^{-k^2 t}\sin(kx)$$

The principle of superposition, then says that

$$\boldsymbol{u}(0) = \sum c_k \boldsymbol{v}_k \implies \boldsymbol{u}(t) = \sum c_k e^{\lambda_k t}\boldsymbol{v}_k$$

and, similarly,

$$u(x,0) = \sum b_k \sin kx \implies u(x,t) = \sum b_k e^{-k^2 t}\sin kx$$

More generally, we will get formats for solutions of the form

$$u(x,t) = \sum b_k e^{-\beta k^2 t} \sin(\alpha k x) \quad \text{or cosines}$$

The scaling will change if the units are different (inches versus meters in x; seconds versus hours in t) and depending on physical constants like the conductivity factor in front of the $(\partial/\partial x)^2$ term, or if the interval is $0 < x < L$ instead of $0 < x < \pi$. Also, we'll see an example with cosines below.

The final issue is how to find the coefficients c_k or b_k. If we have a practical way to find the coefficients c_k in

$$\boldsymbol{u}(0) = \sum c_k \boldsymbol{v}_k,$$

then we say we have decoupled the system. The modes $e^{\lambda_k t}\boldsymbol{v}_k$ evolve according to separate equations $\dot{w}_k = \lambda_k w_k$.

Recall that the dot product of vectors is given, for example, by

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot 2 + (2)(-1) + 3 \cdot 0 = 0$$

When the dot product is zero the vectors are perpendicular. We can also express the length squared of a vector in terms of the dot product:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad \boldsymbol{v} \cdot \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1^2 + 2^2 + 3^2 = (\text{length})^2 = \|\boldsymbol{v}\|^2$$

There is one favorable situation in which it's easy to calculate the coefficients c_k, namely if the eigenvectors $\boldsymbol{v}_k$ are perpendicular to each other

$$\boldsymbol{v}_k \perp \boldsymbol{v}_\ell \iff v_k \cdot \boldsymbol{v}_\ell = 0$$

This happens, in particular, if the matrix A is symmetric. In this case we also normalize the vectors so that their length is one:

$$\|\boldsymbol{v}_k\|^2 = \boldsymbol{v}_k \cdot \boldsymbol{v}_k = 1$$

Then

$$\boxed{c_k = \boldsymbol{v}_k \cdot \boldsymbol{u}(0)}$$

The proof is

$$\boldsymbol{v}_k(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n) = 0 + \cdots + 0 + c_k\boldsymbol{v}_k \cdot \boldsymbol{v}_k + 0 + \cdots = c_k\,.$$

The same mechanism is what makes it possible to compute Fourier coefficients. We have

$$v_k \perp v_\ell \iff \int_0^\pi v_k(x) v_\ell(x)\,dx = 0$$

and

$$\int_0^\pi v_k(x)^2\,dx = \int_0^\pi \sin^2(kx)\,dx = \frac{\pi}{2}$$

To compensate for the length not being 1 we divide by the factor $\pi/2$. It follows that

$$b_k = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin(kx)\,dx$$

The analogy between these integrals and the corresponding dot products is very direct. When evaluating integrals, it makes sense to think of functions as a vectors

$$\vec{f} = [f(x_1), f(x_2), \ldots, f(x_N)]; \quad \vec{g} = [g(x_1), g(x_2), \ldots, g(x_N)].$$

The Riemann sum approximation to an integral is written

$$\int_0^{\pi} f(x)g(x)\,dx \approx \sum_j f(x_j)g(x_j)\Delta x = \vec{f}\cdot\vec{g}\Delta x$$

We have not explained the factor Δx, but this is a normalizing factor that works out after taking into account proper units and dimensional analysis. *To repeat, functions are vectors: we can take linear combinations of them and even use dot products to find their "lengths" and the angle between two of them, as well as distances between them.*

Example 1. Zero temperature ends. We return to the problem from PDE.1, in which the initial conditions and end point conditions were

$$u(x,0) = 1 \quad 0 < x < \pi; \quad u(0,t) = u(\pi,t) = 0 \quad t > 0.$$

Our goal is to express

$$1 = \sum_1^{\infty} b_k \sin(kx), \quad 0 < x < \pi$$

The physical problem does not dictate any value for the function $u(x,0)$ outside $0 < x < \pi$. But if we want it to be represented by this sine series, it's natural to consider the odd function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Moreover, because the sine functions are periodic of period 2π, it's natural to extend f to have period 2π. In other words, $f(x) = Sq(x)$, the square wave. We computed this series in L26 (same formula as above for b_k) and found

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Therefore the solution is

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + e^{-3^2 t}\frac{\sin 3x}{3} + \cdots\right)$$

Example 2. Insulated Ends.

When the ends of the bar are insulated, we have the usual heat equation (taken here for simplicity with conductivity 1 and on the interval $0 < x < \pi$) given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0,$$

with the new feature that the heat flux across 0 and π is zero. This is expressed by the equations

$$\text{insulated ends : } \quad \boxed{\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad t > 0}$$

Separation of variables $u(x,t) = v(x)w(t)$ yields a new eigenfunction equation:

$$v''(x) = \lambda v(x), \qquad \boxed{v'(0) = v'(\pi) = 0}$$

whose solution are

$$v_k(x) = \cos(kx), \quad k = 0,\, 1,\, 2 \ldots$$

Note that the index starts at $k = 0$ because $\cos 0 = 1$ is a nonzero function. The eigenvalues are $\lambda_k = -k^2$, but now the first eigenvalue is

$$\lambda_0 = 0.$$

This will make a difference when we get to the physical interpretation. Since $\dot{w}_k(t) = -k^2 w_k(t)$, we have

$$w_k(t) = e^{-k^2 t}$$

and the normal modes are

$$e^{-k^2 t}\cos(kx), \quad k = 0,\, 1,\, \ldots$$

The general solution has the form (or format)

$$\boxed{u(x,t) = \frac{a_0}{2} e^{0t} + \sum_1^\infty a_k e^{-k^2 t}\cos(kx)}$$

(Here we have anticipated the standard Fourier series format by treating the constant term differently.)

Let us look at one specific case, namely, initial conditions

$$u(x,0) = x, \quad 0 < x < \pi$$

We can imagine an experiment in which the temperature of the bar is 0 on one end and 1 on the other. After a fairly short period, it will have stabilized to the equilibrium distribution x. Then we insulate both ends (cease to provide heat or cooling that would maintain the ends at 0 and 1 respectively). What happens next?

To find out we need to express x as a cosine series. So we extend it evenly to

$$g(x) = |x|, \quad |x| < \pi, \quad \text{with period } 2\pi$$

This is a triangular wave and we calculated its series using $g'(x) = Sq(x)$ as

$$g(x) = \frac{a_0}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

The constant term is not determined by $g'(x) = Sq(x)$ and must be calculated separately. Recall that

$$a_0 = \frac{2}{\pi}\int_0^\pi g(x)\cos 0\, dx = \frac{2}{\pi}\int_0^\pi x\, dx = \left.\frac{x^2}{\pi}\right|_0^\pi = \pi$$

Put another way,

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} g(x)\,dx = \text{average}(g) = \frac{\pi}{2}$$

Thus, putting it all together,

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\left(e^{-t}\cos x + e^{-3^2 t}\frac{\cos 3x}{3^2} + \cdots\right)$$

Lastly, to check whether this makes sense physically, consider what happens as $t \to \infty$. In that case,

$$u(x,t) \to \frac{\pi}{2}$$

In other words, when the bar is insulated, the temperature tends to a constant equal to the average of the initial temperature.

18.03 PDE.3: The Wave Equation

1. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
2. Normal Modes.
3. Wave fronts and wave speed (d'Alembert solution).
4. Real life waves.

I showed you an elastic band which oscillated. In the literature this is usually referred to as a vibrating string. Let $u(x,t)$ represent the vertical displacement of the string. In other words, for each fixed time t, the graph of the string is $y = y(x) = u(x,t)$.

Consider, as in the case of the heat equation, the equilibrium position in which the elastic string is not moving. In that case, it's in a straight line. If the string is concave down (curved above the equilibrium) then the elasticity pulls the string back down in the middle. This tendency is a restoring force like that of a spring. Since force is proportional to acceleration ($F = ma$), and acceleration is $\partial^2 u/\partial t^2$, we have

$$\frac{\partial^2 u}{\partial x^2} < 0 \implies \frac{\partial^2 u}{\partial t^2} < 0.$$

Similarly for concave up configurations,

$$\frac{\partial^2 u}{\partial x^2} > 0 \implies \frac{\partial^2 u}{\partial t^2} > 0.$$

The simplest rule that realizes this effect is proportionality,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

We wrote the constant first as $c > 0$ and rewrote it later as $c^2 > 0$ when it turned out that the right units of c are meters per second so that it represents a speed.

Let us take $c = 1$ for simplicity and fix the ends of the string at 0 and π. The normal modes have the form

$$u(x,t) = v(x)w(t), \quad v(0) = v(\pi) = 0.$$

Substituting into the equation, we find

$$\ddot{w}(t)v(x) = w(t)v''(x),$$

which leads (via the separation of variables method, $\ddot{w}(t)/w(t) = v''(x)/v(x) = \lambda$) to the equations

$$v''(x) = \lambda v(x), \quad v(0) = v(\pi) = 0.$$

These are the same as for the heat equation with fixed ends, and we already found a complete list of solutions (up to multiples)

$$v_n(x) = \sin(nx), \quad n = 1, 2, 3, \dots .$$

We also have $v_n''(x) = -n^2 v_n(x)$, so that $\lambda_n = -n^2$. What is different this time is that the equation for w_n is second order

$$\ddot{w}_n(t) = -n^2 w_n(t)$$

A second order equation has a two dimensional family of solutions. In this case, they are

$$w_n(t) = a\cos(nt) + b\sin(nt)$$

This highlights the main conceptual differences between the heat and wave equations. These wave equation solutions are oscillatory in t not exponential. Also the extra degree of freedom means we have to specify not only the initial position $u(x,0)$, but also the initial velocity $(\partial/\partial t)u(x,0)$.

We will take the simplest initial velocity, namely, initial velocity 0 (also the most realistic choice when we pluck a string). Thus we impose the conditions

$$0 = \dot{w}_n(0) = -an\sin(0) + bn\cos 0 = bn \quad\Longrightarrow\quad b = 0$$

and (for simplicity) $w_n(0) = a = 1$. Now $a = 1$ and $b = 0$, so that the normal modes are

$$u_n(x,t) = \cos(nt)\sin(nx)$$

The principle of superposition says that if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad 0 < x < \pi$$

then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos(nt)\sin(nx), \quad 0 < x < \pi$$

solves the wave equation with constant $c = 1$, initial condition $u(x,0) = f(x)$ and initial velocity $(\partial/\partial t)u(x,0) = 0$ and endpoint conditions $u(0,t) = u(\pi,t) = 0$, $t > 0$. (Actually, the wave equation is reversible, and these equations are satisfied for $-\infty < t < \infty$.)

Notice that there are now **two** inputs at time $t = 0$, the initial position $f(x)$ and the initial velocity which we have set equal to 0 for simplicity.[1] This is consistent with the fact that the equation is second order in the t variable.

Wave fronts. D'Alembert figured out another formula for solutions to the one (space) dimensional wave equation. This works for initial conditions $v(x)$ is defined for all x, $-\infty < x < \infty$. The solution (for $c = 1$) is

$$u_1(x,t) = v(x-t)$$

We can check that this is a solution by plugging it into the equation,

$$\frac{\partial^2}{\partial t^2}u(x,t) = (-1)^2 v''(x-t) = v''(x-t) = \frac{\partial^2}{\partial x^2}u(x,t).$$

Similarly, $u_2(x,t) = v(x+t)$ is a solution.

We plot the behavior of this solution using a space-time diagram and taking the simplest initial condition, namely the step function,

$$v(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

The solution

$$u_1(x,t) = v(x-t)$$

[1] If the initial velocity is not zero, one can write a series solution involving, in addition, the other solution to the equation for $w_n(t)$, namely $\sin(nt)$

takes on only two values, 0 and 1. Therefore, we can draw a picture showing how the solution behaves by drawing the (x,t) plane and dividing the plane into the region where $u_1(x,t) = 1$ versus $u_1(x,t) = 0$. This kind of space-time diagram is often used to describe the behavior of waves.[2] We have

$$u_1(x,t) = v(x-t) = 1 \iff x - t < 0 \iff t > x,$$

and

$$u_1(x,t) = v(x-t) = 0 \iff x - t > 0 \iff t < x.$$

The divider between the places where $u_1 = 1$ and $u_1 = 0$ is known as the *wave front* and it is located at the line of slope 1,

$$t = x$$

We drew this, indicating $u_1 = 1$ above the line $t = x$ and $u_1 = 0$ below.

The only feature we want to extract from this picture is that as time increases, the wave moves at a constant speed 1. An observer at $x = 10$ will see the wave front pass (the value of $u(x,t)$ switch from 0 to 1) at time $t = 10$. If we were to change the constant c we would obtain a solution $v(x-ct)$ whose wave front travels at the speed c.

In order to understand what we are looking at in simulations and real life, we need to enforce both initial conditions, position and velocity. If

$$u(x,t) = av(x-t) + bv(x+t)$$

and $u(x,0) = v(x)$, $(\partial/\partial t)u(x,0) = 0$, then we have

$$u(x,0) = av(x) + bv(x) = v(x) \implies a + b = 1;$$

$$(\partial/\partial t)u(x,0) = -av'(x) + bv'(x) = 0 \implies -a + b = 0.$$

Hence,

$$a = b = \frac{1}{2}; \quad u(x,t) = \frac{1}{2}(v(x-t) + v(x+t))$$

is the solution of interest to us. Plotting the three regions $u = 1$, $u = 1/2$ and $u = 0$, we see the plane divided by a V shape with 1 on the left 1/2 in the middle and 0 on the right. This says that there is a wave front travelling both left and right from the source. This is like what happens with point sources in higher dimensions. In two dimensions, a pebble dropped in a quiet pond will send a disturbance (wave front) outward in all directions with equal speed, forming a circular wave front. The space-time picture of this wave front looks like an ice-cream cone. In three dimensions, a source of sound our light will send out wave in all directions. The wave front is an expanding sphere. In one dimension, the geometry is less evident. What is happening is that there are only two possible directions (left and right) instead of a whole circle or sphere of directions.

The next step is to note that it is not realistic for a string to have a jump discontinuity like the step function. But any feature of the graph will travel at the wave speed. For example, if we strech a string to a triangular shape with a kink, then the kink will travel (in both directions) at the wave speed. We looked at this in the applet, and also saw that when the kink hits the fixed ends it bounces off and returns. (The kinks always go at the speed c; take a look in the applet, in which you can adjust the wave speed.)

Real life waves. Finally, we looked at a slow motion film of an vibrating elastic band at

[2]In the general theory of relativity, certain rescaled space-time diagrams are used to keep track of light as it travels into black holes. In that case, they are called Penrose diagrams.

http://www.acoustics.salford.ac.uk/feschools/waves/quicktime/elastic2512K_Stream.mov

The video shows the kink(s) in the band propagating at a steady speed and bouncing off the ends. This resembles what we saw in the applet. Then we witnessed a new phenomenon: *damping*. As the system loses energy, all the modes are damped.

The damped wave equation is

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad b > 0$$

This introduces the factor $e^{-bt/2}$ in the solutions.

Closer examination indicates (not entirely clear without some more detailed numerical simulation!) that the main mode(s) are revealed more quickly than would be the case using a linear damped wave equation. The higher frequency modes are being damped more quickly than the lower frequency ones. A scientist or engineer might say that this system is exhibiting some *nonlinearity* in its response. The modes of a linear wave equation would all have the same damping constant b. This suggests that one can't explain fully this rubber band using linear differential equations alone. It satisfies some *nonlinear* differential equation that shares many features with the linear equation, including the wave speed and the normal modes.

18.03 EXERCISES

1. First-order ODE's

1A. Introduction; Separation of Variables

1A-1. Verify that each of the following ODE's has the indicated solutions (c_i, a are constants):

a) $y'' - 2y' + y = 0, \qquad y = c_1 e^x + c_2 x e^x$

b) $xy' + y = x \sin x, \qquad y = \dfrac{\sin x + a}{x} - \cos x$

1A-2. On how many arbitrary constants (also called *parameters*) does each of the following families of functions depend? (There can be less than meets the eye. . . ; a, b, c, d, k are constants.)

a) $c_1 e^{kx}$ b) $c_1 e^{x+a}$ c) $c_1 + c_2 \cos 2x + c_3 \cos^2 x$ d) $\ln(ax + b) + \ln(cx + d)$

1A-3. Write down an explicit solution (involving a definite integral) to the following initial-value problems (IVP's):

a) $y' = \dfrac{1}{y^2 \ln x}, \quad y(2) = 0$ b) $y' = \dfrac{ye^x}{x}, \quad y(1) = 1$

1A-4. Solve the IVP's (initial-value problems):

a) $y' = \dfrac{xy + x}{y}, \quad y(2) = 0$ b) $\dfrac{du}{dt} = \sin t \cos^2 u, \quad u(0) = 0$

1A-5. Find the general solution by separation of variables:

a) $(y^2 - 2y)\, dx + x^2 dy = 0$ b) $x\dfrac{dv}{dx} = \sqrt{1 - v^2}$

c) $y' = \left(\dfrac{y - 1}{x + 1}\right)^2$ d) $\dfrac{dx}{dt} = \dfrac{\sqrt{1 + x}}{t^2 + 4}$

1B. Standard First-order Methods

1B-1. Test the following ODE's for exactness, and find the general solution for those which are exact.

a) $3x^2 y\, dx + (x^3 + y^3)\, dy = 0$ b) $(x^2 - y^2)\, dx + (y^2 - x^2)\, dy = 0$

c) $ve^{uv} du + ye^{uv} dv = 0$ d) $2xy\, dx - x^2 dy = 0$

1B-2. Find an integrating factor and solve:

a) $2x\, dx + \dfrac{x^2}{y}\, dy = 0$ b) $y\, dx - (x + y)\, dy = 0, \quad y(1) = 1$

c) $(t^2 + 4)\, dt + t\, dx = x\, dt$ d) $u(du - dv) + v(du + dv) = 0. \quad v(0) = 1$

1B-3. Solve the homogeneous equations

a) $y' = \dfrac{2y-x}{y+4x}$ b) $\dfrac{dw}{du} = \dfrac{2uw}{u^2-w^2}$ c) $xy\,dy - y^2dx = x\sqrt{x^2-y^2}\,dx$

1B-4. Show that a change of variable of the form $u = \dfrac{y}{x^n}$ turns $y' = \dfrac{4+xy^2}{x^2y}$ into an equation whose variables are separable, and solve it.

(Hint: as for homogeneous equations, since you want to get rid of y and y', begin by expressing them in terms of u and x.)

1B-5. Solve each of the following, finding the general solution, or the solution satisfying the given initial condition.

a) $xy' + 2y = x$

b) $\dfrac{dx}{dt} - x\tan t = \dfrac{t}{\cos t}, \quad x(0) = 0$

c) $(x^2-1)y' = 1 - 2xy$

d) $3v\,dt = t(dt - dv), \quad v(1) = \frac{1}{4}$

1B-6. Consider the ODE $\dfrac{dx}{dt} + ax = r(t)$, where a is a positive constant, and $\lim\limits_{t\to\infty} r(t) = 0$. Show that if $x(t)$ is any solution, then $\lim\limits_{t\to\infty} x(t) = 0$. (Hint: use L'Hospital's rule.)

1B-7. Solve $y' = \dfrac{y}{y^3+x}$. Hint: consider $\dfrac{dx}{dy}$.

1B-8. The **Bernoulli** equation. This is an ODE of the form $y' + p(x)y = q(x)y^n, \quad n \neq 1.$ Show it becomes linear if one makes the change of dependent variable $u = y^{1-n}$.

(Hint: begin by dividing both sides of the ODE by y^n.)

1B-9. Solve these Bernoulli equations using the method decribed in 1B-8:

a) $y' + y = 2xy^2$

b) $x^2y' - y^3 = xy$

1B-10. The **Riccati** equation. After the linear equation $y' = A(x) + B(x)\,y$, in a sense the next simplest equation is the Riccati equation

$$y' = A(x) + B(x)y + C(x)y^2,$$

where the right-hand side is now a quadratic function of y instead of a linear function. In general the Riccati equation is not solvable by elementary means. However,

a) show that if $y_1(x)$ is a solution, then the general solution is

$$y = y_1 + u,$$

where u is the general solution of a certain Bernoulli equation (cf. 1B-8).

b) Solve the Riccati equation $y' = 1 - x^2 + y^2$ by the above method.

1B-11. Solve the following second-order autonomous equations ("autonomous" is an important word; it means that the independent variable does not appear explicitly in the equation — it does lurk in the derivatives, of course.)

a) $y'' = a^2y$ b) $yy'' = y'^2$ c) $y'' = y'(1+3y^2), \quad y(0) = 1,\ y'(0) = 2$

1B-12. For each of the following, tell what type of ODE it is — i.e., what method you would use to solve it. (Don't actually carry out the solution.) For some, there are several methods which could be used.

1. $(x^3 + y)\,dx + x\,dy = 0$
2. $\dfrac{dy}{dt} + 2ty - e^{-t} = 0$
3. $y' = \dfrac{x^2 - y^2}{5xy}$
4. $(1 + 2p)\,dq + (2 - q)\,dp = 0$
5. $\cos x\,dy = (y \sin x + e^x)\,dx$
6. $x(\tan y)y' = -1$
7. $y' = \dfrac{y}{x} + \dfrac{1}{y}$
8. $\dfrac{dv}{du} = e^{2u+3v}$
9. $xy' = y + xe^{y/x}$
10. $xy' - y = x^2 \sin x$
11. $y' = (x + e^y)^{-1}$
12. $y' + \dfrac{2y}{x} - \dfrac{y^2}{x} = 0$
13. $\dfrac{dx}{dy} = -x\left(\dfrac{2x^2y + \cos y}{3x^2y^2 + \sin y}\right)$
14. $y' + 3y = e^{-3t}$
15. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$
16. $\dfrac{y' - 1}{x^2} = 1$
17. $xy' - 2y + y^2 = x^4$
18. $y'' = \dfrac{y(y+1)}{y'}$
19. $t\dfrac{ds}{dt} = s(1 - \ln t + \ln s)$
20. $\dfrac{dy}{dx} = \dfrac{3 - 2y}{2x + y + 1}$
21. $x^2y' + xy + y^2 = 0$
22. $y' \tan(x + y) = 1 - \tan(x + y)$
23. $y\,ds - 3s\,dy = y^4\,dy$
24. $du = -\dfrac{1 + u\cos^2 t}{t\cos^2 t}\,dt$
25. $y' + y^2 + (2x + 1)y + 1 + x + x^2 = 0$
26. $y'' + x^2y' + 3x^3 = \sin x$

1C. Graphical and Numerical Methods

1C-1. For each of the following ODE's, draw a direction field by using about five isoclines; the picture should be square, using the intervals between -2 and 2 on both axes. Then sketch in some integral curves, using the information provided by the direction field. Finally, do whatever else is asked.

a) $y' = -\dfrac{y}{x}$; solve the equation exactly and compare your integral curves with the correct ones.

b) $y' = 2x + y$; find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, not from a picture).

c) $y' = x - y$; same as in (b).

d) $y' = x^2 + y^2 - 1$

e) $y' = \dfrac{1}{x + y}$; use the interval -3 to 3 on both axes; draw in the integral curves that pass respectively through $(0, 0)$, $(-1, 1)$, $(0, -2)$. Will these curves cross the line $y = -x - 1$? Explain by using the Intersection Principle (Notes G, (3)).

1C-2. Sketch a direction field, concentrating on the first quadrant, for the ODE

$$y' = \frac{-y}{x^2 + y^2} \ .$$

Explain, using it and the ODE itself how one can tell that the solution $y(x)$ satisfying the initial condition $y(0) = 1$

a) is a decreasing function for $y > 0$;
b) is always positive for $x > 0$.

1C-3. Let $y(x)$ be the solution to the IVP $y' = x - y, \quad y(0) = 1.$

a) Use the Euler method and the step size $h = .1$ to to find an approximate value of $y(x)$ for $x = .1, .2, .3$. (Make a table as in notes G).
Is your answer for $y(.3)$ too high or too low, and why?

b) Use the Modified Euler method (also called Improved Euler, or Heun's method) and the step size $h = .1$ to determine the approximate value of $y(.1)$. Is the value for $y(.1)$ you found in part (a) corrected in the right direction — e.g., if the previous value was too high, is the new one lower?

1C-4. Use the Euler method and the step size .1 on the IVP $y' = x + y^2, \quad y(0) = 1,$ to calculate an approximate value for the solution $y(x)$ when $x = .1, .2, .3$. (Make a table as in Notes G.) Is your answer for $y(.3)$ too high or too low?

1C-5. Prove that the Euler method converges to the exact value for $y(1)$ as the progressively smaller step sizes $h = 1/n, \quad n = 1, 2, 3, \ldots$ are used, for the IVP

$$y' = x - y, \qquad y(0) = 1 \ .$$

(First show by mathematical induction that the approximation to y(1) gotten by using the step size $1/n$ is

$$y_n = 2(1 - h)^n - 1 + nh \ .$$

The exact solution is easily found to be $y = 2e^{-x} + x - 1$.)

1C-6. Consider the IVP $y' = f(x), \quad y(0) = y_0.$

We want to calculate $y(2nh)$, where h is the step size, using n steps of the Runge-Kutta method.

The exact value, by Chapter D of the notes, is $y(2nh) = y_0 + \displaystyle\int_0^{2nh} f(x)\, dx$.

Show that the value for $y(2nh)$ produced by Runge-Kutta is the same as the value for $y(2nh)$ obtained by using Simpson's rule to evaluate the definite integral.

1C-7. According to the existence and uniqueness theorem, under what conditions on $a(x), b(x)$, and $c(x)$ will the IVP

$$a(x)\, y' + b(x)\, y = c(x), \quad y(x_0) = y_0$$

have a unique solution in some interval $[x_0 - h, x_0 + h]$ centered around x_0?

1D. Geometric and Physical Applications

1D-1. Find all curves $y = y(x)$ whose graphs have the indicated geometric property. (Use the geometric property to find an ODE satisfied by $y(x)$, and then solve it.)

a) For each tangent line to the curve, the segment of the tangent line lying in the first quadrant is bisected by the point of tangency.

b) For each normal to the curve, the segment lying between the curve and the x-axis las constant length 1.

c) For each normal to the curve, the segment lying between the curve and the x-axis is bisected by the y-axis.

d) For a fixed a, the area under the curve between a and x is proportional to $y(x) - y(a)$.

1D-2. For each of the following families of curves,

(i) find the ODE satisfied by the family (i.e., having these curves as its integral curves);
(ii) find the orthogonal trajectories to the given family;
(iii) sketch both the original family and the orthogonal trajectories.

a) all lines whose y-intercept is twice the slope
b) the exponential curves $y = ce^x$
c) the hyperbolas $x^2 - y^2 = c$
d) the family of circles centered on the y-axis and tangent to the x-axis.

1D-3. Mixing A container holds V liters of salt solution. At time $t = 0$, the salt concentration is c_0 g/liter. Salt solution having concentration c_1 is added at the rate of k liters/min, with instantaneous mixing, and the resulting mixture flows out of the container at the same rate. How does the salt concentration in the tank vary with time?

Let $x(t)$ be the *amount* of salt in the tank at time t. Then $c(t) = \dfrac{x(t)}{V}$ is the concentration of salt at time t.

a) Write an ODE satisfied by $x(t)$, and give the initial condition.

b) Solve it, assuming that it is pure water that is being added. (Lump the constants by setting $a = k/V$.)

c) Solve it, assuming that c_1 is constant; determine $c(t)$ and find $\lim_{t\to\infty} c(t)$. Give an intuitive explanation for the value of this limit.

d) Suppose now that c_1 is not constant, but is decreasing exponentially with time:

$$c_1 = c_0 e^{-\alpha t}, \qquad \alpha > 0.$$

Assume that $a \neq \alpha$ (cf. part (b)), and determine $c(t)$, by solving the IVP. Check your answer by putting $\alpha = 0$ and comparing with your answer to (c).

1D-4. Radioactive decay A radioactive substance **A** decays into **B**, which then further decays to **C**.

a) If the decay constants of **A** and **B** are respectively λ_1 and λ_2 (the decay constant is by definition ($\ln 2$/half-life)), and the initial amounts are respectively A_0 and B_0, set up an ODE for determining $B(t)$, the amount of **B** present at time t, and solve it. (Assume $\lambda_1 \neq \lambda_2$.)

b) Assume $\lambda_1 = 1$ and $\lambda_2 = 2$. Tell when $B(t)$ reaches a maximum.

1D-5. Heat transfer According to Newton's Law of Cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature.

At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is 80^oC. If the room temperature is 20^oC, when will the water have cooled to 60^oC? (Set up and solve an ODE for $T(t)$.)

1D-6. Motion A mass m falls through air under gravity. Find its velocity $v(t)$ and its terminal velocity (that is, $\lim_{t\to\infty} v(t)$) assuming that

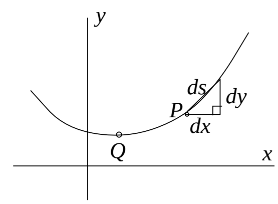
a) air resistance is kv (k constant; this is valid for small v);

b) air resistance is kv^2 (k constant; this is valid for high v).

Call the gravitational constant g. In part (b), lump the constants by introducing a parameter $a = \sqrt{gm/k}$.

1D-7. A loaded cable is hanging from two points of support, with Q the lowest point on the cable. The portion QP is acted on by the total load W on it, the constant tension T_Q at Q, and the variable tension T at P. Both W and T vary with the point P.

Let s denote the length of arc QP.



a) Show that $\dfrac{dx}{T_Q} = \dfrac{dy}{W} = \dfrac{ds}{T}$.

b) Deduce that if the cable hangs under its own weight, and $y(x)$ is the function whose graph is the curve in which the cable hangs, then

(i) $y'' = k\sqrt{1+y'^2}$, $\quad$ k constant

(ii) $y = \sqrt{s^2+c^2} + c_1$, $\quad$ c, c_1 constants

c) Solve the suspension bridge problem: the cable is of negligible weight, and the loading is of constant horizontal density. ("Solve" means: find $y(x)$.)

d) Consider the "Marseilles curtain" problem: the cable is of negligible weight, and loaded with equally and closely spaced vertical rods whose bottoms lie on a horizontal line. (Take the x-axis as the line, and show $y(x)$ satisfies the ODE $y'' = k^2y$.)

1E. First-order autonomous ODE's

1E-1. For each of the following autonomous equations $dx/dt = f(x)$, obtain a qualitative picture of the solutions as follows:

(i) draw horizontally the axis of the dependent variable x, indicating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points; label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn in dashed lines;

(ii) use the information in the first picture to make a second picture showing the tx-plane, with a set of typical solutions to the ODE: the sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

a) $x' = x^2 + 2x$
b) $x' = -(x-1)^2$
c) $x' = 2x - x^2$
d) $x' = (2-x)^3$

2. Higher-order Linear ODE's

2A. Second-order Linear ODE's: General Properties

2A-1. On the right below is an abbreviated form of the ODE on the left:

$$(*) \qquad y'' + p(x)y' + q(x)y = r(x) \qquad\qquad Ly = r(x) \ ;$$

where L is the *differential operator*:

$$L = D^2 + p(x)D + q(x) \ .$$

a) If u_1 and u_2 are any two twice-differentiable functions, and c is a constant, then

$$L(u_1 + u_2) = L(u_1) + L(u_2) \qquad \text{and} \qquad L(c\,u) = c\,L(u).$$

Operators which have these two properties are called **linear** . Verify that L is linear, i.e., that the two equations are satisfied.

b) Show that if y_p is a solution to (*), then all other solutions to (*) can be written in the form

$$y = y_c + y_p \ ,$$

where y_c is a solution to the *associated homogeneous equation* $Ly = 0$.

2A-2.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1 e^x + c_2 e^{2x}$.

b) Verify for this ODE that the IVP consisting of the ODE together with the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_0' \qquad y_0,\ y_0' \ \text{ constants}$$

is always solvable.

2A-3.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1 x + c_2 x^2$.

b) Show that there is no solution to the ODE you found in part (a) which satisfies the initial conditions $y(0) = 1, \quad y'(0) = 1$.

c) Why doesn't part (b) contradict the existence theorem for solutions to second-order linear homogeneous ODE's? (Book: Theorem 2, p. 110.)

2A-4. Consider the ODE $y'' + p(x)y' + q(x)y = 0$.

a) Show that if p and q are continuous for all x, a solution whose graph is tangent to the x-axis at some point must be identically zero, i.e., zero for all x.

b) Find an equation of the above form having x^2 as a solution, by calculating its derivatives and finding a linear equation connecting them. Why isn't part (a) contradicted, since the function x^2 has a graph tangent to the x axis at 0?

2A-5. Show that the following pairs of functions are linearly independent, by calculating their Wronskian.

a) $e^{m_1 x}$, $e^{m_2 x}$, $m_1 \neq m_2$ b) e^{mx}, xe^{mx} (can $m = 0$?)

2A-6. Consider $y_1 = x^2$ and $y_2 = x|x|$. (Sketch the graph of y_2.)

a) Show that $W(y_1, y_2) \equiv 0$ (i.e., is identically zero).

b) Show that y_1 and y_2 are not linearly dependent on any interval (a, b) containing 0. Why doesn't this contradict theorem 3b, p. 116 in your book?

2A-7. Let y_1 and y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$.

a) Prove that $\dfrac{dW}{dx} = -p(x)W$, where $W = W(y_1, y_2)$, the Wronskian.

b) Prove that if $p(x) = 0$, then $W(y_1, y_2)$ is always a constant.

c) Verify (b) by direct calculation for $y'' + k^2 y = 0$, $k \neq 0$, whose general solution is $y_1 = c_1 \sin kx + c_2 \cos kx$.

2B. Reduction of Order

2B-1. Find a second solution y_2 to $y'' - 2y' + y = 0$, given that one solution is $y_1 = e^x$, by three methods:

a) putting $y_2 = ue^x$ and determining $u(x)$ by substituting into the ODE;

b) determining $W(y_1, y_2)$ using Exercise 2A-7a, and from this getting y_2;

c) by using the general formula $y_2 = y_1 \displaystyle\int \frac{1}{y_1^2} e^{-\int p\,dx} dx$.

d) If you don't get the same answer in each case, account for the differences. (What is the most general form for y_2?)

2B-2. In Exercise 2B-1, prove that the general formula in part (c) for a second solution gives a function y_2 such that y_1 and y_2 are linearly independent. (Calculate their Wronskian.)

2B-3. Use the method of reduction of order (as in 2B-1a) to find a second solution to

$$x^2 y'' + 2xy' - 2y = 0 \ ,$$

given that one solution is $y_1 = x$.

2B-4. Find the general solution on the interval $(-1, 1)$ to the ODE

$$(1 - x^2)y'' - 2xy' + 2y = 0,$$

given that $y_1 = x$ is a solution.

2C. Second-order Linear ODE's with Constant Coefficients

2C-1. Find the general solution, or the solution satisfying the given initial conditions, to each of the following:

a) $y'' - 3y' + 2y = 0$

b) $y'' + 2y' - 3y = 0; \quad y(0) = 1,\ y'(0) = -1$

c) $y'' + 2y' + 2y = 0$

d) $y'' - 2y' + 5y = 0; \quad y(0) = 1,\ y'(0) = -1$

e) $y'' - 4y' + 4y = 0; \quad y(0) = 1,\ y'(0) = 1$

2C-2. Show by using the Wronskian criterion that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are linearly independent. Are there any restrictions on the constants a and b ?

2C-3. Consider $y'' + cy' + 4y = 0$, c constant. For each statement below, tell for what value(s) of c it holds (indicate reasoning):

a) the equation has oscillatory solutions

b) all solutions are damped oscillations

2C-4. **Euler's equidimensional equation** is the ODE

$$x^2y'' + pxy' + qy = 0; \qquad p,\ q \text{ constants.}$$

a) Show that setting $x = e^t$ changes it into an equation with constant coefficients.

b) Use this to find the general solution to $x^2y'' + xy' + y = 0$.

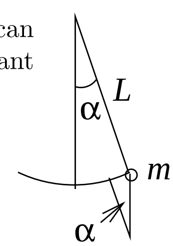
2C-5. The equation $mx'' + cx' + kx = 0$ represents the motion of a damped spring-mass system. (The independent variable is the time t.)

How are the constants m, c, k related if the system is critically damped (i.e., just on the edge of being oscillatory)?

2C-6. Show that the angle α of the pendulum swinging with small amplitude (so you can use the approximation $\sin\alpha \approx \alpha$) approximately obeys a second-order ODE with constant coefficients. Use

$$L = \text{length}, \qquad m = \text{ mass}, \qquad \text{damping } = mc\frac{d\alpha}{dt}, \quad \text{for some constant } c$$

If the motion is undamped, i.e., $c = 0$, express the period in terms of L, m, and the gravitational constant g.



2C-7. For each of the following, tell what you would use as the trial solution in determining a particular solution by the method of undetermined coefficients

a) $y'' + 2y' + 2y = x + e^x$

b) $y'' - 4y' = \cos 2x$

c) $y'' + 4y = \ 3\cos 2x$

d) $y'' - 2y' + y = 3e^x$

e) $y'' - 3y' + 2y = e^{-x} + 3e^{2x}$

f) $y'' - 6y' + 9y = 2xe^{3x}$

2C-8. Find the general solution, or the solution satisfying the given initial conditions:

a) $y'' - 6y' + 5y = e^x$

b) $y'' + 4y = 2\cos x, \quad y(0) = 0,\ y'(0) = 1$

c) $y'' + y' + y = 2x\,e^x$

d) $y'' - y = x^2, \quad y(0) = 0,\ y'(0) = -1$

2C-9. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Prove that if y_i is a particular solution when $r = r_i(x)$, $(i = 1, 2)$, then $y_1 + y_2$ is a particular solution when $r = r_1 + r_2$. (Use the ideas of Exercise 2A-1.)

b) Use part (a) to find a particular solution to $y'' + 2y' + 2y = 2x + \cos x$.

2C-10. A series RLC-circuit is modeled by either of the ODE's (the second equation is just the derivative of the first)

$$Lq'' + Rq' + \frac{q}{C} = \mathcal{E},$$

$$Li'' + Ri' + \frac{i}{C} = \mathcal{E}',$$

where $q(t)$ is the charge on the capacitor, and $i(t)$ is the current in the circuit; $\mathcal{E}(t)$ is the applied electromotive force (from a battery or generator), and the constants L, R, C are respectively the inductance of the coil, the resistance, and the capacitance, measured in some compatible system of units.

a) Show that if $R = 0$ and $\mathcal{E} = 0$, then $q(t)$ varies periodically, and find the period. (Assume $L \neq 0$.)

b) Assume $\mathcal{E} = 0$; how must R, L, C be related if the current oscillates?

c) If $R = 0$ and $\mathcal{E} = E_0 \sin \omega t$, then for a certain ω_0, the current will have large amplitude whenever $\omega \approx \omega_0$. What is the value of ω_0. (Indicate reason.)

2D. Variation of Parameters

2D-1. Find a particular solution by variation of parameters:

a) $y'' + y = \tan x$

b) $y'' + 2y' - 3y = e^{-x}$

c) $y'' + 4y = \sec^2 2x$

2D-2. Bessel's equation of order p is $x^2y'' + xy' + (x^2 - p^2)y = 0$.

For $p = \frac{1}{2}$, two independent solutions for $x > 0$ are

$$y_1 = \frac{\sin x}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{\cos x}{\sqrt{x}}\ , \quad x > 0.$$

Find the general solution to

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})\, y = x^{3/2} \cos x\ .$$

2D-3. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Show that the particular solution obtained by variation of parameters can be written as the definite integral

$$y \;=\; \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W(y_1(t), y_2(t))}\, r(t)\, dt\ .$$

(Write the functions v_1 and v_2 (in the Variation of Parameters formula) as definite integrals.)

b) If instead the particular solution is written as an indefinite integral, there are arbitrary constants of integration, so the particular solution is not precisely defined. Explain why this doesn't matter.

2D-4. When *must* you use variation of parameters to find a particular solution, rather than the method of undetermined coefficients?

2E. Complex Numbers

All references are to Notes C: Complex Numbers

2E-1. Change to polar form: a) $-1+i$ b) $\sqrt{3}-i$.

2E-2. Express $\dfrac{1-i}{1+i}$ in the form $a+bi$ by two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

2E-3.* Show the distance between any two complex points z_1 and z_2 is given by $|z_2 - z_1|$.

2E-4. Prove two laws of complex conjugation:

for any complex numbers z and w, a) $\overline{z+w} = \overline{z} + \overline{w}$ b) $\overline{zw} = \overline{z}\,\overline{w}$.

2E-5.* Suppose $f(x)$ is a polynomial with *real* coefficients. Using the results of 2E-4, show that if $a+ib$ is a zero, then the complex conjugate $a-ib$ is also a zero. (Thus, complex roots of a real polynomial occur in conjugate pairs.)

2E-6.* Prove the formula $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$ by using the definition (Euler's formula (9)), and the trigonometric addition formulas.

2E-7. Calculate each of the following two ways: by changing to polar form, and also by using the binomial theorem.

a) $(1-i)^4$ b) $(1+i\sqrt{3})^3$

2E-8.* By using Euler's formula and the binomial theorem, express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

2E-9. Express in the form $a+bi$ the six sixth roots of 1.

2E-10. Solve the equation $x^4 + 16 = 0$.

2E-11.* Solve the equation $x^4 + 2x^2 + 4 = 0$, expressing the four roots in both the polar form and the Cartesian form $a+bi$.

2E-12.* Calculate A and B explicitly in the form $a+bi$ for the cubic equation on the first page of Notes C, and then show that $A+B$ is indeed real, and a root of the equation.

2E-13.* Prove the law of exponentials (16), as suggested there.

2E-14. Express $\sin^4 x$ in terms of $\cos 4x$ and $\cos 2x$, using (18) and the binomial theorem. Why would you not expect $\sin 4x$ or $\sin 2x$ in the answer?

2E-15. Find $\displaystyle\int e^{2x}\sin x\,dx$ by using complex exponentials.

2E-16. Prove (18): a) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, b) $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.

2E-17.* Derive formula (20): $D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}$ from the definition of complex exponential and the derivative formula (19): $D(u+iv) = Du + iDv$.

2E-18.* Find the three cube roots of unity in the $a+bi$ form by locating them on the unit circle and using elementary geometry.

2F. Linear Operators and Higher-order ODE's

2F-1. Find the general solution to each of the following ODE's:

a) $(D-2)^3(D^2+2D+2)\,y=0$

b) $(D^8+2D^4+1)\,y=0$

c) $y^{(4)}+y=0$

d) $y^{(4)}-8y''+16y=0$

e) $y^{(6)}-y=0$ (use 2E-9)

f) $y^{(4)}+16y=0$ (use 2E-10)

2F-2. Find the solution to $y^{(4)}-16y=0$, which in addition satisfies the four side conditions $y(0)=0$, $y'(0)=0$, $y(\pi)=1$, and $|y(x)|<K$ for some constant K and all $x>0$.

2F-3. Find the general solution to

a) $(D^3-D^2+2D-2)\,y=0$

b) $(D^3+D^2-2)\,y=0$

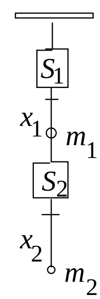
c) $y^{(3)}-2y'-4=0$

d) $y^{(4)}+2y''+4y=0$

(By high-school algebra, if m is a zero of a polynomial $p(D)$, then $(D-m)$ is a factor of $p(D)$. If the polynomial has integer coefficients and leading coefficient 1, then any integer zeros of $p(D)$ must divide the constant term.)

2F-4. A system consisting of two coupled springs is modeled by the pair of ODE's (we take the masses and spring constants to be 1; in the picture the S_i are springs, the m_i are the masses, and x_i represents the distance of mass m_i from its equilibrium position (represented here by a short horisontal line)):

$$x_1''+2x_1-x_2=0, \qquad x_2''+x_2-x_1=0\ .$$



a) Eliminate x_1 to get a 4th order ODE for x_2.

b) Solve it to find the general solution.

2F-5. Let $y=e^{2x}\cos x$. Find y'' by using operator formulas.

2F-6. Find a particular solution to

a) $(D^2+1)y=4e^x$

b) $y^{(3)}+y''-y'+2y=2\cos x$

c) $y''-2y'+4y=e^x\cos x$

d) $y''-6y'+9y=e^{3x}$

(Use the methods in Notes O; use complex exponentials where possible.)

2F-7. Find a particular solution to the general first-order linear equation with constant coefficients, $y'+ay=f(x)$, by assuming it is of the form $y_p=e^{-ax}u$, and applying the exponential-shift formula.

2G. Stability of Linear ODE's with Constant Coefficients

2G-1. For the equation $y'' + 2y' + cy = 0$, c constant,

(i) tell which values of c correspond to each of the three cases in Notes S, p.1;

(ii) for the case of two real roots, tell for which values of c both roots are negative, both roots are positive, or the roots have different signs.

(iii) Summarize the above information by drawing a c-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.

(iv) Finally, use this information to mark the interval on the c-axis for which the corresponding ODE is stable. (The stability criterion using roots is what you will need.)

2G-2. Prove the stability criterion (coefficient form) (Notes S,(8)), in the direction $\Longrightarrow$.

(You can assume that $a_0 > 0$, after multiplying the characteristic equation through by -1 if necessary. Use the high-school algebra relations which express the coefficients in terms of the roots.)

2G-3. Prove the stability criterion in the coefficient form (Notes S,(8)) in the direction $\Longleftarrow$. Use the quadratic formula, paying particular attention to the case of two real roots.

2G-4.* *Note: in what follows, formula references (11), (12), etc. are to Notes S.*

(a) Prove the higher-order stability criterion in the coefficient form (12).

(You can use the fact that a real polynomial factors into linear and quadratic factors, corresponding respectively to its real roots and its pairs of complex conjugate roots. You will need (11) and the stability criterion in the coefficient form for second-order equations.)

(b) Prove that the converse to (12) is true for those equations all of whose characteristic roots are real.

(Use an indirect proof — assume it is false and derive a contradiction.)

(c) To illustrate that the converse to (12) is in general false, show by using the criterion (11) that the equation $y''' + y'' + y' + 6y = 0$ is not stable. (Find a root of the characteristic equation by inspection, then use this to factor the characteristic polynomial.)

2G-5.* (a) Show when n = 2, the Routh-Hurwitz conditions (Notes S, (13)) are the same as the conditions given for second-order ODE's in (8).

(b) For the ODE $y''' + y'' + y' + cy = 0$, use the Routh-Hurwitz conditions to find all values of c for which the ODE is stable.

2G-6.* Take as the input $r(t) = At$, where A is a constant, in the ODE

$$ay'' + by' + cy = r(t), \qquad a,\ b,\ c \text{ constants}, \quad t = \text{time}. \tag{1}$$

a) Assume $a, b, c > 0$ and find by undetermined coefficients the steady-state solution. Express it in the form $K(t-d)$, where K and d are constants depending on the parameter A and on the coefficients of the equation.

b) We may think of d as the "time-delay". Going back to the two physical interpretations of (1) (i.e., springs and circuits), for each interpretation, express d in terms of the usual constants of the system (m-b-k, or R-L-C, depending on the interpretation).

2H. Impulse Response and Convolution

2H-1. Find the unit impulse response $w(t)$ to $y'' - k^2 y = f(t)$.

2H-2.* a) Find the unit impulse response $w(t)$ to $y'' - (a+b)y' + aby = f(t)$.

b) As $b \to a$, the associated homogeneous system turns into one having the repeated characteristic root a, and te^{at} as its weight function, according to Example 2 in the Notes. So the weight function $w(t)$ you found in part (a) should turn into $t\,e^{at}$, even though the two functions look rather different.

Show that indeed, $\lim\limits_{b\to a} w(t) \;=\; t\,e^{at}$. (Hint: write $b = a+h$ and find $\lim_{h\to 0}$.)

2H-3. a) Use (10) in Notes I to solve $y'' + 4y' + 4y = f(x)$, $y(0) = y'(0) = 0$, $x \geq 0$, where $f(x) = e^{-2x}$.
Check your answer by using the method of undetermined coefficients.

b)* Build on part (a) by using (10) to solve the IVP if $f(x) = \begin{cases} e^{-2x}, & 0 \leq x \leq 1; \\ 0, & x > 1\ . \end{cases}$

2H-4. Let $\phi(x) = \displaystyle\int_0^x (2x+3t)^2 dt$. Calculate $\phi'(x)$ two ways:

a) by using Leibniz' formula
b) directly, by calculating $\phi(x)$ explicitly, and differentiating it.

2H-5.* Using Leibniz' formula, verify directly that these IVP's have the solution given:

a) $y'' + k^2 y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \dfrac{1}{k}\displaystyle\int_0^x \sin k(x-t)\, f(t)\, dt.$

b) $y'' - 2ky' + k^2 y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \displaystyle\int_0^x (x-t)e^{k(x-t)}\, f(t)\, dt.$

2H-6.* Find the following convolutions, as explicit functions $f(x)$:

a) $e^{ax} * e^{ax} = xe^{ax}$ (cf. (15)) b) $1 * x$ c) $x * x^2$

2H-7.* Give, with reasoning, the solution to Example 7.

2H-8.* Show $y' + ay = r(x)$, $y(0) = 0$ has the solution $y_p = e^{-ax} * r(x)$ by
a) Leibniz' formula
b) solving the IVP by the first-order method, using a definite integral (cf. Notes D).

2H-9.* There is an analogue of (10) for the IVP with non-constant coefficients:

$$(*) \qquad y'' + p(x)y' + q(x)y = f(x), \quad y(0) = y'(0) = 0.$$

It assumes you know the complementary function: $y_c = c_1 u(x) + c_2 v(x)$. It says

$$y(x) \;=\; \int_0^x g(x,t)\, f(t)\, dt, \qquad \text{where } g(x,t) = \frac{\begin{vmatrix} u(t) & v(t) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}}.$$

By using Leibniz' formula, prove this solves the IVP (*).

3. Laplace Transform

3A. Elementary Properties and Formulas

3A-1. Show from the definition of Laplace transform that $\mathcal{L}(t) = \dfrac{1}{s^2}$, $s > 0$.

3A-2. Derive the formulas for $\mathcal{L}(e^{at}\cos bt)$ and $\mathcal{L}(e^{at}\sin bt)$ by assuming the formula

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$$

is also valid when α is a complex number; you will also need

$$\mathcal{L}(u + iv) = \mathcal{L}(u) + i\mathcal{L}(v),$$

for a complex-valued function $u(t) + iv(t)$.

3A-3. Find $\mathcal{L}^{-1}\big(F(s)\big)$ for each of the following, by using the Laplace transform formulas. (For (c) and (e) use a partial fractions decomposition.)

a) $\dfrac{1}{\frac{1}{2}s+3}$ b) $\dfrac{3}{s^2+4}$ c) $\dfrac{1}{s^2-4}$ d) $\dfrac{1+2s}{s^3}$ e) $\dfrac{1}{s^4-9s^2}$

3A-4. Deduce the formula for $\mathcal{L}(\sin at)$ from the definition of Laplace transform and the formula for $\mathcal{L}(\cos at)$, by using integration by parts.

3A-5. a) Find $\mathcal{L}(\cos^2 at)$ and $\mathcal{L}(\sin^2 at)$ by using a trigonometric identity to change the form of each of these functions.

b) Check your answers to part (a) by calculating $\mathcal{L}(\cos^2 at) + \mathcal{L}(\sin^2 at)$. By inspection, what should the answer be?

3A-6. a) Show that $\mathcal{L}\left(\dfrac{1}{\sqrt{t}}\right) = \sqrt{\dfrac{\pi}{s}}$, $s > 0$, by using the well-known integral

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

(Hint: Write down the definition of the Laplace transform, and make a change of variable in the integral to make it look like the one just given. Throughout this change of variable, s behaves like a constant.)

b) Deduce from the above formula that $\mathcal{L}(\sqrt{t}) = \dfrac{\sqrt{\pi}}{2s^{3/2}}$, $s > 0$.

3A-7. Prove that $\mathcal{L}(e^{t^2})$ does not exist for any interval of the form $s > a$. (Show the definite integral does not converge for any value of s.)

3A-8. For what values of k will $\mathcal{L}(1/t^k)$ exist? (Write down the definition of this Laplace transform, and determine for what k it converges.)

3A-9. By using the table of formulas, find: a) $\mathcal{L}(e^{-t}\sin 3t)$ b) $\mathcal{L}(e^{2t}(t^2 - 3t + 2))$

3A-10. Find $\mathcal{L}^{-1}\big(F(s)\big)$, if $F(s) =$

a) $\dfrac{3}{(s-2)^4}$ b) $\dfrac{1}{s(s-2)}$ c) $\dfrac{s+1}{s^2-4s+5}$

3B. Derivative Formulas; Solving ODE's

3B-1. Solve the following IVP's by using the Laplace transform:

a) $y' - y = e^{3t}, \quad y(0) = 1$

b) $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1$

c) $y'' + 4y = \sin t, \quad y(0) = 1,\ y'(0) = 0$

d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0,\ y'(0) = 1$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1,\ y'(0) = 0$.

3B-2. Without referring to your book or to notes, derive the formula for $\mathcal{L}(f'(t))$ in terms of $\mathcal{L}(f(t))$. What are the assumptions on $f(t)$ and $f'(t)$?

3B-3. Find the Laplace transforms of the following, using formulas and tables:

a) $t\cos bt$ b) $t^n e^{kt}$ (two ways) c) $e^{at} t \sin t$

3B-4. Find $\mathcal{L}^{-1}(F(s))$ if $F(s) =$ a) $\dfrac{s}{(s^2+1)^2}$ b) $\dfrac{1}{(s^2+1)^2}$

3B-5. Without consulting your book or notes, derive the formulas

a) $\mathcal{L}\big(e^{at} f(t)\big) = F(s-a)$ b) $\mathcal{L}\big(t\, f(t)\big) = -F'(s)$

3B-6. If $y(t)$ is a solution to the IVP $y'' + ty = 0, \quad y(0) = 1,\ y'(0) = 0$, what ODE is satisfied by the function $Y(s) = \mathcal{L}(y(t))$?

(The solution $y(t)$ is called an *Airy function*; the ODE it satisfies is the *Airy equation*.)

3C. Discontinuous Functions

3C-1. Find the Laplace transforms of each of the following functions; do it as far as possible by expressing the functions in terms of known functions and using the tables, rather than by calculating from scratch. In each case, sketch the graph of $f(t)$. (Use the unit step function $u(t)$ wherever possible.)

a) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$ b) $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

c) $f(t) = |\sin t|, \quad t \ge 0.$

3C-2. Find $\mathcal{L}^{-1}$ for the following: a) $\dfrac{e^{-s}}{s^2+3s+2}$ b) $\dfrac{e^{-s} - e^{-3s}}{s}$ (sketch answer)

3C-3. Find $\mathcal{L}(f(t))$ for the square wave $f(t) = \begin{cases} 1, & 2n \le t \le 2n+1,\ n = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$

a) directly from the definition of Laplace transform;

b) by expressing $f(t)$ as the sum of an infinite series of functions, taking the Laplace transform of the series term-by-term, and then adding up the infinite series of Laplace transforms.

3C-4. Solve by the Laplace transform the following IVP, where $h(t) = \begin{cases} 1, & \pi \le t \le 2\pi, \\ 0, & \text{otherwise} \end{cases}$

$$y'' + 2y' + 2y = h(t), \quad y(0) = 0,\ y'(0) = 1;$$

write the solution in the format used for $h(t)$.

3C-5. Solve the IVP: $y'' - 3y' + 2y = r(t)$, $y(0) = 1,\ y'(0) = 0$, where $r(t) = u(t)\,t$, the ramp function.

3D. Convolution and Delta Function

3D-1. Solve the IVP: $y'' + 2y' + y = \delta(t) + u(t-1)$, $y(0) = 0,\ y'(0^-) = 1$.

Write the answer in the "cases" format $y(t) = \begin{cases} \cdots, & 0 \le t \le 1 \\ \cdots, & t > 1 \end{cases}$

3D-2. Solve the IVP: $y''+y = r(t)$, $y(0) = 0,\ y'(0) = 1$, where $r(t) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

Write the answer in the "cases" format (see 3D-1 above).

3D-3. If $f(t+c) = f(t)$ for all t, where c is a fixed positive constant, the function $f(t)$ is said to be *periodic*, with period c. (For example, $\sin x$ is periodic, with period 2π.)

a) Show that if $f(t)$ is periodic with period c, then its Laplace transform is

$$F(s) = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} f(t)\,dt\ .$$

b) Do Exercise 3C-3, using the above formula.

3D-4. Find $\mathcal{L}^{-1}$ by using the convolution: a) $\dfrac{s}{(s+1)(s^2+4)}$ b) $\dfrac{1}{(s^2+1)^2}$

Your answer should not contain the convolution $*$.

3D-5. Assume $f(t) = 0$, for $t \le 0$. Show informally that $\delta(t) * f(t) = f(t)$, by using the definition of convolution; then do it by using the definition of $\delta(t)$.

(See (5), section 4.6 of your book; $\delta(t)$ is written $\delta_0(t)$ there.)

3D-6. Prove that $f(t) * g(t) = g(t) * f(t)$ directly from the definition of convolution, by making a change of variable in the convolution integral.

3D-7. Show that the IVP: $y'' + k^2 y = r(t)$, $y(0) = 0,\ y'(0) = 0$ has the solution

$$y(t) = \frac{1}{k}\int_0^t r(u) \sin k(t-u)\,du\ ,$$

by using the Laplace transform and the convolution.

3D-8. By using the Laplace transform and the convolution, show that in general the IVP (here a and b are constants):

$$y'' + ay' + by = r(t), \qquad y(0) = 0,\ y'(0) = 0,$$

has the solution

$$y(t) = \int_0^t w(t-u) r(u)\,du\ ,$$

where $w(t)$ is the solution to the IVP: $y'' + ay' + by = 0$, $y(0) = 0,\ y'(0) = 1$.

(The function $w(t-u)$ is called the **Green's function** for the linear operator D^2+aD+b.)

4. Linear Systems

4A. Review of Matrices

4A-1. Verify that $\begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \end{pmatrix}$.

4A-2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$, show that $AB \neq BA$.

4A-3. Calculate A^{-1} if $A = \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}$, and check your answer by showing that $AA^{-1} = I$ and $A^{-1}A = I$.

4A-4. Verify the formula given in Notes LS.1 for the inverse of a 2×2 matrix.

4A-5. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find A^3 $(= A \cdot A \cdot A)$.

4A-6. For what value of c will the vectors $\mathbf{x}_1 = (1, 2, c)$, $\mathbf{x}_2 = (-1, 0, 1)$, and $\mathbf{x}_3 = (2, 3, 0)$ be linearly dependent? For this value, find by trial and error (or otherwise) a linear relation connecting them, i.e., one of the form $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$

4B. General Systems; Elimination; Using Matrices

4B-1. Write the following equations as equivalent first-order systems:

a) $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + tx^2 = 0$ b) $y'' - x^2y' + (1 - x^2)y = \sin x$

4B-2. Write the IVP

$$y^{(3)} + p(t)y'' + q(t)y' + r(t)y = 0, \qquad y(0) = y_0,\ y'(0) = y_0',\ y''(0) = y_0''$$

as an equivalent IVP for a system of three first-order linear ODE's. Write this system both as three separate equations, and in matrix form.

4B-3. Write out $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ as a system of two first-order equations.

a) Eliminate y so as to obtain a single second-order equation for x.

b) Take the second-order equation and write it as an equivalent first-order system. This isn't the system you started with, but show a change of variables converts one system into the other.

4B-4. For the system $x' = 4x - y$, $y' = 2x + y$,

a) using matrix notation, verify that $x = e^{3t}$, $y = e^{3t}$ and $x = e^{2t}$, $y = 2e^{2t}$ are solutions;

b) verify that they form a fundamental set of solutions — i.e., that they are linearly independent;

c) write the general solution to the system in terms of two arbitrary constants c_1 and c_2; write it both in vector form, and in the form $x = \dots\,,\ y = \dots\ .$

4B-5. For the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$,

a) show that $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ form a fundamental set of solutions (i.e., they are linearly independent and solutions);

b) solve the IVP: $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

4B-6. Solve the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$ in two ways:

a) Solve the second equation, substitute for y into the first equation, and solve it.

b) Eliminate y by solving the first equation for y, then substitute into the second equation, getting a second order equation for x. Solve it, and then find y from the first equation. Do your two methods give the same answer?

4B-7. Suppose a radioactive substance R decays into a second one S which then also decays. Let x and y represent the amounts of R and S present at time t, respectively.

a) Show that the physical system is modeled by a system of equations

$$\mathbf{x}' = A\mathbf{x}, \qquad \text{where } A = \begin{pmatrix} -a & 0 \\ a & -b \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a,\ b \text{ constants.}$$

b) Solve this sytem by either method of elimination described in 4B-6.

c) Find the amounts present at time t if initially only R is present, in the amount x_0.

Remark. The method of elimination which was suggested in some of the preceding problems (4B-3,6,7; book section 5.2) is always available. Other than in these exercises, we will not discuss it much, as it does not give insights into systems the way the methods will decribe later do.

Warning. Elimination sometimes produces extraneous solutions — extra "solutions" that do not actually solve the original system. Expect this to happen when you have to differentiate both equations to do the elimination. (Note that you also get extraneous solutions when doing elimination in ordinary algebra, too.) If you get more independent solutions than the order of the system, they must be tested to see if they actually solve the original system. (The order of a system of ODE's is the sum of the orders of each of the ODE's in it.)

4C. Eigenvalues and Eigenvectors

4C-1. Solve $\mathbf{x}' = A\mathbf{x}$, if A is: a) $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

(First find the eigenvalues and associated eigenvectors, and from these construct the normal modes and then the general solution.)

4C-2. Prove that $m = 0$ is an eigenvalue of the $n \times n$ constant matrix A if and only if A is a singular matrix ($\det A = 0$). (You can use the characteristic equation, or you can use the definition of eigenvalue.)

4C-3. Suppose a 3×3 matrix is upper triangular. (This means it has the form below, where $*$ indicates an arbitrary numerical entry.)

$$A = \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix}$$

Find its eigenvalues. What would be the generalization to an $n \times n$ matrix?

4C-4. Show that if $\vec{\alpha}$ is an eigenvector of the matrix A, associated with the eigenvalue m, then $\vec{\alpha}$ is also an eigenvector of the matrix A^2, associated this time with the eigenvalue m^2. (Hint: use the eigenvector equation in 4F-3.)

4C-5. Solve the radioactive decay problem (4B-7) using eigenvectors and eigenvalues.

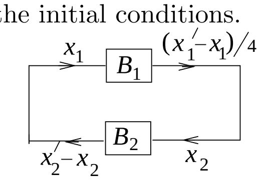
4C-6. Farmer Smith has a rabbit colony in his pasture, and so does Farmer Jones. Each year a certain fraction a of Smith's rabbits move to Jones' pasture because the grass is greener there, and a fraction b of Jones' rabbits move to Smith's pasture (for the same reason). Assume (foolishly, but conveniently) that the growth rate of rabbits is 1 rabbit (per rabbit/per year).

a) Write a system of ODE's for determining how S and J, the respective rabbit populations, vary with time t (years).

b) Assume $a = b = \frac{1}{2}$. If initially Smith has 20 rabbits and Jones 10 rabbits, how do the two populations subsequently vary with time?

c) Show that S and J never oscillate, regardless of a, b and the initial conditions.

4C-7. The figure shows a simple feedback loop.



Black box B_1 inputs $x_1(t)$ and outsputs $\frac{1}{4}(x_1' - x_1)$.

Black box B_2 inputs $x_2(t)$ and outputs $x_2' - x_2$.

If they are hooked up in a loop as shown, and initially $x_1 = 1, x_2 = 0$, how do x_1 and x_2 subsequently vary with time t? (If it helps, you can think of x_1 and x_2 as currents, for instance, or as the monetary values of trading between two countries, or as the number of times/minute Punch hits Judy and vice-versa.)

4D. Complex and Repeated Eigenvalues

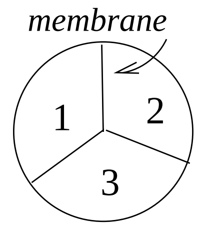
4D-1. Solve the system $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \mathbf{x}$.

4D-2. Solve the system $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}$.

4D-3. Solve the system $\mathbf{x}' = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$.

4D-4. Three identical cells are pictured, each containing salt solution, and separated by identical semi-permeable membranes. Let A_i represent the amount of salt in cell i $(i = 1, 2, 3)$, and let

$$x_i = A_i - A_0$$



be the difference bwetween this amount and some standard reference amount A_0. Assume the rate at which salt diffuses across the membranes is proportional to the difference in concentrations, i.e. to the difference in the two values of A_i on either side, since we are supposing the cells identical. Take the constant of proportionality to be 1.

a) Derive the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

b) Find three normal modes, and interpret each of them physically. (To what initial conditions does each correspond — is it reasonable as a solution, in view of the physical set-up?)

4E. Decoupling

4E-1. A system is given by $x' = 4x + 2y$, $y' = 3x - y$. Give a new set of variables, u and v, linearly related to x and y, which decouples the system. Then verify by direct substitution that the system becomes decoupled when written in terms of u and v.

4E-2. Answer the same questions as in the previous problem for the system in 4D-4. (Use the solution given in the Notes to get the normal modes. In the last part of the problem, do the substitution by using matrices.)

4F. Theory of Linear Systems

4F-1. Take the second-order equation $x'' + p(t)x' + q(t)x = 0$.

a) Change it to a first-order system $\mathbf{x}' = A\mathbf{x}$ in the usual way.

b) Show that the Wronskian of two solutions x_1 and x_2 of the original equation is the same as the Wronskian of the two corresponding solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the system.

4F-2. Let $\mathbf{x}_1 = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ be two vector functions.

a) Prove by using the definition that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

b) Calculate the Wronskian $W(\mathbf{x}_1, \mathbf{x}_2)$.

c) How do you reconcile (a) and (b) with Theorem 5.3 in Notes LS.5?

d) Find a linear system $\mathbf{x}' = A\mathbf{x}$ having $\mathbf{x}_1$ and $\mathbf{x}_2$ as solutions, and confirm your answer to (c). (To do this, treat the entries of A as unknowns, and find a system of equations whose solutions will give you the entries. A will be a matrix function of t, i.e., its entries will be functions of t.)

4F-3. Suppose the 2×2 constant matrix A has two distinct eigenvectors m_1 and m_2, with associated eigenvectors respectively $\vec{\alpha}_1$ and $\vec{\alpha}_2$. Prove that the corresponding vector functions

$$\mathbf{x}_1 = \vec{\alpha}_1 e^{m_1 t}, \qquad \mathbf{x}_2 = \vec{\alpha}_2 e^{m_2 t}$$

are linearly independent, as follows:

a) using the determinantal criterion, show they are linearly independent if and only if $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are linearly independent;

b) then show that $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 = 0 \quad \Rightarrow \quad c_1 = 0,\ c_2 = 0$. (Use the eigenvector equation $(A - m_i I)\vec{\alpha}_i = 0$ in the form: $A\vec{\alpha}_i = m_i\vec{\alpha}_i$.)

4F-4. Suppose $\mathbf{x}' = A\mathbf{x}$, where A is a nonsingular constant matrix. Show that if $\mathbf{x}(t)$ is a solution whose velocity vector $\mathbf{x}'(t)$ is $\mathbf{0}$ at time t_0, then $\mathbf{x}(t)$ is identically zero for all t. What is the minimum hypothesis on A that is needed for this result to be true? Can A be a function of t, for example?

4G. Fundamental Matrices

4G-1. Two independent solutions to $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$.

a) Find the solutions satisfying $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Using part (a), find in a simple way the solution satisfying $x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$.

4G-2. For the system $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$,

a) find a fundamental matrix, using the normal modes, and use it to find the solution satisfying $x(0) = 2,\ y(0) = -1$;

b) find the fundamental matrix normalized at $t = 0$, and solve the same IVP as in part (a) using it.

4G-3.* Same as 4G-2, using the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ instead.

4H. Exponential Matrices

4H-1. Calculate e^{At} if $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Verify directly that $\mathbf{x} = e^{At}\mathbf{x}_0$ is the solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

4H-2. Calculate e^{At} if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; then answer same question as in 4H-1.

4H-3. Calculate e^{At} directly from the infinite series, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; then answer same question as in 4H-1.

4H-4. What goes wrong with the argument justifying the e^{At} solution of $\mathbf{x}' = A\mathbf{x}$ if A is not a constant matrix, but has entries which depend on t?

4H-5. Prove that a) $e^{kIt} = Ie^{kt}$. b) $Ae^{At} = e^{At}A$.

(Here k is a constant, I is the identity matrix, A any square constant matrix.)

4H-6. Calculate the exponential matrix in 4H-3, this time using the third method in the Notes (writing $A = B + C$).

4H-7. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Calculate e^{At} three ways:

a) directly, from its definition as an infinite series;

b) by expressing A as a sum of simpler matrices, as in Notes LS.6, Example 6.3C;

c) by solving the system by elimination so as to obtain a fundamental matrix, then normalizing it.

4I. Inhomogeneous Systems

4I-1. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \end{pmatrix} t$, by variation of parameters.

4I-2. a) Solve $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$ by variation of parameters.

b) Also do it by undetermined coefficients, by writing the forcing term and trial solution respectively in the form:

$$\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{t}; \qquad \mathbf{x}_p = \vec{c}e^{-2t} + \vec{d}e^{t} .$$

4I-3.* Solve $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$ by undetermined coefficients.

4I-4. Solve $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$ by undetermined coefficients.

4I-5. Suppose $\mathbf{x}' = A\mathbf{x} + \mathbf{x}_0$ is a first-order order system, where A is a nonsingular $n \times n$ constant matrix, and $\mathbf{x}_0$ is a constant n-vector. Find a particular solution $\mathbf{x}_p$.

Section 5. Graphing Systems

5A. The Phase Plane

5A-1. Find the critical points of each of the following non-linear autonomous systems.

a) $\begin{aligned} x' &= x^2 - y^2 \\ y' &= x - xy \end{aligned}$ b) $\begin{aligned} x' &= 1 - x + y \\ y' &= y + 2x^2 \end{aligned}$

5A-2. Write each of the following equations as an equivalent first-order system, and find the critical points.

a) $x'' + a(x^2 - 1)x' + x = 0$ b) $x'' - x' + 1 - x^2 = 0$

5A-3. In general, what can you say about the relation between the trajectories and the critical points of the system on the left below, and those of the two systems on the right?

$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$ a) $\begin{aligned} x' &= -f(x,y) \\ y' &= -g(x,y) \end{aligned}$ b) $\begin{aligned} x' &= g(x,y) \\ y' &= -f(x,y) \end{aligned}$

5A-4. Consider the autonomous system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}\ ; \qquad \text{solution : } \quad \mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

a) Show that if $\mathbf{x}_1(t)$ is a solution, then $\mathbf{x}_2(t) = \mathbf{x}_1(t - t_0)$ is also a solution. What is the geometric relation between the two solutions?

b) The existence and uniqueness theorem for the system says that if f and g are continuously differentiable everywhere, there is one and only one solution $\mathbf{x}(t)$ satisfying a given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

Using this and part (a), show that two trajectories cannot intersect anywhere.

(Note that if two trajectories intersect at a point (a, b), the corresponding solutions $\mathbf{x}(t)$ which trace them out may be at (a, b) at different times. Part (a) shows how to get around this difficulty.)

5B. Sketching Linear Systems

5B-1. Follow the Notes (GS.2) for graphing the trajectories of the system $\begin{cases} x' = -x \\ y' = -2y\ . \end{cases}$

a) Eliminate t to get one ODE $\dfrac{dy}{dx} = F(x,y)$. Solve it and sketch the solution curves.

b) Solve the original system (by inspection, or using eigenvalues and eigenvectors), obtaining the equations of the trajectories in parametric form: $x = x(t), y = y(t)$. Using these, put the direction of motion on your solution curves for part (a). What new trajectories are there which were not included in the curves found in part (a)?

c) How many trajectories are needed to cover a typical solution curve found in part (a)? Indicate them on your sketch.

d) If the system were $x' = x$, $y' = 2y$ instead, how would your picture be modified? (Consider both parts (a) and (b).)

5B-2. Answer the same questions as in 5B-1 for the system $x' = y$, $y' = x$. (For part (d), use $-y$ and $-x$ as the two functions on the right.)

5B-3. Answer the same question as in 5B-1a,b for the system $x' = y$, $y' = -2x$.

For part (b), put in the direction of motion on the curves by making use of the vector field corresponding to the system.

5B-4. For each of the following linear systems, carry out the graphing program in Notes GS.4; that is,

(i) find the eigenvalues of the associated matrix and from this determine the geometric type of the critical point at the origin, and its stability;

(ii) if the eigenvalues are real, find the associated eigenvectors and sketch the corresponding trajectories, showing the direction of motion for increasing t; then draw in some nearby trajectories;

(iii) if the eigenvalues are complex, obtain the direction of motion and the approximate shape of the spiral by sketching in a few vectors from the vector field defined by the system.

a) $\begin{aligned} x' &= 2x - 3y \\ y' &= x - 2y \end{aligned}$ b) $\begin{aligned} x' &= 2x \\ y' &= 3x + y \end{aligned}$ c) $\begin{aligned} x' &= -2x - 2y \\ y' &= -x - 3y \end{aligned}$

d) $\begin{aligned} x' &= x - 2y \\ y' &= x + y \end{aligned}$ e) $\begin{aligned} x' &= x + y \\ y' &= -2x - y \end{aligned}$

5B-5. For the damped spring-mass system modeled by the ODE

$$mx'' + cx' + kx = 0, \qquad m, c, k > 0 \ ,$$

a) write it as an equivalent first-order linear system;

b) tell what the geometric type of the critical point at $(0, 0)$ is, and determine its stability, in each of the following cases. Do it by the methods of Sections GS.3 and GS.4, and check the result by physical intuition.

(i) $c = 0$ (ii) $c \approx 0$; $m, k \gg 1$. (iii) Can the geometric type be a saddle? Explain.

5C. Sketching Non-linear Systems

5C-1. For the following system, the origin is clearly a critical point. Give its geometric type and stability, and sketch some nearby trajectories of the system.

$$\begin{aligned} x' &= x - y + xy \\ y' &= 3x - 2y - xy \end{aligned}$$

5C-2. Repeat 5C-1 for the system $\begin{cases} x' = x + 2x^2 - y^2 \\ y' = x - 2y + x^3 \end{cases}$

5C-3. Repeat 5C-1 for the system $\begin{cases} x' = 2x + y + xy^3 \\ y' = x - 2y - xy \end{cases}$

5C-4. For the following system, carry out the program outlined in Notes GS.6 for sketching trajectories — find the critical points, analyse each, draw in nearby trajectories, then add some other trajectories compatible with the ones you have drawn; when necessary, put in a vector from the vector field to help.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

5C-5. Repeat 5C-4 for the system $\begin{cases} x' = x - x^2 - xy \\ y' = 3y - xy - 2y^2 \end{cases}$

5D. Limit Cycles

5D-1. In Notes LC, Example 1,

a) Show that $(0, 0)$ is the only critical point (hint: show that if (x, y) is a non-zero critical point, then $y/x = -x/y$; derive a contradiction).

b) Show that $(\cos t, \sin t)$ is a solution; it is periodic: what is its trajectory?

c) Show that all other non-zero solutions to the system get steadily closer to the solution in part (b). (This shows the solution is an asymptotically stable limit cycle, and the only periodic solution to the system.)

5D-2. Show that each of these systems has no closed trajectories in the region R (this is the whole xy-plane, except in part (c)).

a) $\begin{aligned} x' &= x + x^3 + y^3 \\ y' &= y + x^3 + y^3 \end{aligned}$ b) $\begin{aligned} x' &= x^2 + y^2 \\ y' &= 1 + x - y \end{aligned}$ c) $\begin{aligned} x' &= 2x + x^2 + y^2 \\ y' &= x^2 - y^2 \\ R &= \text{half-plane } x < -1 \end{aligned}$

d) $\begin{aligned} x' &= ax + bx^2 - 2cxy + dy^2 \\ y' &= ex + fx^2 - 2bxy + cy^2 \end{aligned}$: find the condition(s) on the six constants that guarantees no closed trajectories in the xy-plane

5D-3. Show that Lienard's equation (Notes LC, (6)) has no periodic solution if either

a) $u(x) > 0$ for all x b) $v(x) > 0$ for all x .

(Hint: consider the corresponding system, in each case.)

5D-4.* a) Show van der Pol's equation (Notes LC.4) satisfies the hypotheses of the Levinson-Smith theorem (this shows it has a unique limit cycle).

b) The corresponding system has a unique critical point at the origin; show this and determine its geometric type and stability. (Its type depends on the value of the parameter).

5D-5.* Consider the following system (where $r = \sqrt{x^2+y^2}$):
$$\begin{aligned} x' &= -y + xf(r) \\ y' &= x + yf(r) \end{aligned}$$

a) Show that if $f(r)$ has a positive zero a, then the system has a circular periodic solution.

b) Show that if $f(r)$ is a decreasing function for $r \approx a$, then this periodic solution is actually a stable limit cycle. (Hint: how does the direction field then look?)

5E. Structural stability; Volterra's Principle

5E-1. Each of the following systems has a critical point at the origin. For this critical point, find the geometric type and stability of the corresponding linearized system, and then tell what the possibilities would be for the corresponding critical point of the given non-linear system.

a) $x' = x - 4y - xy^2, \quad y' = 2x - y + x^2y$

b) $x' = 3x - y + x^2 + y^2, \quad y' = -6x + 2y + 3xy$

5E-2. Each of the following systems has one critical point whose linearization is not structurally stable. In each case, sketch several pictures showing the different ways the trajectories of the non-linear system might look.

Begin by finding the critical points and determining the type of the corresponding linearized system at each of the critical points.

a) $x' = y, \quad y' = x(1-x)$

b) $x' = x^2 - x + y, \quad y' = -yx^2 - y$

5E-3. The main tourist attraction at Monet Gardens is Pristine Acres, an expanse covered with artfully arranged wildflowers. Unfortunately, the flower stems are the favorite food of the Kandinsky borer; the flower and borer populations fluctuate cyclically in accordance with Volterra's predator-prey equations. To boost the wildflower level for the tourists, the director wants to fertilize the Acres, so that the wildflower growth will outrun that of the borers.

Assume that fertilizing would boost the wildflower growth rate (in the absence of borers) by 25 percent. What do you think of this proposal?

Using suitable units, let x be the wildflower population and y be the borer population.

Take the equations to be $x' = ax - pxy, \qquad y' = -by + qxy,$ where a, b, p, q are positive constants.

6. Power Series

6A. Power Series Operations

6A-1. Find the radius of convergence for each of the following:

a) $\displaystyle\sum_0^\infty n\,x^n$ b) $\displaystyle\sum_0^\infty \frac{x^{2n}}{n2^n}$ c) $\displaystyle\sum_1^\infty n!\,x^n$ d) $\displaystyle\sum_0^\infty \frac{(2n)!}{(n!)^2}\,x^n$

6A-2. Starting from the series $\displaystyle\sum_0^\infty x^n = \frac{1}{1-x}$ and $\displaystyle\sum_0^\infty \frac{x^n}{n!} = e^x$,

by using operations on series (substitution, addition and multiplication, term-by-term differentiation and integration), find series for each of the following

a) $\displaystyle\frac{1}{(1-x)^2}$ b) xe^{-x^2} c) $\tan^{-1} x$ d) $\ln(1+x)$

6A-3. Let $\displaystyle y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$. Show that

a) y is a solution to the ODE $y'' - y = 0$ b) $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$.

6A-4. Find the sum of the following power series (using the operations in 6A-2 as a help):

a) $\displaystyle\sum_0^\infty x^{3n+2}$ b) $\displaystyle\sum_0^\infty \frac{x^n}{n+1}$ c) $\displaystyle\sum_0^\infty nx^n$

6B. First-order ODE's

6B-1. For the nonlinear IVP $y' = x + y^2$, $y(0) = 1$, find the first four nonzero terms of a series solution $y(x)$ two ways:

a) by setting $y = \sum_0^\infty a_n x^n$ and finding in order $a_0, a_1, a_2, \ldots$, using the initial condition and substituting the series into the ODE;

b) by differentiating the ODE repeatedly to obtain $y(0), y'(0), y''(0), \ldots$, and then using Taylor's formula.

6B-2. Solve the following linear IVP by assuming a series solution

$$y = \sum_0^\infty a_n x^n ,$$

substituting it into the ODE and determining the a_n recursively by the method of undetermined coefficients. Then sum the series to obtain an answer in closed form, if possible (the techniques of 6A-2,4 will help):

a) $y' = x + y$, $y(0) = 0$ b) $y' = -xy$, $y(0) = 1$ c) $(1-x)y' - y = 0$, $y(0) = 1$

6C. Solving Second-order ODE's

6C-1. Express each of the following as a power series of the form $\sum_N^\infty b_n x^n$. Indicate the value of N, and express b_n in terms of a_n.

a) $\sum_1^\infty a_n x^{n+3}$ b) $\sum_0^\infty n(n-1)a_n x^{n-2}$ c) $\sum_1^\infty (n+1)a_n x^{n-1}$

6C-2. Find two independent power series solutions $\sum a_n x^n$ to $y'' - 4y = 0$, by obtaining a recursion formula for the a_n.

6C-3. For the ODE $y'' + 2xy' + 2y = 0$,

a) find two independent series solutions y_1 and y_2;

b) determine their radius of convergence;

c) express the solution satisfying $y(0) = 1$, $y'(0) = -1$ in terms of y_1 and y_2;

d) express the series in terms of elementary functions (i.e., sum the series to an elementary function).

(One of the two series is easily recognizable; the other can be gotten using the operations on series, or by using the known solution and the method of reduction of order—see Exercises 2B.)

6C-4. Hermite's equation is $y'' - 2xy' + ky = 0$. Show that if k is a positive even integer $2m$, then one of the power series solutions is a polynomial of degree m.

6C-5. Find two independent series solutions in powers of x to the Airy equation: $y'' = xy$.

Determine their radius of convergence. For each solution, give the first three non-zero terms and the general term.

6C-6. Find two independent power series solutions $\sum a_n x^n$ to

$$(1 - x^2)y'' - 2xy' + 6y = 0 \ .$$

Determine their radius of convergence R. To what extent is R predictable from the original ODE?

6C-7. If the recurrence relation for the a_n has three terms instead of just two, it is more difficult to find a formula for the general term of the corresponding series. Give the recurrence relation and the first three nonzero terms of two independent power series solutions to

$$y'' + 2y' + (x - 1)y = 0 \ .$$

7. Fourier Series

Based on exercises in Chap. 8, Edwards and Penney, Elementary Differential Equations

7A. Fourier Series

7A-1. Find the smallest period for each of the following periodic functions:

a) $\sin \pi t/3$ b) $|\sin t|$ c) $\cos^2 3t$

7A-2. Find the Fourier series of the function $f(t)$ of period 2π which is given over the interval $-\pi < t \leq \pi$ by

a) $f(t) = \begin{cases} 0, & -\pi < t \leq 0; \\ 1, & 0 < t \leq \pi \end{cases}$ b) $f(t) = \begin{cases} -t, & -\pi < t < 0; \\ t, & 0 \leq t \leq \pi \end{cases}$

7A-3. Give another proof of the orthogonality relations $\displaystyle\int_{-\pi}^{\pi} \cos mt\, \cos nt\, dt = \begin{cases} 0, & m \neq n; \\ \pi, & m = n\ . \end{cases}$

by using the trigonometric identity: $\cos A \cos B = \frac{1}{2}\big(\cos(A+B) + \cos(A-B)\big)$.

7A-4. Suppose that $f(t)$ has period P. Show that $\int_I f(t)\, dt$ has the same value over any interval I of length P, as follows:

a) Show that for any a, we have $\displaystyle\int_P^{a+P} f(t)\, dt = \int_0^a f(t)\, dt$. (Make a change of variable.)

b) From part (a), deduce that $\displaystyle\int_a^{a+P} f(t)\, dt = \int_0^P f(t)\, dt$.

7B. Even and Odd Series; Boundary-value Problems

7B-1. a) Find the Fourier cosine series of the function $1-t$ over the interval $0 < t < 1$, and then draw over the interval $[-2,2]$ the graph of the function $f(t)$ which is the sum of this Fourier cosine series.

b) Answer the same question for the Fourier sine series of $1-t$ over the interval $(0,1)$.

7B-2. Find a formal solution as a Fourier series, for these boundary-value problems (you can use any Fourier series derived in the book's Examples):

a) $x'' + 2x = 1, \quad x(0) = x(\pi) = 0;$
b) $x'' + 2x = t, \quad x'(0) = x'(\pi) = 0$ (use a cosine series)

7B-3. Assume $a > 0$; show that $\displaystyle\int_{-a}^{0} f(t)\, dt = \pm \int_0^a f(t)\, dt$, according to whether $f(t)$ is respectively an even function or an odd function.

7B-4. The Fourier series of the function $f(t)$ having period 2, and for which $f(t) = t^2$ for $0 < t < 2$, is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n}\ .$$

Differentiate this series term-by-term, and show that the resulting series does not converge to $f'(t)$.

7C. Applications to resonant frequencies

7C-1. For each spring-mass system, find whether pure resonance occurs, without actually calculating the solution.

a) $2x'' + 10x = F(t)$; $F(t) = 1$ on $(0, 1)$, $F(t)$ is odd, and of period 2;
b) $x'' + 4\pi^2 x = F(t)$; $F(t) = 2t$ on $(0, 1)$, $F(t)$ is odd, and of period 2;
c) $x'' + 9x = F(t)$; $F(t) = 1$ on $(0, \pi)$, $F(t)$ is odd, and of period 2π.

7C-2. Find a periodic solution as a Fourier series to $x'' + 3x = F(t)$, where $F(t) = 2t$ on $(0, \pi)$, $F(t)$ is odd, and has period 2π.

7C-3. For the following two lightly damped spring-mass systems, by considering the form of the Fourier series solution and the frequency of the corresponding undamped system, determine what term of the Fourier series solution should dominate — i.e., have the biggest amplitude.

a) $2x'' + .1x' + 18x = F(t)$; $F(t)$ is as in 7C-2.
b) $3x'' + x' + 30x = F(t)$; $F(t) = t - t^2$ on $(0, 1)$, $F(t)$ is odd, with period 2.

18.03 Exercises

8: Extra Problems

8A. Bifurcation Diagrams

8A-1. Suppose that a population of variable size (in some suitable units) $P(t)$ follows the growth law $\dfrac{dP}{dt} = -P^3 + 12P^2 - 36P + r$, where r is a constant replenishment rate. Without solving the DE explicitly:

a) Let $r = 0$. Find all critical points and classify each according to its stability type using a phase-line diagram. Sketch some representative integral curves.

b) What is the smallest value of r such that the population always stabilizes at a size greater than 4, no matter what the size of the initial population?

c) Sketch the P vs. r bifurcation diagram.

8B. Frequency Response

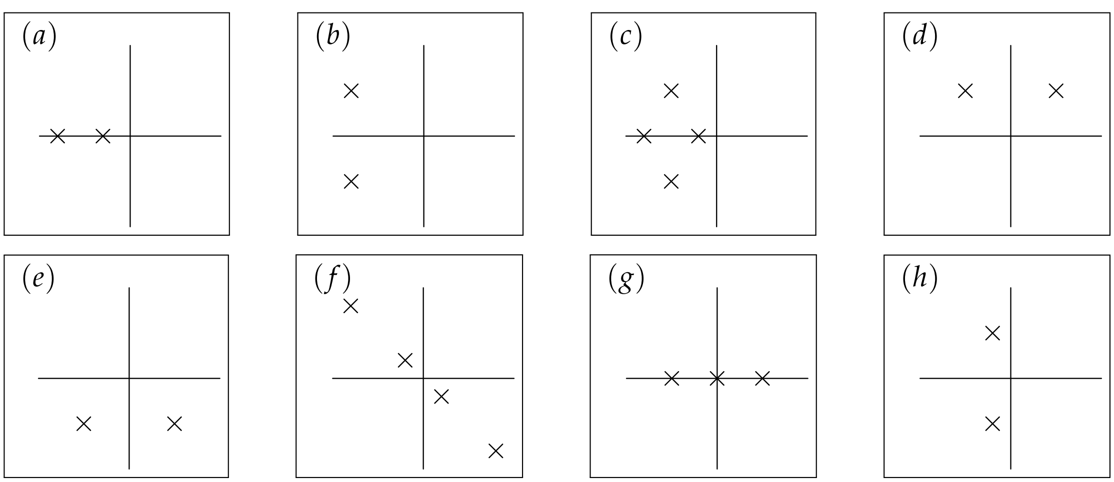
8B-1. For each of the following systems, use your calculus graphing skills to plot the graph of the amplitude response (i.e. gain vs. ω). If there is a resonant frequency say what it is.

a) $x'' + x' + 7x = F_0 \cos \omega t$.

b) $x'' + 8x' + 7x = F_0 \cos \omega t$.

8C. Pole Diagrams

8C-1. Consider the following pole diagrams for some linear time invariant systems $P(D)x = f(t)$.



a) Which of the systems are stable?

b) For which systems are *all* of the non-zero solutions to the homogeneous equation oscillatory?

c) For which systems are *none* of the non-zero solutions to the homogeneous equation oscillatory?

d) For which systems does $P(D)$ have real coefficients?

e) Comparing b and c, for which one does the weight function decay faster. (Assume both plots are on the same scale.)

f) Give the order of each of the systems.

g) Give a possible polynomial $P(D)$ that would have pole diagram (a). Do the same thing for (b) and (c).

h) Comparing (b) and (h) which has the largest possible response to input of the form $\cos \omega t$?

9. 18.03 Linear Algebra Exercises

9A. Matrix Multiplication, Rank, Echelon Form

9A-1. Which of the following matrices is in row-echelon form?

$$\text{(i)}\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(ii)}\ \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} \qquad \text{(iii)}\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{(iv)}\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(v)}\ \begin{bmatrix} 0 \end{bmatrix}$$

9A-2. Find the reduced echelon form of each of the following matrices.

$$\text{(i)}\ \begin{bmatrix} 4 \end{bmatrix} \qquad \text{(ii)}\ \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \text{(iii)}\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{(iv)}\ \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

9A-3. Write a row vector with norm 1 (the square root of the sum of the squares of the entries).

9A-4. Write a column vector with 4 entries whose entries add to zero.

9A-5. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a vector $\mathbf{v}$ such that $A\mathbf{v}$ is the third column of A.

(b) Find a vector $\mathbf{w}$ such that $\mathbf{w}A$ is the third row of A.

9A-6. Find the following matrix products, and their ranks.

$$\text{(a)}\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$$

9B. Column Space, Null Space, Independence, Basis, Dimension

9B-1. Write a matrix equation that shows that the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix},$$

are linearly dependent (or, more properly, form a linearly dependent set).

9B-2 (a) Find a basis for the null spaces of the following matrices.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(First find the reduced echelon form; then set each free variable equal to 1 and the others to zero, one at a time.)

Note: The second matrix is the *transpose* of the first: that is the rows become the columns and the columns become the rows.

(b) Find the genaral solutions to $A\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and $A^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

9B-3 Find a basis for each of the following subspaces of $\mathbb{R}^4$. Do this in (ii) and (iii) by expressing the subspace as the null space of an appropriate matrix, and finding a basis for that null space by finding the reduced echelon form. In each case, state the dimension of this subspace.

(a) All vectors whose entries are all the same.

(b) All vectors whose entries add to zero.

(c) All vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 + x_2 = 0$ and $x_1 + x_3 + x_4 = 0$.

9B-4 (a) For which numbers c and d does the column space of the matrix

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

have dimension 2?

(b) Find numbers c and d such that the null space of the matrix

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

is 3-dimensional.

9C. Determinants and Inverses

Summary of properties of the determinant
(0) $\det A$ is a number determined by a square matrix A.
(1) $\det I = 1$.
(2) Adding a multiple of one row to another does not change the determinant.
(3) Multiplying a row by a number a multiplies the determinant by a.
(4) $\det(AB) = \det(A)\det(B)$.
(5) A is invertible exactly when $\det A \neq 0$.
Also, if you swap two rows you reverse the sign of the determinant.

9C-1 Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(a) Compute $R(\alpha)R(\beta)$. The claim is that it is $R(\gamma)$ for some angle γ.

(b) Compute $\det R(\theta)$ and $R(\theta)^{-1}$.

9C-2 Compute the determinants of the following matrices, and if the determinant is nonzero find the inverse.

(a) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

9D. Eigenvalues and Eigenvectors

9D-1 (a) Find the eigenvalues and eigenvectors of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$.

(b) In LA.5 it was pointed out that the eigenvalues of AA are the squares of the eigenvalues of A. It's not generally the case that the eigenvalues of a product are the products of the eigenvalues, though: find the eigenvalues of AB.

(c) If you know the eigenvalues of A, what can you say about the eigenvalues of cA (where c is some constant, and cA means A with all entries multiplied by c)?

(d) In (c) you have computed the eigenvalues of $A + A$ (think about it!). On the other hand, check the eigenvalues of $A + B$ for the matrices A and B in (i).

9D-2 Find the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ of each of the matrices in **9C-2**.

9D-3 Suppose A and B are square matrices with eigenvalues $\lambda_1, \ldots \lambda_m$ and $\mu_1 \ldots \mu_n$. What are the eigenvalues of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$?

9D-4 Suppose A and B are $n \times n$ matrices. Express the eigenvalues of the $2n \times 2n$ matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ in terms of the eigenvalues of an $n \times n$ matrix constructed out of A and B.

9E. Two Dimensional Linear Dynamics

9E-1 Diffusion: A door is open between rooms that initially hold $v(0) = 30$ people and $w(0) = 10$ people. People tend to move to the less crowded. Let's suppose that the movement is proportional to $v - w$:

$$\dot{v} = w - v \quad , \quad \dot{w} = v - w$$

(a) Write this system as a matrix equation $\dot{\mathbf{u}} = A\mathbf{u}$: What is A?

(b) Find the eigenvalues and eigenvectors of this matrix.

(c) What are v and w at $t = 1$

(d) What are v and w at $t = \infty$? (Some parties last that long!)

9E-2 (a) Find all the solutions to $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{u}$ which trace out the circle of radius 1.

(b) Find all solutions to $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}$ whose trajectories pass through the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

F. Normal modes

9F-1 Find the normal modes of the equation $\dfrac{d^4x}{dt^4} = cx$. (In this 1D example, a normal mode is just a periodic solution. Constant functions are periodic (of any period; they just don't have a *minimal* period).) Your answer will depend upon c.

9G. Diagonalization, Orthogonal Matrices

9G-1 Suppose that A is a 10×10 matrix of rank 1 and trace 5. What are the ten eigenvalues of A? (Remember, eigenvalues can be repeated! and the trace of a matrix, defined as the sum of its diagonal entries, is equally well the sum of its eigenvalues (taken with repetition).)

9G-2 (a) Diagonalize each of the following matrices: that is, find an invertible S and a diagonal Λ such that the matrix factors as $S\Lambda S^{-1}$.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

(b) Write down diagonalizations of A^3 and A^{-1}.

9G-3 A matrix S is *orthogonal* when its columns are orthogonal to each other and all have length (norm) 1. This is the same as saying that $S^T S = I$. Think about why this is true!

Write the symmetric matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ as $S\Lambda S^{-1}$ with Λ diagonal and S orthogonal.

9H. Decoupling

9H-1 Decouple the rabbit model from LA.7: what linear combinations of the two rabbit populations grow purely exponentially? At what rates? Does the hedge between the two fields have any impact on the combined population?

9I. Matrix Exponential

9I-1 The two farmers from **LA.7 and problem 9H-1** want to be able to predict what their rabbit populations will be after one year, for any populations today. They hire you as a consultant. Write down an explicit matrix which tells them how to compute $\begin{bmatrix} x(t_0+1) \\ y(t_0+1) \end{bmatrix}$ in terms of $\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix}$. You can leave the matrix written as a product of explicit matrices if that saves you some work.

9I-2 Compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$.

9J. Inhomogeneous Systems

9J-1 Find particular solutions to the following systems using the exponential response formula (That is, guess a solution of the form $e^{at}\mathbf{v}$.)

(a) $\dot{\mathbf{u}} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \mathbf{u} + e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) $\dot{\mathbf{u}} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{u} + \cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

9J-2 Farmer Jones gets his gun. As he gets madder he shoots faster. He bags at the rate of e^t rabbits/year rabbits at time t. We'll see what happens.

In our work so far we have found S and Λ so that the rabbit matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ factors as $S\Lambda S^{-1}$. So $e^{At} = Se^{\Lambda t}S^{-1}$. The final S^{-1} adjusts the fundamental matrix

$\Phi(t) = Se^{\Lambda t}$ to give it the right initial condition. If you don't care about the initial condition, $\Phi(t)$ is good enough.

Use it in a variation of parameters solution of the equation $\dot{\mathbf{u}} = A\mathbf{u} - \begin{bmatrix} e^t \\ 0 \end{bmatrix}$ with initial condition $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Begin by finding some particular solution (without regard to initial condition). If you are careful, and keep your matrices factorized, you can use matrices with just one term (rather than a sum) in each entry.

10. 18.03 PDE Exercises

10A. Heat Equation; Separation of Variables

10A-1 Solve the boundary value problem for the temperature of a bar of length 1 following the steps below.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1,\ t > 0 \tag{10A-1.1}$$

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{10A-1.2}$$

$$u(x,0) = x \qquad 0 < x < 1 \tag{10A-1.3}$$

(i) Separation of variables. Find all solutions of the form $u(x,t) = v(x)w(t)$ to (10A-1.1) and (10A-1.2) (**not** (10A-1.3)). Write the list of possible answers in the form

$$u_k(x,t) = v_k(x)w_k(t)$$

Note that your answer is ambiguous up to a multiple, so we just pick the simplest $v_k(x)$ and the simplest $w_k(t)$ (so that $w_k(0) = 1$). With $w_k(0) = 1$, we see that

$$u_k(x,0) = v_k(x)$$

Thus, we have succeeded in solving our problem when the initial condition is $v_k(x)$.

(ii) Write the initial condition (10A-1.3) as a sum of $v_k(x)$ — a Fourier series.

$$x = \sum b_k v_k(x), \qquad 0 < x < 1.$$

Hints: How should you extend the function x outside the range $0 < x < 1$? What is the period? What is the parity (odd/even)? Graph the extended function. Once you have figured out what it is, you will be able to find the series in your notes.

(iii) Superposition principle. Write the solution to (10A-1.1), (10A-1.2), and (10A-1.3) in the form

$$u(x,t) = b_1 u_1(x,t) + b_2 u_2(x,t) + \cdots,$$

with explicit formulas for b_k and u_k.

(iv) Find $u(1/2, 1)$ to one significant figure.

10A-2 Use the same steps as in 10A-1 to solve the boundary value problem for the temperature of a bar of length 1:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x} \qquad 0 < x < 1, t > 0 \tag{10A-2.1}$$

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{10A-2.2}$$

$$u(x,0) = 1 \qquad 0 < x < 1 \tag{10A-2.3}$$

10A-3 Consider the boundary value problem with inhomogeneous boundary condition give by:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x} \qquad 0 < x < 1, t > 0 \tag{10A-3.1}$$

$$u(0,t) = 1 \quad u(1,t) = 0 \qquad t > 0 \tag{10A-3.2}$$

$$u(x,0) = 1 \qquad 0 < x < 1 \tag{10A-3.3}$$

(a) In temperature problems a steady state solution u_{st} is constant in time:

$$\frac{\partial u}{\partial t} = 0$$

It follows that $u_{st} = U(x)$, a function depending only on x. Find the steady state solution $u_{st}(x,t) = U(x)$ to (10A-3.1) and (10A-3.2).

(b) Find the partial differential equation, endpoint, and initial conditions satisfied by $\tilde{u}(x,t) = u(x,t) - U(x)$. Then write down the formula for $\tilde{u}$. [Hint: We already know how to solve the problem with zero boundary conditions.]

(c) Superposition principle. Now that we have found $\tilde{u}$ and U, what is u?

(d) Estimate, to two significant figures, the time T it takes for the solution to be within 1% of its steady state value at the midpoint $x = 1/2$. In other words, find T so that

$$|u(1/2,t) - U(1/2)| \leq \frac{1}{100}U(1/2) \quad \text{for } t \geq T.$$

10B. Wave Equation

10B-1 (a) Find the normal modes of the wave equation on $0 \leq x \leq \pi/2$, $t \geq 0$ given by

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u, \quad u(0,t) = u(\pi/2,t) = 0, t > 0$$

(b) If the solution in part (a) represents a vibrating string, then what frequencies will you hear if it is plucked?

(c) If the length of the string is longer/shorter what happens to the sound?

(d) When you tighten the string of a musical instrument such as a guitar, piano, or cello, the note gets higher. What has changed in the differential equation?

Section 1 SOLUTIONS

SOLUTIONS - SECTION 1

1A-1 a) $y = c_1e^x + c_2xe^x$

$\times -2 \quad y' = (c_1+c_2)e^x + c_2xe^x$

$y'' = (c_1+2c_2)e^x + c_2xe^x$

Add $\quad y'' - 2y' + y = 0$ ✓ (easily checked)

b) $y' = -\dfrac{(\sin x + a)}{x^2} + \dfrac{\cos x}{x} + \sin x$

$\dfrac{y}{x} = \dfrac{\sin x + a}{x^2} - \dfrac{\cos x}{x}$

$\therefore \quad y' + \dfrac{y}{x} = \sin x$

1A-2 a) c_1e^{kx} and $c_1'e^{k'x}$ are the same only if $c_1 = c_1'$, $k = k'$

b) let $k = c_1e^a$

then $y = ke^x$

c) $\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$

$\therefore y = c_1 + c_2(2\cos^2 x - 1) + c_3\cos^2 x$
$= (c_1 - c_2) + (2c_2 + c_3)\cos^2 x$
$= k_1 + k_2\cos^2 x$

d) $y = \ln(ax+b)(cx+d)$
$= \ln(acx^2 + (ad+bc)x + bd)$

$\therefore y = \ln(k_1x^2 + k_2x + k_3)$

1A-3 a) Separating variables gives

$y^2\,dy = \dfrac{dx}{\ln x}$ Integrate both sides from 2 to x:

$\dfrac{y^3}{3}\Big]_2^x = \displaystyle\int_2^x \frac{dt}{\ln t}$ . . Now use $y(2)=0$:

$\dfrac{y(x)^3}{3} - \dfrac{0^3}{3} = \displaystyle\int_2^x \frac{dt}{\ln t}$

$\therefore \quad y = \left[3\displaystyle\int_2^x \frac{dt}{\ln t}\right]^{1/3}$.

b) Separate variables: $\dfrac{dy}{y} = \dfrac{e^x}{x}dx$

Can either use same method as in (a), or else: integrate both sides, using a definite integral as the antiderivative on the right:

$\ln y + c = \displaystyle\int_1^x \frac{e^t}{t}dt$ ⊛

Evaluate c by using $y(1) = 1$. This gives

$\underset{0}{\ln y(1)} + c = \displaystyle\int_1^1 \frac{e^t}{t}dt = 0$

$\therefore c = 0$

So from ⊛ $\quad y = e^{\int_1^x e^t/t\,dt}$

1A-4 a) $\dfrac{y\,dy}{y+1} = x\,dx$ Integrate, noting that $\dfrac{y}{y+1} = 1 - \dfrac{1}{y+1}$

$\therefore \quad dy - \dfrac{dy}{y+1} = x\,dx$

$y - \ln(y+1) = c + \frac{1}{2}x^2$ Put $x=2$ to evaluate c: $[y(2)=0]$

$0 - \ln(1) = c + \frac{1}{2}\cdot 2^2$

$\therefore \quad c = -2$

Soln: $\boxed{y - \ln(y+1) = \frac{1}{2}x^2 - 2}$

b) $\sec^2 u\,du = \sin t\,dt$

$\therefore \quad \tan u = -\cos t + c$ Put $t=0$: $u(0)=0$

$\therefore \quad \tan 0 = -1 + c$

so $c = 1$

Soln: $\boxed{u = \tan^{-1}(1 - \cos t)}$

1A-5

a) $\dfrac{dy}{y^2-2y} = -\dfrac{dx}{x^2}$ Integrate left side by partial fractions

$$\frac{1}{2}\frac{dy}{y-2} - \frac{1}{2}\frac{dy}{y} = \frac{-dx}{x^2}$$

$$\frac{1}{2}\ln\left(\frac{y-2}{y}\right) = c_1 + \frac{1}{x}$$ Multiply by 2, exponentiate

$$= 1-\frac{2}{y} \rightarrow \frac{y-2}{y} = c_2 e^{2/x}$$

algebra; (replace left side by $1-\frac{2}{y}$)

$$\therefore \quad y = \frac{2}{1-c_2 e^{2/x}}$$

b) $$\frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

$$\sin^{-1} v = \ln x + c$$

$$v = \sin(\ln x + c)$$

c) $$\frac{dy}{(y-1)^2} = \frac{dx}{(x+1)^2}$$

$$-\frac{1}{y-1} = \frac{-1}{x+1} + c$$

Solve for y by ordinary algebra.

$$y = 1 + \frac{x+1}{1-c(x+1)}$$

d) $$\frac{dx}{\sqrt{1+x}} = \frac{dt}{t^2+4}$$

$$2\sqrt{1+x} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

$$\therefore \quad x = \frac{1}{4}\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c\right)^2 - 1$$

These problems all take for granted that you know the standard integration formulas and methods from 18.01. Review them if you are having trouble.

You need also the laws of exponentials and logarithms.

1B-1

a) $\dfrac{\partial M}{\partial y} = 3x^2 = \dfrac{\partial N}{\partial x}$ $\therefore$ exact. what's $f(x,y)$?

$$\frac{\partial f}{\partial x} = 3x^2 y \quad \therefore f = x^3 y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + g'(y) = x^3 + y^3. \quad \therefore g = \tfrac{1}{4}y^4 + c$$

so that $f = x^3 y + \frac{1}{4}y^4 + c$.

Solution: $\boxed{x^3 y + y^4/4 = c_1}$

b) $\dfrac{\partial M}{\partial y} = -2y$, $\dfrac{\partial N}{\partial x} = -2x$ not exact.

c) $\dfrac{\partial M}{\partial v} = e^{uv} + v e^{uv} = \dfrac{\partial N}{\partial u}$ $\therefore$ exact

$$\frac{\partial f}{\partial u} = v e^{uv}, \quad \therefore f = e^{uv} + g(v)$$

$$\frac{\partial f}{\partial v} = u e^{uv} + g'(v) = u e^{uv}. \quad \therefore g = c$$

so $f = e^{uv} + c$. Soln: $e^{uv} = c_1$

or taking ln of both sides:

$$\boxed{uv = c_2}$$

d) $\dfrac{\partial M}{\partial y} = 2x$, $\dfrac{\partial N}{\partial x} = -2x$ not exact.

1B-2

a) Multiply by y — this gives

$$2xy\,dx + x^2 dy = 0$$

or $d(x^2 y) = 0$ $\quad \therefore x^2 y = c$

so $\boxed{y = c/x^2}$

b) Integrating factor is $\dfrac{1}{y^2}$:

$$\frac{y\,dx - x\,dy}{y^2} - \frac{dy}{y} = 0$$

$$d\left(\frac{x}{y}\right) - d(\ln y) = 0$$

$$(*) \quad \frac{x}{y} - \ln y = c.$$

Evaluate c by setting $x = 1$ (so $y(1) = 1$)

$$\therefore \frac{1}{1} - \underbrace{\ln 1}_{0} = c, \quad \text{so } c = 1$$

$$\therefore x - y\ln y = y$$

or $\boxed{x = y(\ln y + 1)}$

1B-2

c) Divide by t^2 (so integrating factor is $1/t^2$)

$$\left(1+\frac{4}{t^2}\right)dt = \frac{x\,dt - t\,dx}{t^2}$$

$$\therefore\ d\left(t-\frac{4}{t}\right) = d\left(\frac{-x}{t}\right)$$

$$t - 4/t = \frac{-x}{t} + c$$

$$\therefore\ \boxed{x = 4 - t^2 + ct}$$

d) $\frac{1}{u^2+v^2}$ is an integrating factor:

$$\frac{u\,du + v\,dv}{u^2+v^2} + \frac{v\,du - u\,dv}{u^2+v^2} = 0$$

$$\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = c$$

when $u=0$, $v=1$; $\tfrac{1}{2}\ln 1 + \tan^{-1}(0) = c$ $\quad\therefore c = 0$

$$\boxed{\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = 0}$$

(substitute $r = \sqrt{u^2+v^2}$, $\theta = \tan^{-1}\frac{u}{v}$ to get polar coords)

equation becomes $\ln r + \theta = 0$

or $\boxed{r = e^{-\theta}}$

1B-3

a) $z = y/x$ $\quad\therefore y = zx,\ y' = z'x + z$

Substituting:

$$z'x + z = \frac{2z-1}{z+4}, \quad\therefore z'x = -\frac{(z+1)^2}{z+4}$$

Sep. variables:

$$\frac{z+4}{(z+1)^2}\,dz = \frac{-dx}{x}$$

For ease, write $z+1 = u$

$$\left(\frac{u+3}{u^2}\right)du = \frac{-dx}{x}$$

integrate:

$$\ln u - \frac{3}{u} = -\ln x + c$$

To improve this:

$$\ln u + \ln x = \frac{3}{u} + c$$

Combine + exponentiate: $ux = ke^{3/u}$

Finally: $u = z+1 = \frac{y}{x} + 1 = \frac{y+x}{x}$

$$\therefore\ \boxed{y + x = ke^{3x/(y+x)}}$$

b) let $z = \frac{w}{u}$, so $w = zu$, $w' = z'u + z$

Substituting:

$$z'u + z = \frac{2z}{1-z^2}$$

$$\therefore\ z'u = \frac{z(1+z^2)}{1-z^2}, \text{ after a little algebra}$$

Separate variables:

$$(*)\quad \frac{1-z^2}{z(1+z^2)}\,dz = \frac{du}{u}.$$

Use partial fractions on the left; result

$$\frac{1-z^2}{z(1+z^2)} = \frac{1}{z} + \frac{-2z}{z^2+1}$$

Integrating $(*)$:

$$\ln z - \ln(z^2+1) = \ln u + c$$

Combine and exponentiate both sides:

$$\frac{z}{z^2+1} = ku$$

Finally, put $z = w/u$; result is

$$\boxed{\frac{w}{w^2+u^2} = k}$$

as the solution (you could also solve for u in terms of w)

c) Put $z = y/x$; so $y = zx$, $y' = z'x + z$

Here $\frac{dy}{dx} = \frac{y^2 + x\sqrt{x^2-y^2}}{xy}$ $\quad$ Substitute $y = zx$

$$z'x + z = \frac{z^2 + \sqrt{1-z^2}}{z}$$

$$\therefore\ z'x = \frac{\sqrt{1-z^2}}{z}$$

Separate variables

$$\frac{z\,dz}{\sqrt{1-z^2}} = \frac{dx}{x}$$

$$-\sqrt{1-z^2} = \ln x + c$$

$$\boxed{\sqrt{1 - y^2/x^2} = c_1 - \ln x}$$

This can be solved explicitly for y: square both sides, etc...

$$\boxed{y = x\sqrt{1 - (c_1 - \ln x)^2}}$$

1B-4

$$y = ux^n$$

$$\therefore\ y' = x^n u' + n x^{n-1} u$$

$$x^n u' + n x^{n-1} u = \frac{4 + x^{2n+1} u^2}{x^{n+2} u}$$

$$\therefore\ u' = \frac{4 + (1-n) x^{2n+1} u^2}{x^{2n+2} u}$$

If $n = 1$, we can separate vars:

$$u\,du = \frac{4\,dx}{x^4}$$

$$\therefore\ \frac{u^2}{2} = \frac{-4}{3}\cdot\frac{1}{x^3} + c$$

Since $n = 1$, $u = y/x$

$$\therefore\ \boxed{y^2 = -\frac{8}{3x} + 2cx^2}$$

1B-5

a) $y' + \frac{2}{x} y = 1$ when written in normal form for linear eq'n.

Integ. factor: $e^{\int 2/x\,dx} = e^{2\ln x} = x^2$

$$\therefore\ x^2 y' + 2xy = x^2$$

or $(x^2 y)' = x^2$

$$x^2 y = \tfrac{1}{3} x^3 + c$$

$$\boxed{y = \frac{x}{3} + \frac{c}{x^2}}.$$

b) In standard form;

integ. factor is $e^{\int -\tan t\,dt} = e^{\ln(\cos t)}$

$$= \cos t$$

$$\therefore\ \cos t \frac{dx}{dt} - x \sin t = t$$

or $(x\cos t)' = t$

$$x \cos t = \frac{t^2}{2} + c$$

Since $x(0) = 0$, putting $t = 0$ shows $c = 0$.

$$\therefore\ \boxed{x = \frac{t^2}{2}\sec t}$$

1B-5

c) $(x^2-1)y' + 2xy = 1$ LHS is already exact!

$$[(x^2-1)y]' = 1$$

$$(x^2-1)y = x + c$$

$$\therefore\ y = \frac{x+c}{x^2-1}$$

d) Writing it in standard linear form

$$\frac{dv}{dt} + \frac{3v}{t} = 1$$

Integrating factor: $e^{\int 3/t\,dt} = e^{3\ln t} = t^3$

$$\therefore\ t^3 v' + 3t^2 v = t^3$$

$$(t^3 v)' = t^3$$

$$t^3 v = \tfrac{1}{4} t^4 + c$$

$v(1) = \frac{1}{4} \Rightarrow c = 0$ (put $t=1$)

$$\therefore\ \boxed{v = \tfrac{1}{4} t}$$

1B-6

The integrating factor for this linear equation is $e^{\int a\,dt} = e^{at}$

$$(xe^{at})' = e^{at} r(t)$$

$$x = e^{-at}\left[\int_0^t e^{as} r(s)\,ds\right] + c$$

$$x = \frac{\int_0^t e^{as} r(s)\,ds}{e^{at}} + \frac{c}{e^{at}}$$

To find $\lim_{t\to\infty} x(t)$, use L'Hospital's rule, (∞/∞) [note that $c/e^{at} \to 0$] differentiating top and bottom

$$\therefore\ \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{e^{at} r(t)}{a e^{at}} = \lim_{t\to\infty} \frac{r(t)}{a}$$

$$= 0 \text{ by hypothesis}$$

[Where did we need the hypothesis $a > 0$?]

[We used, in connection with L'H rule, the result $\frac{d}{dt}\int_0^t e^{as} r(s)\,ds = e^{at} r(t)$.

This follows from the 2nd Fundamental Theorem of calculus].

1B-7

$$\frac{dy}{dx} = \frac{y}{y^3+x} \Rightarrow \frac{dx}{dy} = \frac{y^3+x}{y}$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = y^2$$

This is now a linear equation in x.

Integ. factor: $e^{-\int \frac{dy}{y}} = e^{-\ln y} = y^{-1}$

$\therefore$ multiply by $\frac{1}{y}$:

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = y$$

or $$\frac{d}{dy}\left(\frac{x}{y}\right) = y$$

$$\frac{x}{y} = \frac{y^2}{2} + c$$

$$\boxed{x = y^3/2 + cy}$$

1B-8

The systematic procedure — it always works, though it's a bit longer in this case -: since we want to substitute for y, y', begin by expressing them in terms of u.
(Don't just differentiate $u = y^{1-n}$ as is).

$$y = u^{1/1-n}$$

$$y' = \frac{1}{1-n}u^{\left(\frac{1}{1-n}-1\right)} \cdot u' = \frac{1}{1-n}u^{\frac{n}{1-n}}u'$$

Substitute into the ODE:

$$\frac{1}{1-n}u^{\frac{n}{1-n}}u' + pu^{\frac{1}{1-n}} = qu^{\frac{n}{1-n}}$$

Divide through by $u^{\frac{n}{1-n}}$:

$$\boxed{\frac{1}{1-n}u' + pu = q}$$

[Note: in this particular case, it's actually easier just to bumble around, but in general, this only leads to a mess.

However: $y' + py = qy^n$

Divide: $$\frac{y'}{y^n} + \frac{p}{y^{n-1}} = q \quad (*)$$

Put $u = y^{1-n} = \frac{1}{y^{n-1}}$

$$u' = (1-n)\cdot\frac{1}{y^n}\cdot y'$$

$\therefore$ (*) becomes

$$\frac{u'}{1-n} + pu = q, \text{ as before.}]$$

1B-9

$n=2$, so $u = y^{1-2} = y^{-1}$ (by problem 13)

Since we want to substitute for y, y', express them in terms of u and u':

$$y = \frac{1}{u}, \quad y' = -\frac{1}{u^2}\cdot u'$$

$\therefore$ the ODE becomes

$$-\frac{u'}{u^2} + \frac{1}{u} = 2x\cdot\frac{1}{u^2}$$

or $\boxed{u' - u = -2x}$ in standard linear eqn form.

Integ. factor: $e^{\int -dx} = e^{-x}$

Eq'n becomes

$$(e^{-x}u)' = -2xe^{-x} \leftarrow \text{integrate by parts}$$

$$\therefore e^{-x}u = 2xe^{-x} + 2e^{-x} + c$$

$$u = 2x + 2 + Ce^x$$

$$\therefore \boxed{y = \frac{1}{2x+2+Ce^x}}$$

1B-9

$y' - y$ Here $n=3$, so by prob. 13,

$$u = y^{1-3} = y^{-2}$$

As above, calculate y, y' in terms of u and u' (not other way around)

$$y = \frac{1}{\sqrt{u}}, \quad y' = -\frac{1}{2}u^{-3/2}\cdot u'.$$

Substitute into the ODE:

$$-x^2\cdot\frac{u'}{2u^{3/2}} - \frac{1}{u^{3/2}} = \frac{x}{u^{1/2}}$$

$$\therefore \boxed{u' + \frac{2u}{x} = -\frac{2}{x^2}}$$

This is linear ODE; integ. factor is

$$e^{\int\frac{2dx}{x}} = e^{2\ln x} = x^2$$

ODE becomes

$$x^2u' + 2xu = -2$$

$$(x^2u)' = -2$$

$$x^2u = -2x + c$$

$$u = \frac{c-2x}{x^2}$$

$$\boxed{y = \frac{\pm x}{\sqrt{c-2x}}}$$

1B-10

a) $y = y_1 + u$

$y' = y_1' + u' = A + By_1 + Cy_1^2 + u'$

Substituting into the ODE:

$$A + By_1 + Cy_1^2 + u' = A + B(y_1+u) + C(y_1+u)^2$$

After some algebra,

$$u' = Bu + 2Cy_1u + Cu^2$$

$$\therefore\ u' - (B + 2Cy_1)u = Cu^2$$

This is a Bernouilli eq'n (problem 13) with $n = 2$.

b) By inspection, $y_1 = x$ is a sol'n to the ODE. $\therefore$ put $y = x + u$

$y' = 1 + u'$;

Substitution into the ODE gives

$$1 + u' = 1 - x^2 + (x+u)^2$$

$$\therefore\ \boxed{u' - 2xu = u^2},$$

a Bernouilli equation with $n = 2$.

Put $w = u^{1-2} = u^{-1}$

$\therefore\ u = \frac{1}{w},\quad u' = \frac{-w'}{w^2}$

Substituting,

$$\frac{-w'}{w^2} - \frac{2x}{w} = \frac{1}{w^2}$$

or $\boxed{w' + 2xw = -1}$

Linear ODE with integrating factor $e^{\int 2x\,dx} = e^{x^2}$

$$\therefore\ (e^{x^2}w)' = -e^{x^2}$$

$$e^{x^2}w = -\int e^{x^2}dx + C$$

$$\boxed{w = -e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}}$$

Finally:

$$y = x + u = x + \frac{1}{w}$$

$$\therefore\ y = x + \frac{e^{x^2}}{C - \int e^{x^2}dx}$$

(Actually, no value for C gives the original sol'n $y = x$; we have to take "$C = \infty$", or simply add $y = x$ to the above family.

1B-11

a) $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = \frac{dz}{dy}\cdot z$$

Substitute into the ODE:

$$\frac{dz}{dy}\cdot z = a^2y\ ; \quad \text{Sep. vars:}$$

$$z\,dz = a^2y\,dy$$

$$z^2 = a^2y^2 + K$$

$$z = \sqrt{a^2y^2+K}$$

$$\therefore\ y' = \sqrt{a^2y^2+K}$$

Separate variables again:

$$\frac{dy}{\sqrt{\cancel{a^2}y^2+K/a^2}} = a\,dx$$

Look this integral up!

$$\cosh^{-1}\left(\frac{ay}{\sqrt{K}}\right) = ax + C$$

$$y = \frac{\sqrt{K}}{a}\cosh(ax + C)$$

$$\therefore\ \boxed{y = C_1\cosh(ax + C)}$$

$\boxed{1B-11}$ (b)

Let $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot z$$

Substituting, $y\cdot\frac{dz}{dy}\cdot z = z^2$

$\therefore \frac{dz}{z} = \frac{dy}{y}$ $\quad\therefore \ln z = \ln y + \text{const.}$

$\therefore \quad z = y' = Ky$

Then $\frac{dy}{y} = K\,dx$

$\therefore \quad \ln y = Kx + c$

or $y = e^{Kx+c}$ is the solution

$\boxed{1B-11}$ (c)

Let $y' = z$

$$y'' = \frac{dz}{dy}\cdot z$$

Substituting, $\frac{dz}{dy}\cdot z = z(1+3y^2)$

$\therefore \quad dz = (1+3y^2)dy$

$\therefore \quad z = y + y^3 + c$ $\quad$ Using the initial conditions, $c = 0$

$\therefore \frac{dy}{y+y^3} = dx$ $\quad$ (remember: $z = \frac{dy}{dx}$)

Integrating by partial fractions:

$$\frac{1}{y+y^3} = \frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1}$$

$$\therefore \frac{dy}{y} - \frac{y\,dy}{y^2+1} = dx$$

$$\ln y - \tfrac{1}{2}\ln(y^2+1) = x + c$$

Exponentiating both sides,

$$\frac{y}{\sqrt{y^2+1}} = Ke^x$$

Using the initial conditions,

$$\frac{1}{\sqrt{2}} = K$$

$\therefore$ soln: $\boxed{\dfrac{y}{\sqrt{y^2+1}} = \dfrac{e^x}{\sqrt{2}}}$ $\quad$ (can solve for y in terms of x, if desired)

$$y = \frac{e^x}{\sqrt{2-e^{2x}}}$$ $\quad$ (by squaring both sides)

$\boxed{1B-12}$

1. Exact; also linear (divide by dx)

2. Linear; (integ. factor is e^{t^2})

3. Homogeneous: put $z = y/x$, get an ODE for z where you separate variables.

4. Separate variables; also linear in q and linear in p.

5. Exact; also linear.

6. Separate variables.

7. Bernoulli equation: $n = -1$
put $u = y^{1-(-1)} = y^2 \ldots$

8. Separate variables: $\frac{dv}{e^{3v}} = e^{2u}du$

9. Divide by x – this make it homogeneous, so put $z = y/x \ldots$

10. Linear equation (integ. factor is $\frac{1}{x^2}$)

11. Think of y as <u>indep't</u> variable, x as dep't variable; then equation is $\frac{dx}{dy} = x + e^y$, which is linear in x.

12. Separate variables; also a Bernoulli equation (exercise 13)

13. When written in the form
$P(x,y)dx + Q(x,y)dy = 0$, it becomes exact.

14. Linear, with int. factor e^{3x}

15. Divide by x — it becomes homogeneous, so put $z = y/x$, etc.

16. Separate variables

17. Riccati equation (exercise 15a)
A particular sol'n is $y_1 = x^2$;
make the substitution $u = y - y_1$,
get Bernoulli eq'n in u $(n=2)$, etc.

18. Autonomous – x missing.
Put $y' = v$, $y'' = v\frac{dv}{dy}$; separate variables

19. homogeneous – put $z = s/t$
($\ln s - \ln t = \ln s/t$, notice)

20. Exact when written as $Pdy + Qdx = 0$

21. Bernoulli eqn with $n=2$. (ex. 13)

22. Make change of variable
$u = x + y$
(so $u' = 1 + y'$)
Then you can separate variable

23. Becomes linear if you think of y as indept variable, s as dependent variable.

24. Linear (u dep't variable
t indep't variable)

25. $y_1 = -x$ is a particular sol'n.
Riccati equation (ex. 15a) –
put $u = y - y_1$, ...

OR BETTER:
write as $y' + (x+y)^2 + (x+y) + 1 = 0$.
and put $u = x + y$
$u' = 1 + y'$
leads to separation of variables.

26. Put $y' = v$ (so $y'' = v'$)
Get a first order linear eq'n in v.

1C-1

(a)

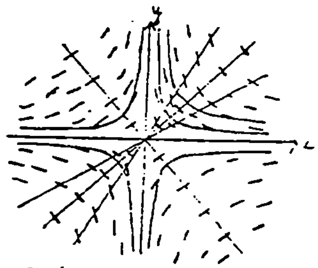
Isoclines: $-\frac{y}{x} = c$

Exact solution:

$\frac{dy}{y} = \frac{-dx}{x}$

$\therefore \ln y = -\ln x + K'$

$\therefore y = \frac{K'}{x}$



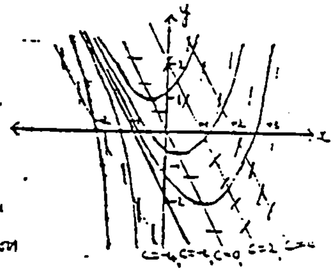
(Sol'n curves are hyperbolas)

(b) Isoclines:

$2x + y = c$.

This is a solution if $y' = -2 = c$;

i.e. $y + 2x + 2 = 0$ is an isocline which is a solution

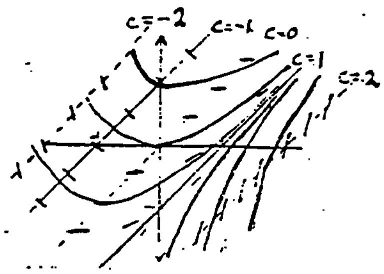


(c) Isoclines:

$x - y = c$

This is a solution if $y' = 1 = c$;

i.e., $x - y = 1$ is an isocline which is a solution.

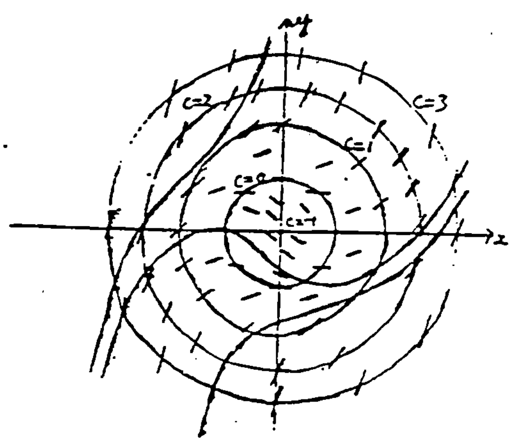


1C-1

d)

Isoclines: $x^2 + y^2 - 1 = c$

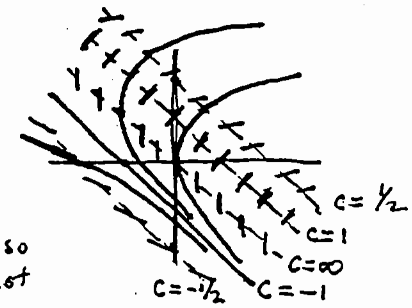
i.e. circles centre $(0,0)$, radius $\sqrt{1+c}$



1C-1

e) Isolines $x+y=1/c$

or $y=-x+\frac{1}{c}$



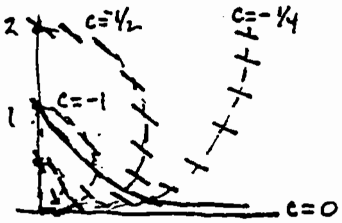
$y=-x-1$ is an integral curve, so other sol'ns cannot cross it.

1C-2

Isoclines: $x^2+y^2+\frac{y}{c}=0$, or completing the square:

$$x^2+\left(y+\frac{1}{2c}\right)^2=\left(\frac{1}{2c}\right)^2$$

(Circles, center at $(0,-1/2c)$.)



a) decreasing, since $y'=\frac{-y}{x^2+y^2}<0$ when $y>0$

b) soln must have $y>0$ for $x>0$ since it cannot cross the integral curve $y=0$.

1C-3

a) Using $\Delta y_n = hf(x_n,y_n)=h(x_n-y_n)$,

get $y_{n+1}=y_n+h(x_n-y_n)$.

Table entries:

x	0	.1	.2	.3
y	1	.9	.82	.758

For example,
$$y_1 = y_0+h(x_0-y_0) = 1+.1(-1)=.9$$
$$y_2 = y_1+h(x_1-y_1) = .9+.1(.1-.9)=.82$$
$$y_3 = .82+.1(.2-.82)=.758$$

some isolines $x-y=c$ are drawn.

sol'n curve through $(0,1)$ is convex (= "concave up"); thus Euler's method gives too low a result:

1C-4

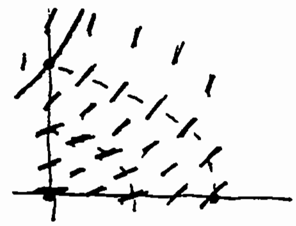
Euler method formula:

$$y_{n+1}=y_n+hf(x_n,y_n)$$

x_n	y_n	$f(x_n,y_n)$	$hf(x_n,y_n)$
0	1	1	.1
.1	1.1	1.31	.131
.2	1.231	1.72	.172
.3	1.403		

$h=.1$

$f(x,y)=x+y^2$



isoclines $x+y^2=c$ (parabolas)))

Solution curve through $(0,1)$ is convex ("concave up"),

∴ Euler method gives too low a result (same reasoning as in 19a)

1C-3

b) $\Delta y_n = \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},\bar{y}_{n+1})\right]$

$y_{n+1}-y_n$

For this ODE, $f(x,y)=x-y$

Thus ($\bar{y}_{n+1}$ is the value given by the next step of Euler's method).

So, $y_0=1$, $\bar{y}_1=.9$ (from part a)

$$\therefore\ y_1-y_0=\frac{.1}{2}\left[f(0,1)+f(.1,.9)\right]$$
$$=\frac{.1}{2}\left[-1-.8\right]=-.09$$
$$\therefore\ y_1=y_0-.09$$
$$y_1=1-.09=\boxed{.91}$$

This does correct the Euler value ($\bar{y}_1=.9$) in the right direction, since we predicted it would be too low. (.910 is actually the correct value of the sol'n to 3 places.)

$\boxed{\text{1C-5}}$

By the formula in 19a,

$$y_n = y_{n-1} + h(x_{n-1} - y_{n-1})$$
$$= (1-h)y_{n-1} + h x_{n-1}.$$

But for $x_0 = 0$, we get $x_1 = h$, $x_2 = 2h$, and in general $x_{n-1} = (n-1)h$.

$$\therefore \quad \boxed{y_n = (1-h)\,y_{n-1} + h^2(n-1)} \quad (**)$$

We prove by induction that the explicit formula for y_n is:

$$(*) \quad \boxed{y_n = 2(1-h)^n - 1 + nh}$$

a) it's true if $n=0$, since $y_0 = 2(1-h)^0 - 1 + 0 = 1$ ✓

b) if true for y_n, it's true for y_{n+1}: since, using $(*)$,

$$y_{n+1} = (1-h)y_n + h^2 n$$
$$= 2(1-h)^{n+1} + (1-h)(-1+nh) + h^2 n$$
$$\therefore \quad y_{n+1} = (1-h)^{n+1} - 1 + (n+1)h \quad ✓$$

[Note: $(**)$ is called a "difference equation" — there are standard ways to solve such things; here $(*)$ is the solution].

Continuing, in our case $h = 1/n$

$$\therefore \quad y_n = 2\left(1 - \tfrac{1}{n}\right)^n - 1 + 1$$
$$= 2\left(1 - \tfrac{1}{n}\right)^n.$$

$$\lim_{n\to\infty} y_n = 2e^{-1} \qquad \left(\begin{array}{l}\text{since} \\ \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e; \\ \text{put } k = -n\end{array}\right)$$

The exact sol'n to the equation is

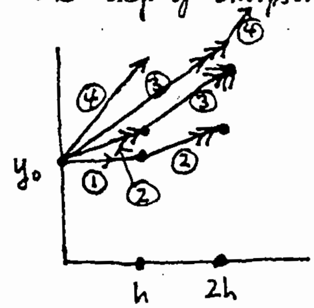
$$y = 2e^{-x} - 1 + x.$$

so $y(1) = 2e^{-1} - 1 + 1 = 2e^{-1}$,

which checks.

$\boxed{\text{1C-6}}$

It suffices to prove this is true for one step of the Runge-Kutta method and one step of Simpson's rule.



We calculate, in R-K method, the 4 slopes marked ① → ④

Then we use a weighted average of them to find $y(2h)$:

$$y_{2h} = y_0 + 2h\cdot\left(\frac{① + 2\cdot② + 2\cdot③ + ④}{6}\right)$$

Since the ODE is simply:

$$y' = f(x),$$

from the picture

slope ① $= f(0)$
slope ② $= f(h)$
slope ③ $= f(h)$
slope ④ $= f(2h)$

$$\therefore \quad y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right)$$

Contrast this with the exact formula: $\quad y_{2h} = y_0 + \int_0^{2h} f(x)\,dx.$

Evaluating the integral approximately by one step of Simpson's rule:

$$y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right),$$

same as what Runge-Kutta gives.

1C-7

The existence and uniqueness theorem requires the equation to be written in the form

$$y' = f(x,y).$$

Doing this, we get

$$y' = -\frac{b(x)}{a(x)}\,y + \frac{c(x)}{a(x)}.$$

The conditions then are:

"$f(x,y)$ continuous" which will be so if

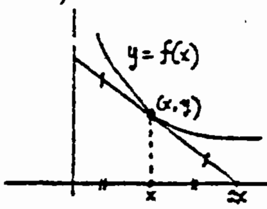
$\boxed{a(x),\ b(x),\ c(x) \text{ continuous (on an interval } [x_0-h,\ x_0+h]) \text{ and } a(x) \neq 0 \text{ in this interval.}}$

"$f_y(x,y)$ continuous", which will be so if $\frac{b(x)}{a(x)}$ is continuous, — and this is already implied by the above condition.

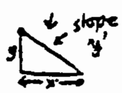
[Note that we must have $a(x) \neq 0$, a condition which is often missed.]

1D-1

a)



If (x,y) is a point on the curve, the geometric condition translates to:

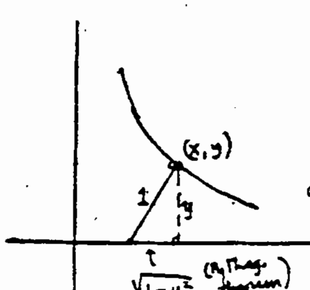


$$\text{slope of tan. line} = \frac{y}{-x}$$

$$\therefore\ y' = -\frac{y}{x}$$

The solution (sep. of vars.) is $\boxed{y = \frac{c}{x}}$ [hyperboles]

b)



Since the normal is ⊥ to the tangent, its slope is the negative reciprocal.

$$\therefore\ \frac{y}{\sqrt{1-y^2}} = -\frac{1}{y'}$$

Solve by sep. of variables: $-\frac{y\,dy}{\sqrt{1-y^2}} = dx$

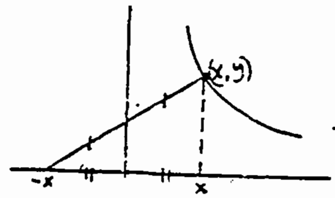
$\therefore\ \sqrt{1-y^2} = x + c$ or

$(x+c)^2 + y^2 = 1$ (Circles, radius 1, centre on x-axis — obvious, what?)

$y = \pm 1$ are also solutions to the problem (above assumed implicitly that $y \neq \pm 1$)

1D-1

(c)



Equating slopes of normal:

$$\frac{y}{2x} = -\frac{1}{y'} \quad \text{(neg. recip. of slope of tangent)}$$

Solve by sep. vars, get $\boxed{\frac{1}{2}y^2 + x^2 = C}$ (ellipses)

(d)

The required property translates mathematically into:

$$\int_a^x y(t)\,dt = k\big(y(x) - y(a)\big)$$

k = constant of proportionality

Differentiate this to get an ODE for $y(x)$:

$$y(x) = k\,y'(x)$$

(by 2nd Fund Thm of Calculus)

solution: $\boxed{y = ce^{x/k}}$ This is the general exponential curve.

1D-2

(a)

(i) The y-intercept of line $y = mx + c$ is $(0, c)$ $\therefore\ c = 2m$

ie $y = mx + 2m = m(x+2)$

Differentiating $\Rightarrow y' = m$

Eliminate m: $\therefore\ \boxed{y' = \frac{y}{x+2}}$ ODE of family

(ii) Orthogonal trajectories satisfy: $-\frac{1}{y'} = \frac{y}{x+2}$

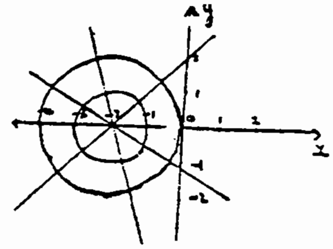
$\Rightarrow \frac{-dx}{dy} = y/(x+2) \Rightarrow y\,dy = -x\,dx + 2\,dx$

$\therefore\ \frac{y^2}{2} + \frac{x^2}{2} + 2x = \text{constant}$

$\therefore\ (x+2)^2 + y^2 = k$

ie Circles centre $(-2, 0)$, variable radii

(iii)



Original family: Lines thro' $(-2,0)$

Orthogonal trajectories: Circles centre $(-2,0)$

1D-2

(b)

$$y = ce^x$$
$$y' = ce^x = y$$

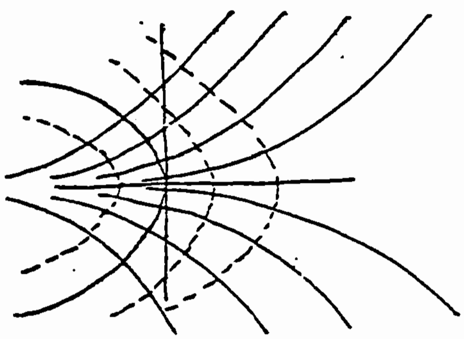
Equation of the orthogonal family:

$$y' = -\frac{1}{y}$$

To find the curves, solve by separation of variables:

$$y\,dy = -dx$$
$$\tfrac{1}{2}y^2 = -x + c \qquad \underline{\text{parabolas}}$$

(all translations of one fixed parabola $\tfrac{1}{2}y^2 = -x$ along the x-axis)



1D-2

(c)

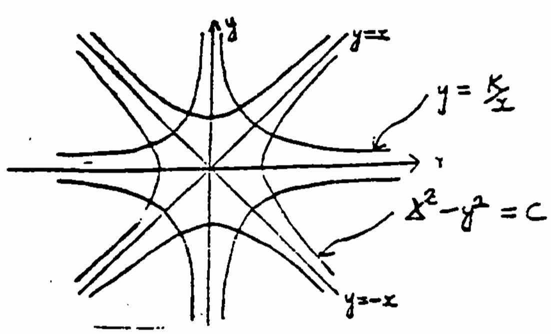
(i) Differentiating gives

$$2x - 2yy' = 0$$
$$\therefore\quad y' = \frac{x}{y} \text{ is required ODE}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{x}{y}$$
$$\therefore\quad \frac{-dy}{y} = \frac{dx}{x}$$
$$\therefore\quad -\ln y = \ln x + c_1$$
$$\therefore\quad y = K/x$$

(iii)



1D-2

(d) Circles with centre on y-axis have equation $x^2 + (y-k)^2 = r^2$

Circle tangent to x-axis $\Rightarrow r = \pm k \quad \therefore\ r^2 = k^2$

$$\therefore\quad x^2 + y^2 - 2yk = 0$$
$$\therefore\quad \frac{x^2+y^2}{2y} = k.$$

Differentiate w.r.t. x:

$$\therefore\quad \frac{2x + 2yy'}{2y} - \frac{(x^2+y^2)y'}{2y^2} = 0$$
$$\therefore\quad 2xy + 2y^2y' - x^2y' - y^2y' = 0$$
$$\text{i.e.}\quad y' = \frac{2xy}{x^2-y^2}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{2xy}{x^2-y^2}$$
$$\text{i.e.}\quad y' = \frac{y^2-x^2}{2xy} \quad \leftarrow \text{a } \underline{\text{homogeneous}} \text{ equation}$$

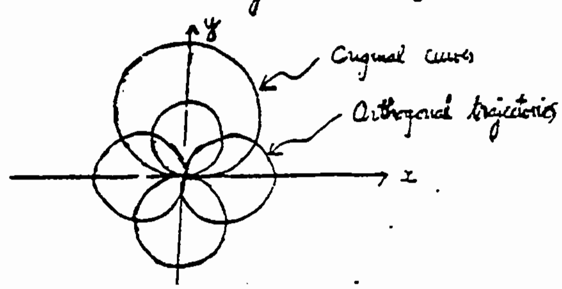
Let $y = zx \quad \therefore\ z = y/x$

Then $y' = xz' + z$

$$\therefore\quad xz' + z = \frac{z^2x^2 - x^2}{2zx^2} = \frac{z^2-1}{2z}$$
$$\therefore\quad xz' = \frac{-(z^2+1)}{2z} \quad \text{i.e.}\quad \frac{2z\,dz}{z^2+1} = \frac{-dx}{x}$$
$$\therefore\quad \ln(z^2+1) = -\ln x + c$$
$$\therefore\quad z^2+1 = \frac{2K}{x} \qquad (2K = e^c)$$
$$\therefore\quad y^2 + x^2 = 2Kx$$

These are circles with centre on the x-axis and tangent to y-axis

(iii)



1D-3

a) $\frac{dx(t)}{dt}$ = rate at which salt flows in − rate of salt outflow

$= \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{in}\end{pmatrix} \cdot \begin{pmatrix}\text{conc.}\\ \text{salt}\\ \text{in added}\\ \text{sol'n}\end{pmatrix} - \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{out}\end{pmatrix} \cdot \begin{pmatrix}\text{conc.}\\ \text{of salt}\\ \text{in tank}\end{pmatrix}$

$$\boxed{x' = kc_1 - k\cdot\frac{x}{V}}$$

b) $x' + ax = 0$ (since $c_1 = 0$)

$x(0) = Vc_0$ $\quad (a = k/V)$

Solution is, by sep. of variables

$$x = Vc_0 e^{-at} \qquad (a = k/V)$$

c) The general case is $\begin{cases} x' + ax = kc_1, \\ x(0) = Vc_0 \end{cases}$ which can be solved by separating variables, or as a linear equation.

Separating variables:

$$\frac{dx}{dt} = kc_1 - ax$$

$$\frac{dx}{kc_1 - ax} = dt$$

$$-\frac{1}{a}\ln(kc_1 - ax) = t + A \quad \text{const. of integration}$$

or $kc_1 - ax = A_1 e^{-at}$ $\quad A_1$ = arbitrary constant

Using the initial condition to find A_1:

$$kc_1 - aVc_0 = A_1 \quad (\text{note that } aV = k)$$

$$\therefore \quad k(c_1 - c_0) = A_1$$

so soln is (note that $k/a = V$)

$$x = Vc_1 - V(c_1 - c_0)e^{-at}$$

or in terms of the concentration $c(t)$:

$$\boxed{c = c_1 - (c_1 - c_0)e^{-at}}$$

As $t \to \infty$, $e^{-at} \to 0$, so $\boxed{c \to c_1}$

d) If $c_1 = c_0 e^{-\alpha t}$, then the ODE (IVP) becomes (as in (c))

$$\begin{cases} x' + ax = kc_0 e^{-\alpha t} \\ x(0) = Vc_0 \end{cases}$$

This must be solved as a linear equation. The integrating factor is e^{at}:

$$x'e^{at} + axe^{at} = kc_0 e^{(a-\alpha)t}$$

or $(xe^{at})' = kc_0 e^{(a-\alpha)t}$ ⊛

Integrating,

$$xe^{at} = \frac{kc_0}{a-\alpha} e^{(a-\alpha)t} + A \quad \text{const. of integ.}$$

Using the initial condition to find A:

$$Vc_0 = A + \frac{kc_0}{a-\alpha}$$

$$\therefore \quad x = \frac{kc_0}{a-\alpha} e^{-\alpha t} + \left(Vc_0 - \frac{kc_0}{a-\alpha}\right)e^{-at}$$

Dividing by V to get concentration:

$$c = \frac{ac_0}{a-\alpha} e^{-\alpha t} + \left(c_0 - \frac{ac_0}{a-\alpha}\right)e^{-at}$$

[If $\alpha = 0$, then $c_1 = c_0$, and this agrees with part (c)]

1D-4

$$\frac{dA}{dt} = -\lambda_1 A, \qquad \lambda_1 = \frac{\ln 2}{\text{half-life}}$$

$\frac{dB}{dt}$ = rate at which B produced by decay of A − rate at which B is lost by decay of B

$$\therefore \quad \frac{dB}{dt} = \lambda_1 A - \lambda_2 B$$

$\therefore$, From the first equation, $A = A_0 e^{-\lambda_1 t}$

$$\therefore \quad \boxed{\frac{dB}{dt} + \lambda_2 B = \lambda_1 A_0 e^{-\lambda_1 t}} \quad \text{ODE for } B(t)$$

Solve it as a linear equation, using $e^{+\lambda_2 t}$ as integrating factor, and $B(0) = B_0$ as initial condition.

Solution is

$$B(t) = \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \left(B_0 - \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}\right)e^{-\lambda_2 t}$$

Taking $\lambda_1 = 1$, $\lambda_2 = 2$,

$$B(t) = A_0 e^{-t} + (B_0 - A_0)e^{-2t}$$

Differentiating to see when $B(t)$ is maximum:

$$0 = B'(t) = -A_0 e^{-t} - 2(B_0 - A_0)e^{-2t}$$

Solving for t: $\frac{A_0}{2(A_0 - B_0)} = e^{-t}$

If $A_0 > 2B_0$, then $t = -\ln\left(\frac{A_0}{2(A_0 - B_0)}\right) > 0$

If $A_0 \le 2B_0$, no solution (the maximum is at $t = 0$).

1D-5

By Newtons cooling law

$$\frac{dT}{dt} = K(T-20)$$

(K a constant of proportionality)

Solving this (by sep. of variables) gives

$$T = \alpha e^{Kt} + 20 \quad (\alpha \text{ another constant})$$

$$T(0) = 100$$

$$\therefore \quad \alpha + 20 = 100$$

$$\therefore \quad \alpha = 80$$

$$T(5) = \alpha e^{5K} + 20 = 80$$

$$\therefore \quad \alpha e^{5K} = 60$$

$$\therefore \quad K = \tfrac{1}{5}\ln\left(\tfrac{60}{\alpha}\right) = \tfrac{1}{5}\ln\left(\tfrac{3}{4}\right) < 0.$$

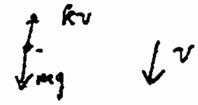
$$\therefore \quad T = 80\, e^{-\frac{1}{5}\ln(\frac{4}{3})t} + 20$$

When $T = 60$ we then find

$$t = \frac{5\ln 2}{\ln(\frac{4}{3})} \simeq 12 \text{ mins.}$$

1D-6

a)



Downwards force $= m\frac{dv}{dt} = mg - kv$

$$\therefore \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

Solving this by separation of variables (or as a linear equation), we get

$$v = \alpha e^{-k/m\, t} + \frac{mg}{k} \qquad (\alpha \text{ constant})$$

Using the initial condition

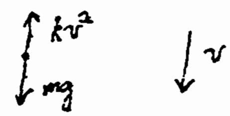
$v(0) = 0$. $\quad \therefore \quad \frac{mg}{k} + \alpha = 0$.

$$\therefore \quad v = \frac{mg}{k}\left(1 - e^{-kt/m}\right) \qquad \text{SOLN.}$$

terminal velocity:

$$\lim_{t\to\infty} v(t) = \frac{mg}{k} \qquad \text{(constant)}$$

b)



Downwards force $= m\frac{dv}{dt} = mg - kv^2$

$$\therefore \quad \frac{dv}{v^2 - \frac{mg}{k}} = -\frac{k}{m}\,dt$$

But

$$\frac{1}{v^2 - \frac{mg}{k}} = \frac{1}{v^2 - a^2} = \frac{1}{2a}\left[\frac{1}{v-a} - \frac{1}{v+a}\right]$$

where $a \equiv \sqrt{\frac{mg}{k}}$

$$\therefore \quad \frac{dv}{v-a} - \frac{dv}{v+a} = -\frac{2ak}{m}\,dt$$

$$\therefore \quad \ln\left|\frac{v-a}{v+a}\right| = C - \frac{2ak}{m}t$$

But $v(0) = 0$ $\quad \therefore \quad \ln 1 = C$, i.e., $C = 0$

$$\therefore \quad \frac{a-v}{a+v} = e^{-\frac{2akt}{m}}$$

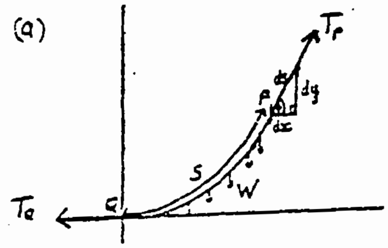
(since L.H.S. > 0 (at least near $t=0$))

$$\therefore \quad v = a\left(\frac{1 - e^{-\frac{2akt}{m}}}{1 + e^{-\frac{2akt}{m}}}\right)$$

$$\therefore \quad \lim_{t\to\infty} v(t) = a = \sqrt{\frac{mg}{k}}$$

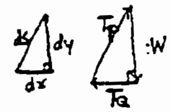
1D-7

(a)



$\tan\phi = \frac{dy}{dx}$

OR: the Δs are similar:



(Δ of forces is closed since cable is in equilibrium)

$\therefore$

$\frac{dx}{T_a} = \frac{dy}{W} = \frac{ds}{T_P}$

(corresponding sides)

Balancing forces horizontally

$$T_a = T_P \cos\phi = T_P \frac{dx}{ds}$$

$$\therefore \quad \frac{ds}{T_P} = \frac{dx}{T_a} \qquad (i)$$

Balancing forces vertically

$$W = T_P \sin\phi = T_P \frac{dy}{ds}$$

$$\therefore \quad \frac{ds}{T_P} = \frac{dy}{W} \qquad (ii) \quad \text{as required.}$$

(b) Suppose the cable hangs under its own weight and has constant density ρ per unit length

Then $W = \rho s$

Now $\frac{dx}{T_a} = \frac{dy}{W} = \frac{dy}{\rho s}$

$$\therefore \quad \frac{dy}{dx} = ks \quad (\text{where } k = \frac{\rho}{T_a} \text{ is a constant})$$

$$\text{Then} \quad \frac{d^2y}{dx^2} = k\frac{ds}{dx} = \frac{k\sqrt{(dx)^2+(dy)^2}}{dx}$$

$$= k\sqrt{1+(y')^2} \quad \text{which proves (i)}$$

Also, $\frac{dy}{W} = \frac{ds}{T_P}$; but $T_P = \sqrt{W^2+T_a^2}$ (from the force triangle)

$$\therefore \quad \frac{dy}{\rho s} = \frac{ds}{\sqrt{\rho^2 s^2 + T_a^2}}$$

$$\therefore \quad \frac{dy}{ds} = \frac{s}{\sqrt{s^2+c^2}} \qquad \text{where } c = T_a/\rho$$

$$\therefore \quad y = \sqrt{s^2+c^2} + c_1,$$

which proves (ii)

(c) Let λ be the constant weight per unit horizontal length

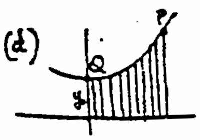
$$\therefore \quad W = \lambda x$$

$$\text{Then} \quad \frac{dy}{dx} = \frac{W}{T_a} = \frac{\lambda x}{T_a}$$

$$\therefore \quad y = \frac{\lambda}{T_a}\frac{x^2}{2} + y_0$$

Thus the cable takes the form of a parabola.

(d)



Here $W = k_1 \cdot (\text{area under } \widehat{QP})$ since rods are equally and closely spaced.

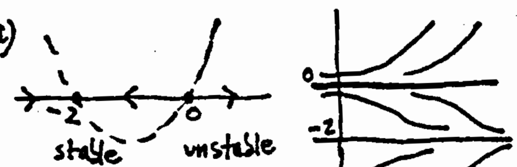
$$\text{so} \quad \frac{dy}{dx} = \frac{W}{T_a} = \frac{k_1}{T_a}\int_0^x y(t)\,dt$$

$$\therefore \quad \frac{d^2y}{dx^2} = k^2 y, \qquad \text{by the 2nd Fund. Thm. of Calculus.} \quad (k^2 = k_1/T_a > 0)$$

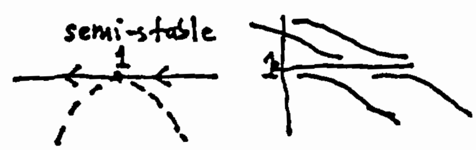
[The curve is once again of the form $y = \cosh(kx) + c_1$]

1E-1

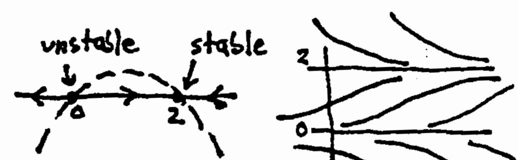
a)



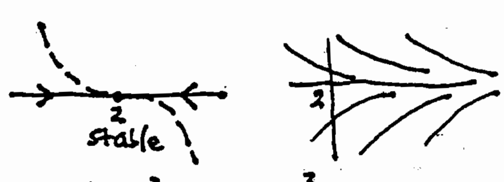
b)



c)



d)



(write: $(2-x)^3 = -(x-2)^3$)

Section II Solutions

2A-1 a) This is true because D^2, pD, and multiplich by q are all linear operators:

$$q(y_1+y_2) = qy_1 + qy_2 \quad (1)$$
$$pD(y_1+y_2) = p(Dy_1+Dy_2)$$
$$\therefore \quad = pDy_1 + pDy_2 \quad (2)$$
$$D^2(y_1+y_2) = D^2y_1 + D^2y_1 \quad (3)$$

Adding ①, ②, ③ gives
$$L(y_1+y_2) = Ly_1 + Ly_2$$
The proof for $L(cy_1) = cLy_1$ is similar.

b) (i) $Ly_h = 0$ since y_h solves the eqn $Ly=0$

$Ly_p = r$ since y_p solves the original eqn.

Adding, using linearity of L: $L(y_h+y_p) = r$ $\quad \therefore y_h+y_p$ is a sol'n.

(ii) if y_1 is any sol'n, then
$$L(y_1-y_p) = Ly_1 - Ly_p = r - r = 0$$
$$\therefore \quad y_1 - y_p = y_h \quad (\text{a sol'n of } Ly=0)$$
$$\therefore y_1 = y_h + y_p.$$

Parts (i) + (ii) together show all sol'ns are of the form $y_h + y_p$.

2A-2 a)
$$y = c_1e^x + c_2e^{2x}$$
$$y' = c_1e^x + 2c_2e^{2x}$$
$$y'' = c_1e^x + 4c_2e^{2x}$$

$$y'-y = c_2e^{2x}$$
$$y''-y' = 2c_2e^{2x}$$
$$\therefore y''-y' = 2(y'-y)$$

or: $\boxed{y''-3y'+2y=0}$.

b) The question is whether we can find values for c_1, c_2 such that
$$c_1e^{x_0} + c_2e^{2x_0} = y_0$$
$$c_1e^{x_0} + 2c_2e^{2x_0} = y_0'$$

These equations can be solved (by Cramer's rule) for c_1, c_2 provided that $\begin{vmatrix} e^{x_0} & e^{2x_0} \\ e^{x_0} & 2e^{2x_0} \end{vmatrix} \neq 0$.
(coefficient determinant)

But this det $= e^{3x_0} \neq 0$ for any x_0. ✓

2A-3 a)
$$y = c_1x + c_2x^2$$
$$y' = c_1 + 2c_2x$$
$$y'' = 2c_2$$

You want to eliminate c_1, c_2. One way —:

$c_2 = y''/2$ from last eqn
$c_1 = y' - y''x$ from 2^{nd} + 3^{rd} eqn.

Substitute into 1^{st} eqn, get
$$y = (y'-y''x)x + \frac{y''}{2}x^2,$$
which by algebra becomes
$$\boxed{x^2y'' - 2xy' + 2y = 0}$$

b) all sol'ns $y = c_1x + c_2x^2$ satisfy $y(0) = 0$

c) This theorem requires that when eqn is written $y'' + p(x)y' + q(x)y = 0$, that p, q be continuous functions.
But here, the ODE in standard form is
$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0;$$
coefficients are discontinuous at $x=0$.

2A-4 a) Suppose y_1 is a solution to
$$y'' + p(x)y' + q(x)y = 0 \quad ⊛$$
tangent to x-axis at the pt. x_0.
Then
$$y_1(x_0) = 0$$
$$y_1'(x_0) = 0.$$
But $y_2(x) \equiv 0$ is another sol'n to ⊛ with this same property:
$$y_2(x_0) = 0$$
$$y_2'(x_0) = 0$$
$\therefore$ by the uniqueness theorem,
$$y_1 \equiv y_2 \text{ for all } x,$$
i.e., $y_1 \equiv 0$.

b)
$$y = x^2$$
$$y' = 2x$$
$$y'' = 2$$

$\therefore \boxed{xy'' - y' = 0}$ is such an equation

or: $\boxed{y'' - \frac{1}{x}y' = 0}$

Part (a) is not contradicted, since the coefficient $\frac{1}{x}$ is discontinuous at $x=0$.

2A-5 a) $W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$

$= (m_2 - m_1)e^{(m_1+m_2)x}$;

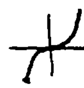
since $e^x \neq 0$ for all x, this is never 0 if $m_1 \neq m_2$. $\therefore$ functions are lin. ind.

b) $W(e^{mx}, xe^{mx}) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & mxe^{mx} + e^{mx} \end{vmatrix}$

$= e^{2mx} \neq 0$ for any x.

(This holds true even if $m = 0$).

$\therefore$ the functions are lin. indep't.

2A-6 (The graph of $x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$: )

a) If $x \geq 0$, $W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} \equiv 0$

if $x \leq 0$, $W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} \equiv 0$

b) Suppose they were linearly dependent on an interval (a, b) containing 0, that is, suppose there are c_1, c_2 such that $\boxed{c_1y_1 + c_2y_2 = 0 \text{ for all } x \in (a,b)}$.

Then if $x \geq 0$, $y_1 = y_2$, $\therefore c_1 = -c_2$

if $x < 0$, $y_1 = -y_2$, $\therefore c_1 = c_2$

Thus $c_1 = 0$ and $c_2 = 0$, so that y_1 and y_2 are not lin. dep't on (a, b).

Since $y_2' = 2x$ for $x > 0$,
$y_2' = -2x$ for $x < 0$

graph of y_2' is 

Thus y_2'' does not exist at $x = 0$, so it cannot be the solution to a 2nd order equation $y'' + p(x)y' + q(x)y = 0$ on the interval (a, b) containing 0.

Thus thm in the book ($W \equiv 0$ for 2 solns to ODE $\Rightarrow$ solns are lin dept) is not contradicted.

2A-7 a) This can be done directly, by differentiating $y_1y_2' - y_1'y_2$. (*see below)

An elegant way to do it is to use the formula for differentiating a determinant: diff. one row at a time, then add:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}' = \begin{vmatrix} u_1' & u_2' \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ v_1' & v_2' \end{vmatrix}$$

(this works for dets. of any size).

Applying this to the Wronskian:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} ;$$

Since y_1 and y_2 solve $y'' = -py' - qy$, we get the above right-hand det.

$$= \begin{vmatrix} y_1 & y_2 \\ -py_1' - qy_1 & -py_2' - qy_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' & -py_2' \end{vmatrix}$$

(adding $q \cdot$ (1st row) to 2nd doesn't change value of the determinant)

$$= -p\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW.$$

(* you also have to use that y_1, y_2 are solns, i.e., that $y_1'' = -py_1' - qy_1$, $y_2'' = -py_2' - qy_2$).

b) From part (a), if $p(x) = 0$, then $\frac{dW}{dx} = 0$, so $W(y_1, y_2) = c$.

c) $y'' + k^2y = 0$ Here $p = 0$

$W(\cos kx, \sin kx)$

$= \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix}$

$= k(\cos^2 kx + \sin^2 kx)$

$= k$, a constant.

2B-1

a) $y_2 = ue^x$

$\underline{x\ -2]}$ $y_2' = u'e^x + ue^x$

$y_2'' = u''e^x + 2u'e^x + ue^x$

Multiply second row by -2 and add:

$y_2'' - 2y_2' + y_2 = u''e^x$ (all other terms cancel out)

If y_2 is a soln to the ODE, the left-hand side must be 0. Therefore we must have

$$u''e^x = 0$$

so $u'' = 0$,

$\therefore\ u = ax + b$

and $\therefore\ y_2 = (ax+b)e^x$

Any of these for which $a \neq 0$ gives a second solution — for ex., $\boxed{y_2 = xe^x}$.

b) From II/7a, $\frac{dW}{dx} = -pW = 2W$

$\therefore\ W(y_1, y_2) = ce^{2x}$, $c \neq 0$

But $W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$

Equating these two expressions for W,

$e^x(y_2' - y_2) = ce^{2x}$

$\therefore\ y_2' - y_2 = ce^x$

(c can have any $\neq 0$ value)

Solving this ODE gives (it's a linear equation)

$y_2 = e^x(x + c_1)$ as a family of second solutions.

c) $y_2 = e^x \int \frac{1}{e^{2x}} e^{-\int -2dx} dx$

$= e^x \int 1 \cdot dx = e^x(x + c)$

[more generally: $e^{\int 2dx} = e^{2x + c_1}$

$\therefore\ y_2 = e^x \int (e^{c_1}) dx$ put $c_2 = e^{c_1}$

$= e^x(c_2 x + c)$]

d) All the solutions are the same — the most general form is

$y_2 = e^x(c_1 x + c_2)$, with $c_1 \neq 0$

(if $c_1 = 0$, we just get y_1 back)

2B-2

$$W(y_1, y_2) = \begin{vmatrix} e^x & e^x(ax+b) \\ e^x & e^x(ax+b) + ae^x \end{vmatrix}$$

$= ae^{2x}$, $\neq 0$ if $a \neq 0$.

[This shows it for the special equation only].

In general:

$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$

$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} dx$

$\therefore\ y_2' = y_1' \int \frac{1}{y_1^2} e^{-\int p dx} + y_1 \cdot \frac{1}{y_1^2} e^{-\int p dx}$

$= y_1' y_2 / y_1 + \frac{1}{y_1} e^{-\int p dx}$

$\therefore\ W(y_1, y_2) = y_1' y_2 + e^{-\int p dx} - y_1' y_2$

$= e^{-\int p dx} \neq 0$

[Note that this same formula for the Wronskian follows from II/7a].

2B-3

let $y_2 = x \cdot u$, so that

$y_2' = u + xu'$, $y_2'' = 2u' + xu''$.

Substituting into $x^2y'' + 2xy' - 2y = 0$ gives after cancellation and dividing by x^2:

$xu'' + 4u' = 0$ Put $v = u'$.

$x\frac{dv}{dx} + 4v = 0$ or $\boxed{\frac{dv}{v} = \frac{-4dx}{x}}$

Solving, $v = \frac{c}{x^4}$, or $u' = \frac{c}{x^4}$

$\therefore\ u = \frac{c}{-3x^3} + c_0 = \frac{c_1}{x^3} + c_0$

$\therefore\ \boxed{y_2 = \frac{c_1}{x^2} + c_0 x}$, a second sol'n (if $c_1 \neq 0$)

[can also use the general formula given in II/8c]

2B-4

Using the general formula [II/8c]:

Find: $e^{-\int p dx}$ $\int p\,dx = \int \frac{-2x}{1-x^2} dx = \ln(1-x^2)$

$= \frac{1}{1-x^2}$

$\therefore\ \int \frac{1}{x^2} e^{-\int p dx} = \int \frac{dx}{x^2(1-x^2)}$

we do this by partial fractions →

(cont'd)

2B-4

(cont'd)

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)}$$
$$= \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\therefore \int \frac{dx}{x^2(1-x^2)} = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)$$
$$= -\frac{1}{x} + \frac{1}{2}\ln\frac{(1+x)}{(1-x)}$$

$$\therefore y_2 = \underset{= x}{y_1} \int \frac{1}{y_1^2} e^{-\int p\,dx} = \boxed{-1 + \frac{x}{2}\ln\frac{1+x}{1-x}}$$

The general solution is now
$$c_1 y_1 + c_2 y_2$$
or $\boxed{c_1 x + c_2\left(-1 + \frac{x}{2}\ln\frac{1+x}{1-x}\right)}$

2C-1

a) Char eq'n: $\lambda^2 - 3\lambda + 2 = 0$
or $(\lambda-1)(\lambda-2) = 0$

roots: $\lambda = 1, 2$

$\therefore \boxed{y = c_1 e^x + c_2 e^{2x}}$

b) Char eq'n: $r^2 + 2r - 3 = 0$
$(r+3)(r-1) = 0$

$\therefore y = c_1 e^x + c_2 e^{-3x}$. Put in initial conditions:

$y(0)=1 \Rightarrow c_1 + c_2 = 1$
$y'(0)=1 \Rightarrow c_1 - 3c_2 = -1$
} solve for c_1, c_2

$c_1 = 1/2, \quad c_2 = 1/2$

$\therefore \boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{-3x}}$

c) Char. eqn $r^2 + 2r + 2 = 0$

By quad. formula: $r = -1 \pm i$

$\therefore y = e^{-x}(c_1\cos x + c_2\sin x)$

[using as y_1, y_2 the real + imaginary parts of the cx. soln $y = e^{(-1+i)x}$
$= e^{-x}(\cos x + i\sin x)$]

2C-1

d) Char. eqn: $r^2 - 2r + 5 = 0$

By quad. fmla: $r = 1 \pm 2i$

Gen'l soln: $y = e^x(c_1\cos 2x + c_2\sin 2x)$

Putting in initial cond'ns (you'll have to find y' first!)

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 1 \Rightarrow c_1 + 2c_2 = -1, \quad \therefore c_2 = -1$

so $y = e^x(\cos 2x - \sin 2x)$

e) Char. eqn: $r^2 - 4r + 4 = 0$
or $(r-2)^2 = 0$; $r = 2$ double root

$\therefore y = e^{2x}(c_1 x + c_2)$

is the general solution. Put in initial conditions:

$y(0) = 1 \Rightarrow c_2 = 1$

$y'(0) = 1 \Rightarrow 2c_2 + c_1 = 1, \quad \therefore c_1 = -1$

so sol'n is: $y = (1-x)e^{2x}$

2C-2

$$W = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(a\sin bx + b\cos bx) \end{vmatrix}$$

$$= \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ -e^{ax}b\sin bx & e^{ax}(b\cos bx) \end{vmatrix},$$

(by subtracting $a\cdot$(1st row) from 2nd row);

$= e^{2ax}(b\cos^2 bx + b\sin^2 bx) = e^{2ax}\cdot b$

$\neq 0$ if $\boxed{b \neq 0}$ (no restriction on a)

2C-3

Char. eqn: $r^2 + cr + 4 = 0$

roots: $r = \dfrac{-c \pm \sqrt{c^2-16}}{2}$

a) has oscillatory solns $\Leftrightarrow$ r is complex (so soln has sin + cos terms);

$\Leftrightarrow c^2 - 16 < 0$, or $\boxed{-4 < c < 4}$

b) if the solutions oscillate, above shows that $r = -\frac{c}{2} \pm i\beta \quad (\beta \neq 0)$

and solns are $y = e^{-\frac{cx}{2}}(c_1\cos\beta x + c_2\sin\beta x)$.

Damped oscillations $\Leftrightarrow c > 0$ (so $y \to 0$ as $t \to \infty$)

$\therefore \boxed{0 < c < 4}$ is condition.

2C-4

a) [Use y' for $\frac{dy}{dx}$, $\dot{y}$ for $\frac{dy}{dt}$.]

We have $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$ $\qquad x = e^t$, $\frac{dx}{dt} = e^t$, $\frac{dt}{dx} = e^{-t}$

$$\therefore\ y' = \dot{y}\, e^{-t}$$

$$y'' = \frac{d}{dt}(\dot{y}e^{-t})\cdot\frac{dt}{dx}$$

$$= (\ddot{y}e^{-t} - \dot{y}e^{-t})e^{-t}$$

$$= (\ddot{y} - \dot{y})e^{-2t}$$

Substituting into the ODE:

$x^2y'' + pxy' + qy = 0$ becomes

$$(\ddot{y} - \dot{y}) + p\dot{y} + qy = 0$$

b) $p = q = 1$, so we get $\ddot{y} + y = 0$, whose solution are $y = c_1\cos t + c_2\sin t$

$\left.\begin{array}{l} x = e^t \\ \therefore t = \ln x \end{array}\right\}$ gives $y = c_1\cos(\ln x) + c_2\sin(\ln x)$

2C-5

Char. eqn is $\boxed{Mr^2 + cr + k = 0}$

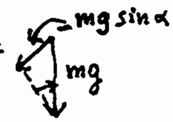
For critical damping, it should have two equal roots; by quadratic formula

$$r = \frac{-c \pm \sqrt{c^2 - 4Mk}}{2M}, \quad \therefore \boxed{c^2 - 4mk = 0} \text{ is condition}$$

(when $c^2 - 4mk < 0$, get oscillations).

2C-6

Force triangle



$\vec{F} = ma$ becomes:

$$\underset{(\text{grav.})}{-mg\sin\alpha} \underset{(\text{air res.})}{- mc\frac{d\alpha}{dt}} = ml\frac{d^2\alpha}{dt^2}$$

$$\therefore \boxed{\ddot{\alpha} + \frac{c}{l}\dot{\alpha} + \frac{g}{l}\sin\alpha = 0}$$

If α small, $\sin\alpha \approx \alpha$

If undamped, $c = 0$, get approx. $\boxed{\ddot{\alpha} + \frac{g}{l}\alpha = 0}$ $\quad$ [char eqn is $r^2 + g/l = 0$]

$\therefore$ Solns are $y = c_1\cos\sqrt{\frac{g}{l}}\,t + c_2\sin\left(\sqrt{\frac{g}{l}}\,t\right)$

$$\text{The period} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}}$$

(so as length increases, so does the period; on the moon, it swings slower (bigger period))
(less g)

2C-7

a) $a + bx + ce^x$ $\qquad$ b) $a\cos 2x + b\sin 2x$

c) $ax\cos 2x + bx\sin 2x$

d) ax^2e^x (1 is a double root of the char. eqn)

e) $ae^{-x} + bxe^{2x}$ (2 is a root of char. eqn)

f) $(ax^3 + bx^2)e^{3x}$ (3 is double root of char. eqn)

2C-8

b) $y_h = a_1\cos 2x + a_2\sin 2x$

To find y_p, use undet. coefficients:

$y_p = c_1\cos x + c_2\sin x$ $\qquad$ $\lfloor \times 4$ (mult. factor)

$\therefore\ y_p'' = -c_1\cos x - c_2\sin x$ $\qquad$ and add:

LHS is by hypothesis: $y_p'' + 4y_p = 2\cos x$

$$2\cos x = 3c_1\cos x + 3c_2\sin x$$

$$\therefore c_1 = 2/3 \quad c_2 = 0$$

So $\boxed{y = a_1\cos 2x + a_2\sin 2x + \frac{2}{3}\cos x}$

$y(0) = 0 \Rightarrow a_1 + 2/3 = 0$ $\qquad$ $\therefore \boxed{\begin{array}{l} a_1 = -2/3 \\ a_2 = 1/2 \end{array}}$

$y'(0) = 1 \Rightarrow 2a_2 = 1$

2C-8

a) $y_h = a_1e^x + a_2e^{5x}$, as usual.

Try $y_p = cxe^x$ $\qquad$ $\lfloor \times 5$

$\therefore\ y_p' = ce^x(x+1)$ $\qquad$ $\lfloor \times -6$ $\quad$ } multiplicn factors

$y_p'' = ce^x(x+2)$ $\qquad$ then add:

$$e^x = e^x(-6c + 2c) + xe^x\underbrace{(5c - 6c + c)}_{0}$$

$$\therefore -4c = 1$$

$$c = -1/4$$

$$\boxed{y = a_1e^x + a_2e^{5x} - \frac{1}{4}xe^x}$$

c) Char eqn: $r^2 + r + 1 = 0$, $r = \frac{-1 \pm \sqrt{-3}}{2}$

$$\therefore y_h = e^{-x/2}\left(a_1\cos\frac{\sqrt{3}}{2}x + a_2\sin\frac{\sqrt{3}}{2}x\right)$$

Try $y_p = c_1xe^x + c_2e^x$ $\qquad$ Add the eqns:

$y_p' = c_1e^x(x+1) + c_2e^x$

$y_p'' = c_1e^x(x+2) + c_2e^x$

$$2xe^x = 3c_1xe^x + (3c_1 + 3c_2)e^x$$

$$\therefore c_1 = 2/3, \quad c_2 = -2/3$$

$$y = e^{-x/2}\left(a_1\cos\frac{\sqrt{3}}{2}x + a_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{2}{3}e^x(x-1)$$

2C-8

d) $y_h = a_1 e^x + a_2 e^{-x}$

Try: $y_p = c_1 x^2 + c_2 x + c_3 \quad \underline{\cdot -1}$

$y_p'' = 2c_1 \qquad$ Add:

$$x^2 = -c_1 x^2 + c_2 x + 2c_1 - c_3$$

$\therefore c_1 = -1, \quad c_2 = 0, \quad 2c_1 - c_3 = 0$

$c_3 = -2$

$$\boxed{y = a_1 e^x + a_2 e^{-x} - x^2 - 2}$$

$y(0) = 0 \Rightarrow a_1 + a_2 - 2 = 0$

$y'(0) = -1 \Rightarrow a_1 - a_2 = -1$

solving, $\boxed{a_1 = 1/2, \quad a_2 = 3/2}$

$\therefore \; y = \frac{1}{2} e^x + \frac{3}{2} e^{-x} - x^2 - 2$

2C-9

a) Write the ODE as $\boxed{Ly = r}$, where L is the linear operator

$$L = D^2 + pD + q$$

• By hypothesis,

$Ly_1 = r_1 \leftarrow$ (i.e., y_1 is a solution to $Ly = r_1$)

$Ly_2 = r_2 \leftarrow$ (similarly)

Adding, $L(y_1 + y_2) = r_1 + r_2$

(using the linearity of L: $L(y_1+y_2) = Ly_1 + Ly_2$)

$\therefore \; y_1 + y_2$ solves $Ly = r_1 + r_2$.

b) First consider $y'' + 2y' + 2y = 2x$

Try $y_1 = c_1 x + c_2 \quad \underline{\cdot 2}$

$y_1' = c_1 \quad \underline{\cdot 2}$

$y_1'' = 0 \quad$ Add

$$2x = 2c_1 x + (2c_2 + 2c_1)$$

$\therefore c_1 = 1, \quad c_2 = -1 \qquad \boxed{y_1 = x - 1}$

Then: $y'' + 2y' + 2y = \cos x$

Try $y_2 = a_1 \cos x + a_2 \sin x \quad \underline{\cdot 2}$

$y_2' = -a_1 \sin x + a_2 \cos x \quad \underline{\cdot 2}$

$y_2'' = -a_1 \cos x - a_2 \sin x \quad$ Add

$$\cos x = \cos x\,(2a_1 + 2a_2 - a_1) + \sin x\,(2a_2 - 2a_1 - a_2)$$

$\therefore \left.\begin{array}{l} a_1 + 2a_2 = 1 \\ -2a_1 + a_2 = 0 \end{array}\right\} \; \therefore \begin{array}{l} a_2 = 2/5 \\ a_1 = 1/5 \end{array} \qquad y_2 = \frac{1}{5}\cos x + \frac{2}{5}\sin x$

2C-10

a) $R = 0, \; E = 0$

Eqn is $Lq'' + \frac{q}{C} = 0$ or $q'' + \frac{q}{LC} = 0$

Solving as usual,

$$\boxed{q = c_1 \cos \frac{1}{\sqrt{LC}} t + c_2 \sin \frac{1}{\sqrt{LC}} t}$$

Period is $2\pi\sqrt{LC}$ ($= 2\pi/\text{frequency}$)

frequency $= 1/\sqrt{LC}$

b) Char. eqn is $Lr^2 + Rr + \frac{1}{C} = 0$

roots: $r = \dfrac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$

oscillates if $\boxed{R^2 - \frac{4L}{C} < 0}$

c) $Li'' + \frac{i}{C} = \omega E_0 \cos \omega t$

Solns of homog. eqn are

$$i = a_1 \cos \frac{1}{\sqrt{LC}} t + a_2 \sin \frac{1}{\sqrt{LC}} t$$

The particular soln i_p will have form $c_1 \cos \omega t + c_2 \sin \omega t$ unless $\omega = \frac{1}{\sqrt{LC}}$, in which case it will be $c_1 t \cos \omega t + c_2 t \sin \omega t$, which gets large as $t \to \infty$.

Thus if $\boxed{\omega \approx \frac{1}{\sqrt{LC}}}$, solns will be large in amplitude

$\therefore$ this is ω_0

—— o ——

> The advantage of this method (divide and conquer!) is that we don't have to assume
> $$y_p = d_1 x + d_2 + d_3 \cos x + d_4 \sin x,$$
> which would give 4 equations in 4 unknowns to solve . . .

Using part (a), the particular solution to

$$y'' + 2y' + 2y = 2x + \cos x$$

is

$$y = y_1 + y_2 = x - 1 + \tfrac{1}{5}\cos x + \tfrac{2}{5}\cos x$$

2D-1

a) $y_h = c_1\cos x + c_2 \sin x$, as usual. ($y_1 = \cos x$, $y_2 = \sin x$)

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $y_p = u_1 y_1 + u_2 y_2$

The equations for variation of pars. are:

$$u_1' \cos x + u_2' \sin x = 0$$
$$u_1'(-\sin x) + u_2' \cos x = \tan x$$

Either by elimination, or by Cramer's rule, we get as sol'n: (the denom. is $W(y_1, y_2)$)

$$u_1' = \frac{-y_2 f(x)}{W(y_1, y_2)} = -\sin x \tan x = \cos x - \sec x$$ (so it can be integrated)

$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \cos x \tan x = \sin x$$

$$\therefore \quad u_1 = \sin x - \ln|\sec x + \tan x|$$ (from tables)

$$u_2 = -\cos x$$

$$\therefore \quad y_p = \big(\sin x - \ln|\sec x + \tan x|\big)\cos x - \cos x \sin x$$

i.e., $$\boxed{y_p = -\cos x\,(\ln|\sec x + \tan x|)}$$

b) Two indept soln's of the assoc. homog. eqn are: $y_1 = e^x$, $y_2 = e^{-3x}$ (method (as usual))

$$W(y_1, y_2) = -4e^{-2x} \quad \left(= \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix}\right)$$

$$y_p = u_1 y_1 + u_2 y_2 \qquad (*)$$

The eqn's for variation of parameters are:

$$u_1' e^x + u_2' e^{-3x} = 0$$
$$u_1' e^x - u_2'(3e^{-3x}) = e^{-x}$$ ← $f(x)$, (from orig. eqn)

Solve them by elimination, or by Cramer's rule; following the latter, we get as sol'n

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{1}{4}e^{-2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x \cdot e^{-x}}{-4e^{-2x}} = -\frac{1}{4}e^{2x}$$

$$\therefore \quad u_1 = -\frac{1}{8}e^{-2x}, \qquad u_2 = -\frac{1}{8}e^{2x}$$

and so $$y_p = -\frac{1}{8}e^{-2x}\cdot e^x - \frac{1}{8}e^{2x}\cdot e^{-3x},$$ by (*);

or: $$\boxed{y_p = -\frac{1}{4}e^{-x}}$$

c) Two indept solns of the assoc. homog. eqn are: $y_1 = \cos 2x$, $y_2 = \sin 2x$ (by the usual method)

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $y_p = u_1 y_1 + u_2 y_2$

Then $$\begin{cases} u_1' \cos 2x + u_2' \sin 2x = 0 \\ u_1'(-2\sin 2x) + u_2'(2\cos 2x) = \sec^2 2x \end{cases}$$

are the eqn's for the method of var. of pars.

Solving them by elimination, or by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-\sin 2x}{2\cos^2 2x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos 2x}{2\cos^2 2x} = \frac{\sec 2x}{2}$$

Integrating,

$$u_1 = -\frac{1}{4}\cdot\frac{1}{\cos 2x}$$

$$u_2 = \frac{1}{4}\ln|\sec 2x + \tan 2x|$$

$$\therefore \quad \boxed{y_p = -\frac{1}{4} + \frac{1}{4}\ln|\sec x + \tan x| \cdot \sin 2x}$$

2D-2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{1}{x},$$ after some calculation.

$$y_p = u_1 y_1 + u_2 y_2$$

Equations for method of var. of pars. are:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = \frac{\cos x}{\sqrt{x}}$$

(note: the ODE must be written $y'' + \frac{1}{x}y' + (-)y = \frac{\cos x}{\sqrt{x}}$) ← $f(x)$

Solving these by Cramer's rule:

$$u_1' = \frac{-y_2 f(x)}{W} = \cos^2 x$$

$$u_2' = \frac{y_1 f(x)}{W} = -\sin x \cos x$$

$$\therefore \quad u_1 = \frac{x}{2} + \frac{\sin 2x}{4}, \qquad u_2 = \frac{\cos 2x}{4}$$

and so (using identities):

$$y_p = \frac{\sin x}{\sqrt{x}}\left(\frac{x}{2} + \frac{2\sin x \cos x}{4}\right) + \frac{\cos x}{\sqrt{x}}\left(\frac{\cos^2 x - \sin^2 x}{4}\right)$$

so $$y_p = \frac{x \sin x}{2\sqrt{x}} + \frac{1}{4}\frac{\cos x}{\sqrt{x}}$$

(The term $\frac{1}{4}\frac{\cos x}{\sqrt{x}}$ is part of the general soln $y = y_p + c_1\frac{\cos x}{\sqrt{x}} + c_2\frac{\sin x}{\sqrt{x}}$; so it can be omitted:

$$\boxed{y_p = \frac{\sqrt{x}\sin x}{2}}$$ is the best answer)

2D-3

a) Let y_1, y_2 be indep't solutions of the associated homogeneous equation.

$$y_p = u_1 y_1 + u_2 y_2, \qquad W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x), & y_2'(x) \end{vmatrix}$$

and the eqn's for the method of var. of pars. are

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$

Solving by Cramer's rule gives

$$u_1' = \frac{-y_2(x)f(x)}{W[y_1(x), y_2(x)]}, \qquad u_2' = \frac{y_1(x)f(x)}{W[y_1(x), y_2(x)]}$$

so that (use definite integrals so as to get a definite function)

$$u_1(x) = \int_a^x \frac{-y_2(t)f(t)\,dt}{W[y_1(t), y_2(t)]}, \qquad u_2(x) = \int_a^x \frac{y_1(t)f(t)\,dt}{W[y_1(t), y_2(t)]}$$

Thus: $\quad y_p(x) = u_1(x)\cdot y_1(x) + u_2(x)y_2(x)$ —

we can put $y_1(x)$ and $y_2(x)$ inside the integral sign because they are "constants" — the integration is with respect to t, not x; then we can add the integrands. The result is:

$$y_p = \int_a^x \frac{-y_1(x)\,y_2(t) \quad + y_2(x)y_1(t)}{W[y_1(t), y_2(t)]}\cdot f(t)\,dt$$

or

$$y_p = \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} f(t)\,dt}{W(y_1(t), y_2(t))}$$

b) The arbitrary constants of integration — call them a_1 and a_2, — will change u_1 and u_2 by an additive constant:

$$u_1 + a_1, \qquad u_2 + a_2$$

leading to the particular soln:

$$y_p = (u_1 + a_1)y_1 + (u_2 + a_2)y_2$$

$$\circledast \quad y_p = \boxed{u_1 y_1 + u_2 y_2} + a_1 y_1 + a_2 y_2$$

The boxed part is the particular solution of part (a); the part added on is in the general soln y_h to the associated homog. eqn, hence the particular soln $\circledast$ is just as good a particular soln as the previous one.

2D-4

It depends on the ODE form — (it must be linear!)

Undetermined coefficients requires

① The ODE is linear, with constant coefficients

② The inhomogeneous term $f(x)$ has a special form: a sum of terms of the form

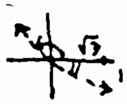
$$(\text{polynomial})\cdot e^{ax}\cdot \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix}.$$

can be 1 $\quad$ a can be 0 $\quad$ b can be 0

If the coeffs. are not constant, or $f(x)$ is not of the above form, you must use variation of parameters to find y_p.

Drawback: you must be able to find y_1, y_2 first — i.e., solve the assoc. homog. eq'n.

(Note that finding y_p by undet. coeffs. does not require you to solve for y_1, y_2 first (unless you are unlucky and $f(x)$ is a soln of the assoc. homog. eqn --- but you can always test this without solving the eqn.)

2E-1 

$-1+i = \sqrt{2}\, e^{i3\pi/4}$

$\sqrt{3}-i = 2e^{-i\pi/6}$

2E-2 $\dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{(1+i)(1-i)} = \dfrac{-2i}{2} = -i$

Other way:

$1-i = \sqrt{2}\, e^{-i\pi/4}$

$1+i = \sqrt{2}\, e^{i\pi/4}$

$\therefore \dfrac{1-i}{1+i} = \dfrac{\sqrt{2}}{\sqrt{2}} \cdot e^{i(-\pi/4-\pi/4)}$

$= e^{-i\pi/2} = -i$

2E-4 $z = a+bi, \quad w = c+di$

$zw = (ac-bd) + i(ad+bc)$

$\therefore \overline{zw} = (ac-bd) - i(ad+bc)$

$\bar{z}\,\bar{w} = (a-bi)(c-di),$ $\quad =$

$= (ac-bd) - i(ad+bc)$

2E-7 a) $(1-i)^4 = 1 + 4(-i) + 6(-i)^2 + 4(-i)^3 + (-i)^4$

$= 1-6+1 + i(-4+4) = \boxed{-4}$

By DeMoivre:

$1-i = \sqrt{2}\, e^{-i\pi/4}$

$(1-i)^4 = (\sqrt{2})^4 e^{-i\pi} = 4\cdot(-1)$

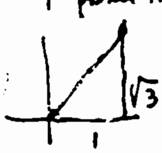
$= -4.$

b) $(1+i\sqrt{3})^3 = 1 + 3(i\sqrt{3}) + 3(i\sqrt{3})^2 + (i\sqrt{3})^3$

$= 1 + 3i\sqrt{3} + 3\cdot -3 + i^3 3\sqrt{3}$

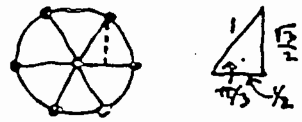
$= -8 + i(3\sqrt{3} - 3\sqrt{3}) = -8$

By polar form:



$1+i\sqrt{3} = 2e^{i\pi/3}$

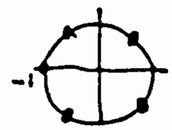
$(1+i\sqrt{3})^3 = 8e^{i\pi} = -8$



2E-9 The sixth roots of 1 are $e^{\frac{i\pi}{3}k}$ where $k = 0, 1, 2, \ldots, 5$

get $\therefore$ $1, -1, \dfrac{\pm 1 \pm i\sqrt{3}}{2}$.

2E-10 $\sqrt[4]{-16} = 2\cdot\sqrt[4]{-1}$.



The 4th roots of -1 are on the picture: $\dfrac{\pm 1 \pm i}{\sqrt{2}}$

$\therefore \sqrt{2}\cdot(\pm 1 \pm i)$ are the roots of $x^4+16=0$.

2E-14 $\sin^4 x = \left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^4$; by bin. thm, this

$= \frac{1}{16}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right)$

$= \frac{1}{16}\left(e^{4ix}+e^{-4ix}\right) - \frac{4}{16}\left(e^{2ix}+e^{-2ix}\right) + \frac{6}{16}\cdot 1$

$= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$.

Since $\sin^4 x$ is an <u>even</u> function, the answer should not contain the <u>odd</u> functions $\sin 4x$, $\sin 2x$.

2E-15 $e^{(2+i)x} = e^{2x}(\cos x + i\sin x)$

So $e^{2x}\sin x = \operatorname{Im} e^{(2+i)x}$.

$\int e^{(2+i)x}\,dx = \dfrac{1}{2+i} e^{(2+i)x}$; $\dfrac{1}{2+i}\cdot\dfrac{2-i}{2-i} = \dfrac{2-i}{5}$;

$= \dfrac{2-i}{5}\left(e^{2x}\cos x + ie^{2x}\sin x\right)$

We want just the imaginary part:

$\therefore \int e^{2x}\sin x\,dx = e^{2x}\left(\frac{2}{5}\sin x - \frac{1}{5}\cos x\right)$

2E-16 $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

since $\cos(-x) = \cos x$ + $\sin(-x) = -\sin x$,

Adding: $\dfrac{e^{ix}+e^{-ix}}{2} = \cos x$

Subtract: $\dfrac{e^{ix}-e^{-ix}}{2i} = \sin x$

2F-1

a) $D^2+2D+2=0$ has roots $-1\pm i$

$\therefore\ y = e^{2x}(c_1+c_2x+c_3x^2) + e^{-x}(c_4\cos x + c_5 \sin x)$

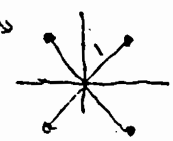
b) $D^8-2D^4+1 = (D^4-1)^2 = [(D^2-1)(D^2+1)]^2$

$= (D-1)^2(D+1)^2(D^2+1)^2$

$\therefore\ y = e^x(c_1+c_2x) + e^{-x}(c_3+c_4x) + \cos x\,(c_5+c_6x) + \sin x\,(c_7+c_8x)$

c) Characteristic eq'n is $\boxed{z^4+1=0}$
Roots are $\sqrt[4]{-1}$

$\frac{1\pm i}{\sqrt{2}}$ and $\frac{-1\pm i}{\sqrt{2}}$



letting $a = 1/\sqrt{2}$, get $\therefore$

$y = e^{ax}(c_1\cos ax + c_2 \sin ax) + e^{-ax}(c_3 \cos ax + c_4 \sin ax)$

d) Char. eq'n is $\boxed{z^4-8z^2+16=0}$
which factors as

$(z^2-4)^2$ or $(z+2)^2(z-2)^2$

$\therefore$ has double roots at $2, -2$

so

$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$

e) $y = c_1e^x + c_2e^{-x} + e^{x/2}(c_3\cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x) + e^{-x/2}(c_5\cos\frac{\sqrt{3}}{2}x + c_6\sin\frac{\sqrt{3}}{2}x)$

[using roots as given in soln to 2E-9]

f) $y = e^{\sqrt{2}x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + e^{-\sqrt{2}x}(c_3\cos\sqrt{2}x + c_4\sin\sqrt{2}x)$

2F-2

$y'''' - 16y = 0$

characteristic equation $\boxed{z^4-16=0}$

roots: $2,\ 2i,\ -2,\ -2i$

(one real root is 2, so the others are all of the form $2\cdot\sqrt[4]{1}$, where $\sqrt[4]{1} = 1, i, -1, -i$)

From roots, general sol'n is

$y = c_1e^{2x} + c_2e^{-2x} + c_3\sin 2x + c_4\cos 2x$

Putting in conditions:

$\boxed{c_1=0}$ since $|y(x)|<K$ for all $x>0$

($\because |c_1e^{2x}| \to \infty$ unless $c_1=0$)

$y(0)=0 \Rightarrow c_2+c_4=0 \quad \therefore c_4=-c_2$

$y'(0)=0 \Rightarrow -2c_2+2c_3=0 \quad \therefore c_3=c_2$

$\therefore$ sol'n is – so far –

$\boxed{y = c_2(e^{-2x}+\sin 2x - \cos 2x)}$

Finally

$y(\pi)=1 \Rightarrow c_2(e^{-2\pi}-1) = 1,$

$\therefore\ \boxed{c_2 = \frac{1}{e^{-2\pi}-1}}$.

2F-3

a) $\boxed{z^3-z^2+2z-2=0}$ is char. eq'n.

1 is a root, $\therefore z-1$ is factor

get $(z-1)(z^2+2)$ roots: $1, i\sqrt{2}, -i\sqrt{2}$

$y = c_1e^x + c_2\cos\sqrt{2}x + c_3\sin\sqrt{2}x$

b) $\boxed{z^3+z^2-2=0} = (z-1)(z^2+2z+2)$

roots $1,\ -1\pm i$

$\therefore\ y = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x$

c) $(D^3-2D-4) = (D-2)(D^2+2D+2)$

$\therefore\ y = c_1e^{2x} + e^{-x}(c_2\cos x + c_3\sin x)$ roots are $-1\pm i$

d) ↓

d): **2F-3** $x^4+2x^2+4=0$; $\therefore\ x^2 = \frac{-2\pm\sqrt{2^2-4\cdot4}}{2} = -1\pm\sqrt{-3} = -1\pm\sqrt{3}\,i$;

changing to polar representation: $= 2e^{2\pi/3\,i},\ 2e^{4\pi/3\,i}$

$\therefore\ x = \sqrt{2}e^{\pi/3\,i},\ \sqrt{2}e^{4\pi/3\,i}$ (square roots of the first ↗)

$= \sqrt{2}e^{2\pi/3\,i},\ \sqrt{2}e^{5\pi/3\,i}$ (" " " " others →)

↖ and ↙ are conjugates

Using therefore just $\sqrt{2}e^{\pi/3\,i}$ and $\sqrt{2}e^{\frac{2\pi i}{3}}$:

$\sqrt{2}e^{\pi/3\,i} = \sqrt{2}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \sqrt{2}(\frac{1}{2}+i\frac{\sqrt{3}}{2})$; similarly, get $\sqrt{2}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})$

leading to: $y = e^{\frac{\sqrt{2}}{2}x}(c_1\cos\frac{\sqrt{6}}{2}x + c_2\sin\frac{\sqrt{6}}{2}x) + e^{-\frac{\sqrt{2}}{2}x}(c_3\cos\frac{\sqrt{6}}{2}x + c_4\sin\frac{\sqrt{6}}{2}x)$

2F-4

$$x_1'' + 2x_1 - x_2 = 0$$
$$x_2'' + x_2 - x_1 = 0$$

Eliminate x_1 by solving for x_1:

$$x_1 = x_2'' + x_2$$

substitute into first equation:

$$(x_2''+x_2)'' + 2(x_2''+x_2) - x_2 = 0$$

or $x_2'''' + 3x_2'' + x_2 = 0$

char. eqn: $\boxed{z^4+3z^2+1=0}$

as quadratic eqn in z^2: solve, get

$z^2 = \frac{-3\pm\sqrt{5}}{2}$: both nos. are real, + negative; call them $-a^2, -b^2$

$\therefore z^2 = -a^2$, $z^2 = -b^2$

$z = \pm ia$, $z = \pm ib$

so $x_2 = c_1\cos at + c_2\sin at + c_3\cos bt + c_4\sin bt$

2F-5

$$D^2 e^{2x}\cos x = e^{2x}(D+2)^2\cos x = e^{2x}(D^2+4D+4)\cos x = e^{2x}(3\cos x - 4\sin x)$$

2F-6

a) By (12) in notes, (see Example 2)

$$y_p = \frac{4}{1^2+1}e^x = 2e^x$$

b) $(D^3+D^2-D+2)y = 2e^{ix}$

$$\therefore y_p = \frac{2e^{ix}}{i^3+i^2-i+2} = \frac{2(1+2i)}{(1-2i)(1+2i)}e^{ix}$$

$\therefore y_p = \frac{2+4i}{5}(\cos x + i\sin x)$ $\therefore Re(y_p) = \boxed{\frac{2}{5}\cos x - \frac{4}{5}\sin x}$

c) $(D^2-2D+4)y = e^{(1+i)x}$

$(1+i)^2 - 2(1+i) + 4 = 2$ $\therefore y_p = \frac{e^{(1+i)x}}{2}$

$Re(y_p) = \boxed{\frac{1}{2}e^x\cos x}$

d) $D^2 - 6D + 9 = (D-3)^2$ $\therefore y_p = cx^2e^{3x}$

$(D-3)^2 y_p = ce^{3x}D^2x^2$ (by exp-shift rule)

$= 2ce^{3x} = e^{3x}$ (from the ODE)

$\therefore c = 1/2$, $y_p = \frac{1}{2}x^2e^{3x}$

2F-7

$(D+a)e^{-ax}u = e^{-ax}Du = f(x)$

$\therefore Du = e^{ax}f(x)$, $u = \int e^{ax}f(x)\,dx$

$y_p = e^{-ax}\int e^{ax}f(x)\,dx$

2G-1

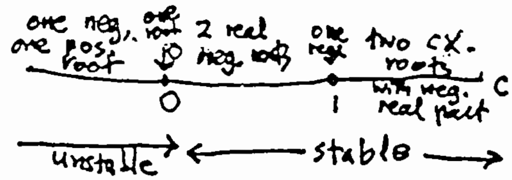
$$y'' + 2y' + cy = 0$$

Char. eqn is:

$$r^2 + 2r + c = 0$$

By quadratic formula:

$$\text{roots} = \frac{-2\pm\sqrt{4-4c}}{2} = -1 \pm \sqrt{1-c}\,.$$



2G-2

$$r^2 + \frac{b}{a}r + \frac{c}{a} = (r-r_1)(r-r_2)$$

$\therefore \frac{b}{a} = -(r_1+r_2)$ (*)

$\frac{c}{a} = r_1r_2$

Real case: $r_1, r_2 < 0 \Rightarrow b/a > 0$, $c/a > 0$

$\therefore a, b, c$ have same sign.

Complex case:

$r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\alpha < 0 \Rightarrow \frac{b}{a} = -2\alpha > 0$

by (*) $\frac{c}{a} = \alpha^2 + \beta^2 > 0$.

2G-3

Assume $a, b, c > 0$ (if not, multiply ODE through by -1).

$r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ are the roots.

If roots are real, $\frac{-b-\sqrt{}}{2a} < 0$

and $-b + \sqrt{b^2-4ac} < 0$, therefore (since $b^2 - 4ac < b^2$).

If roots are complex, $\frac{-b}{2a} < 0$

$\therefore$ in both cases, the char. roots have negative real part.

$\boxed{2H-1}$ $\quad y'' - k^2 y = 0, \quad y(0)=0, \; y'(0)=1$

$y_c = c_1 e^{kx} + c_2 e^{-kx}$

Soln to IVP is

$$w(t) = \frac{e^{kx} - e^{-kx}}{2k} = \frac{\sinh kx}{k}$$

$\boxed{2H-3a}$ $\quad$ By Example 2 (p.2),

$w(x) = x e^{-2x}$

Therefore

$$y(x) = \int_0^x \underbrace{(x-t)e^{-2(x-t)}}_{w(x-t)} \cdot \underbrace{e^{-2t}}_{f(t)} \, dt$$

$$= e^{-2x} \int_0^x (x-t)\, dt$$

$$= e^{-2x}\left(xt - \frac{t^2}{2}\right)\Big|_0^x = \boxed{\frac{x^2}{2} e^{-2x}}$$

By undetermined coeffs, since $y_c = e^{-2x}(c_1 + c_2 x)$, try $c x^2 e^{-2x}$

$$(D+2)^2 c e^{-2x} x^2 = c e^{-2x} D^2 x^2$$

$$= c e^{-2x} \cdot 2$$

From the ODE,

$$c e^{-2x} \cdot 2 = e^{-2x}, \quad \boxed{c = \frac{1}{2}} \checkmark$$

$\boxed{2H-4}$ $\quad$ a) By Leibniz:

$$\phi'(x) = \frac{d}{dx}\int_0^x (2x+3t)^2\, dt =$$

$$= (5x)^2 + \int_0^x 2\cdot(2x+3t)\cdot 2\, dt$$

$$= (5x)^2 + 4\left(2xt + \frac{3t^2}{2}\right)\Big|_0^x = (5x)^2 + 14x^2$$

$$= \boxed{39x^2}$$

b) Directly:

$$\phi(x) = \frac{1}{9}(2x+3t)^3\Big|_0^x = \frac{1}{9}\left((5x)^3 - (2x)^3\right)$$

$$= 13x^3$$

So. $\phi'(x) = 39x^2$. $\checkmark$

Section 3 Solutions

$\boxed{3A\text{-}1}$ $\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$. Integrate by parts:

$$= t\frac{e^{-st}}{-s}\Big]_{t=0}^{\infty} - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

Since $\lim_{t\to\infty} te^{-st} = 0$ $\boxed{\text{if } s>0,}$ the left-hand term is 0 at both endpoints. Integrating the right-hand term:

$$= -\frac{e^{-st}}{(-s)(-s)}\Big]_0^\infty = 0 - \left(\frac{-1}{s^2}\right) = \frac{1}{s^2},$$

$s>0$.

$\boxed{3A\text{-}2}$ $\mathcal{L}\{e^{(a+ib)t}\} = \mathcal{L}\{e^{at}\cos bt\} + i\mathcal{L}\{e^{at}\sin bt\}$ (*)

On the other hand,

$$\mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)};$$ multiplying top + bottom by $(s-a)+ib$:

$$= \frac{(s-a)+ib}{(s-a)^2+b^2} = \frac{s-a}{(\cdots)} + \frac{ib}{(\cdots)} \qquad (**)$$

$$\therefore \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

[by equating real + imag. parts of (*) and (**).]

$\boxed{3A\text{-}3}$ a) $\mathcal{L}^{-1}\left(\frac{1}{\frac{s}{2}+3}\right) = \mathcal{L}^{-1}\left(\frac{2}{s+6}\right) = 2e^{-6t}$

b) $\mathcal{L}^{-1}\left(\frac{3}{s^2+4}\right) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2}\sin 2t$

c) $\mathcal{L}^{-1}$: $\frac{1}{s^2-4} = \frac{1/4}{s-2} - \frac{1/4}{s+2}$ (partial fractions)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

d) $$\frac{1+2s}{s^3} = \frac{1}{s^3} + \frac{2}{s^2}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1+2s}{s^3}\right) = \frac{t^2}{2} + 2t$$

e) $$\frac{1}{s^4-9s^2} = \frac{-1/9}{s^2} + \frac{0}{s} + \frac{1/54}{s-3} + \frac{-1/54}{s+3}$$

$$= \frac{1}{s^2(s-3)(s+3)}$$

(by cover-up method. Find the 0 by putting $s=1$)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^4-9s^2}\right) = -\frac{t}{9} + \frac{1}{54}\left(e^{3t} - e^{-3t}\right)$$

$\boxed{3A\text{-}4}$ $L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,dt$; Integrate by parts:

$$= \sin at \cdot \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty a\cos at \cdot \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{a}{s}\mathcal{L}\{\cos at\}$$

$$= \frac{a}{s}\cdot\frac{s}{s^2+a^2}, \quad s>0$$

$$= \frac{a}{s^2+a^2}, \quad s>0.$$

$\boxed{3A\text{-}5}$ $\mathcal{L}\{\cos^2 at\} = \mathcal{L}\{\tfrac{1}{2} + \tfrac{1}{2}\cos 2at\}$

$$= \mathcal{L}\{\tfrac{1}{2}\} + \tfrac{1}{2}\mathcal{L}\{\cos 2at\}$$

$$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right).$$

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2at\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right)$$

$\mathcal{L}\{\cos^2 at + \sin^2 at\} = \frac{1}{s}$, from the above;

$$\mathcal{L}\{1\} = \frac{1}{s} \;\checkmark.$$

$\boxed{3A\text{-}6}$ a) $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt$, $(s>0)$

Put $x^2 = st$, so $t = \frac{x^2}{s}$

Then the integral becomes (in terms of s, x):

$$= \int_0^\infty e^{-x^2}\frac{\sqrt{s}}{x}\cdot\frac{2x}{s}\,dx$$

$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx = \frac{2}{\sqrt{s}}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi/s}.$$

b) $\mathcal{L}\{\sqrt{t}\} = \int_0^\infty e^{-st}\sqrt{t}\,dt$; integrate by parts:

$$= \sqrt{t}\,\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot\frac{1}{2\sqrt{t}}\,dt$$

$$= 0 + \frac{1}{2s}\int_0^\infty e^{-st}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2s}\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{1}{2s}\cdot\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

$\boxed{3A-7}$ $\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}\cdot e^{t^2}\,dt$
$= \int_0^\infty e^{t^2-st}\,dt$

This integral is infinite for every real value of s, no matter how large, since if $t > s$, $t^2 - st > 0$, and therefore

$$\int_0^\infty e^{t^2-st}\,dt > \int_s^\infty e^{t^2-st}\,dt > \int_s^\infty e^0\,dt, \quad = \infty.$$

$\boxed{3A-8}$ $\mathcal{L}\{\frac{1}{t^k}\} = \int_0^\infty e^{-st}\frac{1}{t^k}\,dt$, $(s>0)$

The trouble here is when $t=0$.
Near $t=0$, $e^{-st} \approx e^0 = 1$.
$\therefore$ the integral is like:

$$\int_0^a e^{-st}\frac{1}{t^k}\,dt \gtrsim \int_0^a \frac{dt}{t^k}$$

and this last integral converges only if $k<1$ $\left[\text{since it} \begin{cases} = \frac{t^{1-k}}{1-k}\Big|_0^a & \text{for } k\neq 1 \\ = \ln x\Big|_0^a & \text{for } k=1 \end{cases}\right]$

[At the upper limit ∞ the original integral always converges, if $s>0$].
$\therefore$ $\mathcal{L}\{\frac{1}{t^k}\}$ exists for $k<1$.

$\boxed{3A-9}$ a) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$
By the exponential-shift formula,
$$\mathcal{L}\{e^{-t}\sin 3t\} = F(s+1) = \frac{3}{(s+1)^2+9}$$

b) $\mathcal{L}\{t^2-3t+2\} = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s} = F(s)$
By exponential-shift rule,
$$\mathcal{L}\{e^{2t}(t^2-3t+2)\} = F(s-2)$$
$$= \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

$\boxed{3A-10}$ $\mathcal{L}^{-1}\{\frac{3}{(s-2)^4}\} = e^{2t}\mathcal{L}^{-1}\{\frac{3}{s^4}\} =$
$= e^{2t}\frac{t^3}{2}$

$\mathcal{L}^{-1}\{\frac{1}{s(s-2)}\} = \mathcal{L}^{-1}\{\frac{1/2}{s-2} - \frac{1/2}{s}\}$,
(by partial fractions)
$= \frac{1}{2}e^{2t} - \frac{1}{2}$.

$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\}$:

Complete the square in the denominator:

$\frac{s+1}{s^2-4s+5} = \frac{s+1}{(s-2)^2+1}$; express top in terms of $s-2$:

$$= \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$

$\therefore \mathcal{L}^{-1}(\cdots) = e^{2t}\cos t + 3e^{2t}\sin t$,
(by the exponential-shift rule).

3B-1 We use throughout the two formulas:

$$\mathcal{L}(y') = -y(0+) + sY \leftarrow (\mathcal{L}(y))$$

and

$$\mathcal{L}(y'') = -y'(0+) - sy(0+) + s^2 Y$$

[The 0+ indicates that if $y(t)$ is discontinuous at 0, we $\lim_{t\to 0+} y(t)$, the righthand limit].

a) $y' - y = e^{3t}, \quad y(0) = 1$

$$(sY - 1) - Y = \frac{1}{s-3}$$

$$(s-1)Y = 1 + \frac{1}{s-3}$$

$$Y = \frac{1}{s-1} + \frac{1}{(s-3)(s-1)}$$

make partial fractions decomp; $= \dfrac{1/2}{s-1} + \dfrac{1/2}{s-3}$

$$\therefore \boxed{y = \tfrac{1}{2}e^t + \tfrac{1}{2}e^{3t}}$$

b) $y'' - 3y' + 2y = 0, \quad y(0) = 1, \ y'(0) = 1$

$$(s^2Y - s - 1) - 3(sY - 1) + 2Y = 0$$

$$\therefore (s^2 - 3s + 2)Y = s - 2$$

$$Y = \frac{1}{s-1}$$

$$\therefore \boxed{y = e^t}$$

c) $y'' + 4y = \sin t, \quad y(0) = 1, \ y'(0) = 0$

$$(s^2Y - s) + 4Y = \frac{1}{s^2+1}$$

$$\therefore Y = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} \quad (*)$$

Apply partial fractions ↑; treat s^2 as a single variable: i.e.,

$$\frac{1}{(u+1)(u+4)} = \frac{1/3}{u+1} - \frac{1/3}{u+4}; \quad \text{now put } u = s^2.$$

$$Y = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + \frac{s}{s^2+4}$$

$$\therefore \boxed{y = \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t + \cos 2t}$$

(*) Note that it's easier not to combine terms at this point

d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0, \ y'(0) = 1$

$$(s^2Y - 1) - 2sY + 2Y = \frac{2}{s-1}$$

$$\therefore (s^2 - 2s + 2)Y = \frac{2}{s-1} + 1 = \frac{s+1}{s-1}$$

$$Y = \frac{s+1}{(s^2-2s+2)(s-1)}$$

By partial fractions:

$$Y = \frac{2}{s-1} + \frac{3-2s}{s^2-2s+2}; \quad \text{complete the square:}$$

$$= \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

(note how we write the 2nd term as an expression in $s-1$; the last term is what's left over.)

$$\therefore y = 2e^t - 2e^t\cos t + e^t \sin t$$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1, \ y'(0) = 0.$

$$s^2Y - s - 2(sY - 1) + Y = \frac{1}{s-1}$$

$$\underset{(s-1)^2}{(s^2 - 2s + 1)}Y = \frac{1}{s-1} + s - 2$$

$$= \frac{1}{s-1} + (s-1) - 1$$

$$\therefore Y = \frac{1}{(s-1)^3} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2}$$

$$\therefore \boxed{y = \tfrac{t^2}{2}e^t + e^t - te^t}$$

3B-2

12. $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

Integ. by parts: $= e^{-st}f(t)\Big]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$

see below: (since $f(t)$ is of exp. order) $\Rightarrow 0 - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$

$$\therefore \mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}.$$

Assumes:

$f'(t)$ piecewise continuous and exponential order (so $\int_0^\infty e^{-st}f'(t)\,dt$ exists) (i.e., $|f'(t)| < Ke^{at}$ if t is large).

$f(t)$ of exponential order, so $\mathcal{L}\{f\}$ exists. (it's continuous, since $f'(t)$ exists).

3B-3

These use the formula:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \underbrace{F(s)}_{=\mathcal{L}\{f(t)\}}$$

a)

$$\mathcal{L}\{t \cos bt\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right) = \frac{b^2 - s^2}{(b^2+s^2)^2}$$

b) $\mathcal{L}\{t^n e^{kt}\}$: by the exp-shift rule,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\therefore \mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}.$$

By the above formula,

$$\mathcal{L}\{t^n e^{kt}\} = (-1)^n \frac{d^n}{ds^n}(s-k)^{-1}$$

$$= (-1)^n \cdot (-1)(-2)\cdots(-n)(s-k)^{-(n+1)}$$

$$= \frac{n!}{(s-k)^{n+1}}, \text{ as before.}$$

c) $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2} \quad \text{by the above fmla.}$$

$$\therefore \mathcal{L}\{t e^{at} \sin t\} = \frac{2(s-a)}{((s-a)^2+1)^2}.$$

3B-4

a) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2}$, as in 13 (c) above

b) $\frac{1}{(s^2+1)^2}$ suggests some combination of $\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$ and $\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$$\mathcal{L}\{t \cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1} - \underbrace{\frac{2}{(s^2+1)^2}}_{\text{what we want}}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \longrightarrow$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}[\sin t - t\cos t]$$

3B-5

a) $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a),$$

since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

b) $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Differentiating under the integral sign, with respect to s : ⊛

$$F'(s) = \int_0^\infty -t e^{-st} f(t)\,dt,$$

since t is a constant with respect to the differentiation;

$$= \mathcal{L}\{-t f(t)\}$$

$$= -\mathcal{L}\{t f(t)\}.$$

[⊛ this is legal if $f(t)$ is continuous and of exponential order].

3B-6

$$y'' + ty = 0, \quad y(0) = 1, \ y'(0) = 0$$

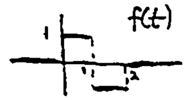
Take the Laplace transform:

$$(s^2 Y - s) - \frac{d}{ds} Y = 0$$

$$\frac{dY}{ds} - s^2 Y = -s,$$

(which is first order, linear).

3C-1

a) 

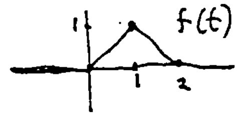
Using $u(t)$: $f(t) = u(t) - 2u(t-1) + u(t-2)$

$\therefore F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}(1-e^{-s})^2$

Directly:

$F(s) = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \frac{1}{s}(1-e^{-s})^2$

(by straight calc.)

b) 

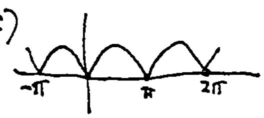
Using $u(t)$: $f(t) = t\,u(t) - u(t-1)\cdot 2(t-1)$
$+ u(t-2)(t-2)$

$\therefore F(s) = \frac{1}{s^2}(1-2e^{-s} + e^{-2s})$ (*)

Directly:

$F(s) = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$ [integrate each $\int$ by parts]

$= \frac{te^{-st}}{-s}\Big]_0^1 - \left[\frac{e^{-st}}{(-s)^2}\right]_0^1 + (2-t)\frac{e^{-st}}{-s}\Big]_1^2 - \frac{-e^{-st}}{(-s)^2}\Big]_1^2$ (which agrees with * after canceling terms)

c) 

$|\sin t| = (-1)^n \sin t,$
$n\pi \le t \le (n+1)\pi.$

This can be done directly,
(adding up the integrals over even + odd intervals):

$F(s) = \int_0^\infty |\sin t| e^{-st}dt = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (-1)^n \sin t \cdot e^{-st}dt$

Change variable: $u = t - n\pi$

$= \sum_{n=0}^\infty \int_0^\pi (-1)^n \sin(u+n\pi) e^{-s(u+n\pi)}du$

$\sin(u+n\pi) = (-1)^n \sin u$; $e^{-sn\pi}$ is a "constant"

$= \sum_{n=0}^\infty e^{-sn\pi} \underbrace{\int_0^\pi \sin u \cdot e^{-su}du}_{\text{call it "K". Then } K = \frac{1+e^{-s\pi}}{1+s^2}}$ (from tables)

$= K \cdot \sum_0^\infty e^{-sn\pi}$; adding up this geometric series gives

$= K \cdot \frac{1}{1-e^{-s\pi}}$ Ans: $\frac{1+e^{-s\pi}}{(1+s^2)(1-e^{-s\pi})}$

3C-2

a) $\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$ (Partial fractions)

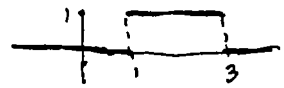
$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-t} - e^{-2t} = f(t)$

$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+3s+2}\right\} = u(t-1)f(t-1)$

$= u(t-1)\left(e^{1-t} - e^{2(1-t)}\right)$

b) $\mathcal{L}^{-1}\left(\frac{e^{-s}-e^{-3s}}{s}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right)$

$= u(t-1) - u(t-3)$



3C-3

a) $L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

$= \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt + \int_4^5 e^{-st}dt + \dots$

$= \frac{e^0 - e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} + \frac{e^{-4s}-e^{-5s}}{s} + \dots$

$= \frac{1}{s}\cdot(e^0 - e^{-s} + e^{-2s} - e^{-3s} + \dots)$

geometric series, whose sum is

$= \frac{1}{s}\cdot\left(\frac{1}{1+e^{-s}}\right)$

b) $f(t) = u(t) - u(t-1) + u(t-2) - \dots$

$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \dots$

$= \frac{1}{s}(e^0 - e^{-s} + e^{-2s} - e^{-3s} \dots)$

$= \frac{1}{s}\cdot\frac{1}{1+e^{-s}}$, as before.

3C-4

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-\pi) - u(t-2\pi)\} = \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

The ODE is: $y'' + 2y' + 2y = h(t)$, $y(0)=0$, $y'(0)=1$;

Laplace Transform is:

$$(s^2Y - 1) + 2(sY) + 2Y = \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

$$(s^2 + 2s + 2)Y = 1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

$$Y = \frac{1}{(s+1)^2 + 1}\left[1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}\right]$$

By partial fractions

$$\frac{1}{(s^2+2s+2)s} = \frac{-s/2 - 1}{s^2+2s+2} + \frac{1/2}{s} = \frac{-\frac{1}{2}(s+1) - \frac{1}{2}}{(s+1)^2 + 1} + \frac{1/2}{s}$$

$$\therefore\ y = e^{-t}\sin t + \frac{1}{2}\Big[1 - e^{-(t-\pi)}\big(\underbrace{\sin(t-\pi)}_{=-\sin t} + \underbrace{\cos(t-\pi)}_{=-\cos t}\big)\Big]u(t-\pi) - \frac{1}{2}\Big[1 - e^{-(t-2\pi)}\big(\underbrace{\sin(t-2\pi)}_{=\sin t} + \underbrace{\cos(t-2\pi)}_{=\cos t}\big)\Big]u(t-2\pi)$$

$$\therefore\ y = \begin{cases} e^{-t}\sin t, & (0 \le t \le \pi) \\ \frac{1}{2} + \left(1 + \frac{e^{\pi}}{2}\right)e^{-t}\sin t + \frac{e^{\pi}}{2}e^{-t}\cos t, & (\pi \le t \le 2\pi) \\ \left(1 + \frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2}\right)e^{-t}\sin t + \left(\frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2}\right)e^{-t}\cos t, & (2\pi \le t) \end{cases}$$

3C-5

$$\mathcal{L}\{\overset{=r(t)}{u(t)\cdot t}\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$y'' - 3y' + 2y = r(t)$, $y(0)=1$, $y'(0)=0$ gives:

$$(s^2Y - s) - 3(sY - 1) + 2Y = \frac{1}{s^2}$$

$$(s^2 - 3s + 2)Y = s - 3 + \frac{1}{s^2}$$

$$Y = \frac{s-3}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)} = \frac{s^3 - 3s^2 + 1}{s^2(s-2)(s-1)}$$

cont'd above ↑

3C-5 (cont'd) By partial fractions

$$Y = \frac{1}{s-1} - \frac{3/4}{s-2} + \frac{3/4}{s} + \frac{1/2}{s^2}$$

$$\therefore\ y = e^{t} - \frac{3}{4}e^{2t} + 3/4 + t/2$$

3D-1

$y'' + 2y' + y = \delta(t) + u(t-1)$, $y(0)=0$, $y'(0)=1$

$$(s^2Y - 1) + 2sY + Y = 1 + \frac{e^{-s}}{s}$$

$$(s^2 + 2s + 1)Y = 2 + \frac{e^{-s}}{s};$$ Divide, use part. fractions:

$$Y = \frac{2}{(s+1)^2} + e^{-s}\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$y = 2te^{-t} + u(t-1)\left[1 - e^{-(t-1)} - (t-1)e^{-(t-1)}\right] = 2te^{-t} + u(t-1)\left[1 - te^{1-t}\right]$$

$$\therefore\ y(t) = \begin{cases} 2te^{-t}, & 0 \le t \le 1 \\ 1 + (2-e)te^{-t}, & t \ge 1 \end{cases}$$

3D-2

$y'' + y = r(t)$, $y(0)=0$, $y'(0)=1$

$$r(t) = \begin{cases} 1, & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases} = 1 - u(t-\pi)$$

$$\therefore\ \mathcal{L}\{r(t)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

So

$$(s^2Y - 1) + Y = \frac{1 - e^{-\pi s}}{s}$$

$$Y = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi s}}{s(s^2+1)}$$

$$\boxed{\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}}$$

$$y = \sin t + 1 - \cos t - \big(1 - \underbrace{\cos(t-\pi)}_{[=-\cos t]}\big)u(t-\pi)$$

$$\therefore\ y = \begin{cases} 1 + \sin t - \cos t, & 0 \le t \le \pi \\ \sin t - 2\cos t, & t \ge \pi \end{cases}$$

$\boxed{\text{3D-3}}$ Also: $\boxed{\text{SEE BELOW}}$

a) $$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \sum_{n=0}^\infty \int_{nc}^{(n+1)c} e^{-st} f(t)\,dt$$

[breaking $[0,\infty)$ up into the intervals $[nc,(n+1)c]$.

Change variable: $u = t - nc$

$$\int_{nc}^{(n+1)c} e^{-st} f(t)\,dt = \int_0^c e^{-s(u+nc)} f(u)\,du,$$

since $f(u+nc) = f(u)$.

Therefore our sum becomes:

$$F(s) = \sum_{n=0}^\infty e^{-snc} \underbrace{\int_0^c e^{-su} f(u)\,du}_{\text{call this } K}$$

$$= K \sum_{n=0}^\infty (e^{-sc})^n \leftarrow \text{a geometric series, whose sum is}$$

$$= K \cdot \frac{1}{1-e^{-sc}}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-sc}} \cdot \int_0^c e^{-su} f(u)\,du$$

(FOR A BETTER WAY, SEE NEXT PAGE)

b) For problem 19, $c = 2$

$$\int_0^2 e^{-su} f(u)\,du = \int_0^1 e^{-su}\,du$$
$$= \frac{1-e^{-s}}{s}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s}$$
$$= \frac{1}{s\cdot(1+e^{-s})}, \text{ as before.}$$

$\boxed{\text{3D-4}}$

a) $$\frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$$

$$\therefore\ \mathcal{L}^{-1}\left(\frac{\cdots}{\cdots}\right) = e^{-t} * \cos 2t$$
$$= \int_0^t e^{-(t-u)} \cos 2u\,du$$
$$= e^{-t} \int_0^t e^u \cos 2u\,du$$
$$= e^{-t}\left[\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}\right]$$
$$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}$$

b) $$\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \sin t * \sin t$$
$$= \int_0^t \sin(t-u)\cdot \sin u\,du$$

Easiest is to use a trig identity:

$$= \int_0^t \frac{1}{2}\left[\cos(t-2u) - \cos t\right] du$$
$$= \frac{\sin t}{2} - \frac{t}{2}\cos t.$$

$\boxed{\text{3D-5}}$

a) $f(t) \underset{\mathcal{L}}{\rightsquigarrow} F(s)$, $\delta(t) \underset{\mathcal{L}}{\rightsquigarrow} 1$

$$\mathcal{L}\{\delta * f\} = 1 \cdot F(s) = F(s)$$
$$\therefore\ \delta * f(t) = f(t)u(t) = f(t),$$

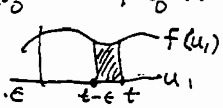
[THIS IS JUST FORMAL] since $f(t) = 0$, $t \le 0$.

b) Using the definition of $*$:

$$\delta * f = \int_0^t \delta(t-u) f(u)\,du$$
$$= \int_{-\infty}^\infty \delta(t-u) f(u)\,du$$
(SHADY) $$= f(t) \int_{-\infty}^\infty \delta(t-u)\,du$$
since $\delta(t-u) = 0$ except if $u = t$
$$= f(t)\cdot 1$$

c) using the "definition" of $\delta(t)$

$$\delta * f(t) = \int_0^t \delta(t-u) f(u)\,du = \int_0^t \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[u(t-u_1) - u(t-u_1-\epsilon)\right] f(t)\,dt$$
$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^t [u(t-u_1) - u(t-u_1-\epsilon)] f(t)\,dt = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_0^t f(u_1)\,du_1 - \int_0^{t-\epsilon} f(u_1)\,du_1\right]$$
(SHADY!)
$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{t-\epsilon}^t f(u_1)\,du = f(t),$$
since area $\approx f(t)\cdot\epsilon$



3D-6

$$(f * g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

let $x = t-u$ (change variable u to the var. x in the integral)
$dx = -du$

limits:
when $u=0$, $x=t$
when $u=t$, $x=0$
$\therefore$ integral becomes:

$$= -\int_t^0 f(x)\,g(t-x)\,dx = \int_0^t g(t-x)\,f(x)\,dx$$

$$= (g*f)(t).$$

3D-7

Taking the Laplace transform:

$$s^2Y + k^2Y = R(s),$$

where $R(s) = \mathcal{L}\{r(t)\}$.

$$\therefore\ Y = \frac{R(s)}{s^2+k^2} = \frac{1}{s^2+k^2}\cdot R(s)$$

$$\therefore\ y = \frac{1}{k}\sin kt * r(t)$$

$$= \frac{1}{k}\int_0^t \sin k(t-u)\cdot r(u)\,du.$$

3D-8

$$y'' + ay' + by = r(t), \quad y(0)=0,\ y'(0)=0$$

$$s^2Y + asY + bY = R(s)$$

$$\therefore\ Y = \frac{1}{s^2+as+b}\cdot R(s)$$

$$\therefore\ y = g(t) * r(t), \quad \text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\}$$

$$y = \int_0^t g(t-u)\,r(u)\,du.$$

To interpret $g(t)$, consider the ODE-IVP

$$y'' + ay' + by = 0, \quad y(0)=0,\ y'(0)=1.$$

then $s^2Y - 1 + asY + bY = 0$

so $Y = \dfrac{1}{s^2+as+b}$

and $y = g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+as+b}\right\}$. Thus $g(t)$ may be interpreted as the soln to this IVP.

(see above for another interp. of g)

3D-8

(continued)

$g(t)$ may also be interpreted as the solution to

$$y'' + ay' + by = \delta(t), \quad y(0)=0,\ y'(0)=0$$

since this leads to

$$s^2Y + asY + bY = 1$$

or $Y = \dfrac{1}{s^2+as+b}$,

so that $y = g(t)$.

3D-3

a.

we have:

$$u(t-c)f(t-c) + f_0(t) = u(t)f(t),$$

where $f_0(t) = \begin{cases} f(t), & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$.

$\therefore$ taking LT's:

$$e^{-cs}F(s) + \int_0^c e^{-st}f(t)\,dt = F(s).$$

Solve for $F(s)$:

$$F(s) = \frac{1}{1-e^{-cs}}\int_0^c e^{-st}f(t)\,dt.$$

$\boxed{4A-1}$ Product is $\begin{bmatrix}1&0&0\\3&-2&-6\end{bmatrix}$

$\boxed{4A-2}$ $AB=\begin{bmatrix}4&1\\-2&-4\end{bmatrix}$ $BA=\begin{bmatrix}-3&1\\5&3\end{bmatrix}$

$\boxed{4A-3}$ $A^{-1}=\frac{1}{-2}\cdot\begin{bmatrix}2&-2\\-3&2\end{bmatrix}$ by the formula

(since $|A|=-2$) $=\begin{bmatrix}-1&1\\3/2&-1\end{bmatrix}$

check: $\begin{bmatrix}2&2\\3&2\end{bmatrix}\begin{bmatrix}-1&1\\3/2&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\boxed{4A-4}$ $\frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\cdot\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$=\frac{1}{|A|}\cdot\begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

(similarly, $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\begin{bmatrix}|A|&0\\0&|A|\end{bmatrix}$)

$\boxed{4A-5}$ $\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}=A^2$

$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1&2\\1&2\end{bmatrix}=\begin{bmatrix}1&2\\2&4\end{bmatrix}$

$\boxed{4A-6}$ Using determinantal criterion for lin. dependence,

we want $0=\begin{vmatrix}1&2&c\\-1&0&1\\2&3&0\end{vmatrix}=4-3c-3$

$\therefore -3c+1=0$

$c=1/3$

Adding: $(1\ \ 2\ \ \overset{1/3}{\overset{\shortparallel}{c}})\times 3$

$-(-1\ \ 0\ \ 1)$

$-(2\ \ 3\ \ 0)\times 2$

$(0\ \ 0\ \ 0)$

$\boxed{4B-1}$ a) $x''+5x'+tx^2=0 \rightarrow \begin{array}{l}x'=y\\ y'=-tx^2-5y\end{array}$

b) $y''-x^2y'+(1-x^2)y=\sin x$

$\rightarrow \begin{array}{l}y'=z\\ z'=(x^2-1)y+x^2z+\sin x\end{array}$

$\boxed{4B-2}$

$y'''+py''+qy'+ry=0$

let $y=y_1$

$y_1'=y_2$

$y_2'=y_3$

$y_3'=-py_3-qy_2-ry_1$

matrix form: $\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}'=\begin{bmatrix}0&1&0\\0&0&1\\-r&-q&-p\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

$\boxed{4B-3}$

$\begin{cases}x'=x+y\\ y'=4x+y\end{cases}$ To eliminate y: $y=x'-x$ from 1st eq'n.

$\therefore (x'-x)'=4x+(x'-x)$ 2nd eq.

or $x''-x'=4x+x'-x$

or $\boxed{x''-2x'-3x=0}$

converting to system:

let $x_1=x$

system is $\begin{cases}x_1'=x_2\\ x_2'=2x_2+3x_1\end{cases}$ → This system is not same as first, but is equivalent to it — just using different dep't variable.

The rel'n between the variables is:

$\begin{array}{l}x_1=x\\ x_2=x+y\end{array}$ or the other way: $\left[\begin{array}{l}x=x_1\\ y=x_2-x_1\end{array}\right.$

If you make this change of vars. the 1st system turns into the second.

$\boxed{4B-4}$

$\begin{bmatrix}1\\1\end{bmatrix}e^{3t}$ and $\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ solve $\vec{x}'=\begin{bmatrix}4&-1\\2&1\end{bmatrix}\vec{x}$:

a) vectorially: $\frac{d}{dt}\begin{bmatrix}1\\1\end{bmatrix}e^{3t}=3\begin{bmatrix}1\\1\end{bmatrix}e^{3t}$ these are equal. Other goes same way.

$\begin{bmatrix}4&-1\\2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}e^{3t}=\begin{bmatrix}3\\3\end{bmatrix}e^{3t}$

components: $\begin{array}{l}x=e^{3t}\\ y=e^{3t}\end{array}$ solves $\begin{cases}x'=4x-y\\ y'=2x+y\end{cases}$: just plug in & check it.

b) linearly indep't: $\begin{vmatrix}e^{3t}&e^{2t}\\e^{3t}&2e^{2t}\end{vmatrix}=e^{5t}\neq 0$.

c) gen soln: $c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t}+c_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ or $\begin{bmatrix}c_1e^{3t}+c_2e^{2t}\\c_1e^{3t}+2c_2e^{2t}\end{bmatrix}$

which is same as: $x=c_1e^{3t}+c_2e^{2t}$

$y=c_1e^{3t}+2c_2e^{2t}$.

4B-5 $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ and $\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}$ solve $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&3\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

(do it same as N3 1a above). Linear indep: $\begin{vmatrix}e^{4t} & e^{-2t}\\ e^{4t} & -e^{-2t}\end{vmatrix} = -2e^{2t}$.

IVP: $\vec{x}(0) = \begin{bmatrix}5\\1\end{bmatrix}$, gives: (since $e^{4t}, e^{-2t} = 1$ when $t=0$)

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} \quad \therefore \begin{matrix}c_1+c_2=5\\c_1-c_2=1\end{matrix} \quad \therefore \begin{matrix}c_1=3\\c_2=2\end{matrix}$$

soln: $3\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + 2\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t} = \vec{x}$. ∴

4B-6 $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$. or $\begin{matrix}x' = x+y\\ y' = y\end{matrix}$

a) From second eqn, $y = c_1e^t$

$\therefore x' - x = c_1e^t$ soln: $x = c_2e^t + c_1te^t$, $y = c_1e^t$

b) Here we eliminate y instead:

$y = x' - x$ (1st eqn.) $\therefore (x'-x)' = x' - x$

$x'' - 2x' + x = 0$ $\therefore x = c_1e^t + c_2te^t$

$(m-1)^2 = 0$ $\therefore y = c_2e^t$ (since $y = x' - x$)

Same as before (just switch c_1 & c_2).

4B-7 $x' = -ax$ (straight decay)

$y' = \underbrace{-by}_{\text{decay rate}} + \underbrace{ax}_{\text{rate at which decay of } x \text{ produces } y}$ matrix: $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}-a&0\\a&-b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Soln by elimination: eliminate x: $x = \frac{1}{a}y' + \frac{b}{a}y$

subst. into 1st eqn, get

$$\frac{1}{a}y'' + \frac{b}{a}y' = -y' - by$$

$$y'' + (b+a)y' + ba\,y = 0 \qquad m^2 + (a+b)m + ab = 0$$

$$m = -a,\ m = -b$$

$$\therefore \begin{cases} y = c_1e^{-at} + c_2e^{-bt}\\ x = c_1\left(-1+\frac{b}{a}\right)e^{-at}\end{cases} \leftarrow \left[\ x = \frac{1}{a}(y' + by) = \frac{1}{a}\begin{pmatrix}-ac_1e^{-at}\\ -bc_2e^{-bt}\\ +bc_1e^{-at}\\ +bc_2e^{-bt}\end{pmatrix}\right.$$

[NOTE: having found y, you can't just say $x' = -ax$, $\therefore x = c_3e^{-at}$, since c_3 is not arbitrary — x must also satisfy the 2nd eqn !!

4C-1 a)

a) $\vec{x}' = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \vec{x}$ eigenvalues: $\begin{vmatrix} -3-m & 4 \\ -2 & 3-m \end{vmatrix} = 0$

$\therefore -(3+m)(3-m) + 8 = 0$

$m^2 - 1 = 0 \qquad m = \pm 1$

Eigenvectors:

if $m = 1$,

$\begin{cases} -4\alpha_1 + 4\alpha_2 = 0 \\ -2\alpha_1 + 2\alpha_2 = 0 \end{cases}$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its mult. are soln: eigenvect.

if $m = -1$:

$-2\alpha_1 + 4\alpha_2 = 0$
$-2\alpha_1 + 4\alpha_2 = 0$

soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigenvector.

$\therefore \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

[NOTE: can also write down char. polyn. using: $m^2 - (a_1 + b_2)m + \det A = 0$.

b) $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$ $\qquad \begin{vmatrix} 4-m & -3 \\ 8 & -6-m \end{vmatrix} = 0$ gives

$m^2 + 2m = 0 \qquad m = -2, \; m = 0$.

$m = 0$:

$4\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 6\alpha_2 = 0$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ eigenvect.

$m = -2$

$6\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 4\alpha_2 = 0$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ eigenv.

$\therefore \vec{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$

4C-1

c) eigenvalues: $\begin{vmatrix} 1-m & -1 & 0 \\ 1 & 2-m & 1 \\ -2 & 1 & -1-m \end{vmatrix} = -(1-m)(2-m)(1+m) + 2 + (m-1) - 1 - m = 0$

$\boxed{(1-m)(2-m)(1+m) = 0}$

eigenvalues $\therefore$ are $m = 1, \; m = 2, \; m = -1$

$m = 1$

$0\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 2\alpha_3 = 0$
$\therefore \alpha_2 = 0 \quad \alpha_3 = -\alpha_1$

soln: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ eigenvector

$m = 2$

$-\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$
$\therefore \alpha_2 = -\alpha_1$
$\alpha_3 = -\alpha_1$

$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ eigenvector

$m = -1$

$+2\alpha_1 - \alpha_2 = 0$
$\alpha_1 + 3\alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 = 0$
$\alpha_2 = +2\alpha_1$
$\alpha_3 = -7\alpha_1$

$\begin{bmatrix} 1 \\ +2 \\ -7 \end{bmatrix}$ eigenvector.

soln:

$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} e^{-t}$

4C-2

Proof #1: $\therefore$ 0 is an eigenvalue if and only if $A\vec{\alpha} = 0\vec{\alpha}$ has a nontriv. soln for $\vec{\alpha}$

$\Leftrightarrow A\vec{\alpha} = \vec{0}$ " " " " "

$\Leftrightarrow \det A = 0$ (see notes p. 2 (5)).

Proof #2: The characteristic equation is $\det(A - mI) = 0$.

If $m = 0$ is a root, this says (substituting $m = 0$) $\det(A) = 0$

4C-3

$\begin{vmatrix} a-m & * & * \\ 0 & b-m & * \\ 0 & 0 & c-m \end{vmatrix} = (a-m)(b-m)(c-m) = 0$

$\therefore m = a, b, c$ are eigenvalues

This always holds: using a Laplace expansion by the minors of first column:

$\begin{vmatrix} a_1-m & * & \cdots & * \\ 0 & a_2-m & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & & a_k-m \end{vmatrix} = (a_1-m) \begin{vmatrix} a_2-m & * & \cdots & * \\ 0 & \ddots & & * \\ 0 & \cdots & 0 & a_k-m \end{vmatrix}$

$= (a_1-m)(a_2-m) \cdots (a_k-m)$

by mathematical induction on the size of matrix (i.e., k)

$\therefore$ eigenvalues are the roots: $m = a_1, a_2, \ldots, a_k$ = diagonal elements.

4C-4

By hypothesis, $A\vec{\alpha} = m\vec{\alpha}$.

Multiply both sides by A:

$A A\vec{\alpha} = m A\vec{\alpha} = m(m\vec{\alpha})$.

$\therefore A^2\vec{\alpha} = m^2\vec{\alpha}$ so $\vec{\alpha}$ is eigenv. of A^2, assoc. to eigenvalue m^2.

[Continuing, one sees that $A^k\vec{\alpha} = m^k\vec{\alpha}$ — the eigenvalues of A^k are the k^{th} powers of the eigenvalues of A].

4C-5

$\vec{x}' = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix} \vec{x}$ Eigenvalues: $-a, -b$ (by previous problem, or directly)

$m = -a$

$a\alpha_1 + (b+a)\alpha_2 = 0$

soln: $\begin{bmatrix} \frac{a-b}{a} \\ 1 \end{bmatrix}$ eigenvector

$m = -b$

$(-a-b)\alpha_1 = 0$
$a\alpha_1 + 0\alpha_2 = 0$

$\therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ soln. eigenvector.

$\therefore \vec{x} = c_1 \begin{bmatrix} \frac{a-b}{a} \\ 1 \end{bmatrix} e^{-at} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-bt}$

when written out with components, this is identical to our earlier solution...

4C-6

$S' = S - aS + bJ$

natural growth rate; fraction that leave for Jones; rabbits coming in from Jones.

$\therefore S' = (1-a)S + bJ$

$J' = aS + (1-b)J.$

$J' = J - bJ + aS$

if $a = b = \frac{1}{2}$,

$$\begin{bmatrix} S \\ J \end{bmatrix}' = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}\begin{bmatrix} S \\ J \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} \frac12 - m & \frac12 \\ \frac12 & \frac12 - m \end{vmatrix} = m^2 - m = 0$

$m = 0,\ m = 1$ eigenvalues:

$\underline{m=0}$: $\frac12\alpha_1 + \frac12\alpha_2 = 0$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{m=1}$: $-\frac12\alpha_1 + \frac12\alpha_2 = 0$, $\frac12\alpha_1 - \frac12\alpha_2 = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$ IVP: $c_1 + c_2 = 20$, $-c_1 + c_2 = 10$ since $t=0$; $c_2 = 15$, $c_1 = 5$

Sol'n: $\therefore \begin{bmatrix} S \\ J \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 15\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$

Nonoscillation: Show this char. polynomial never has complex roots: (Carry this part should come later!!) $m^2 + (a+b-2)m + (1-a-b) = 0$

4C-7

From the "picture":

$\frac14(x_1' - x_1) = x_2$

$x_2' - x_2 = x_1$

$\therefore \begin{cases} x_1' = x_1 + 4x_2 \\ x_2' = x_1 + x_2 \end{cases}$

solving:

eigenvalues: $\begin{vmatrix} 1-m & 4 \\ 1 & 1-m \end{vmatrix} = (1-m)^2 - 4 = 0$ $\therefore 1 - m = \pm 2$ $\therefore m = 3, -1$

$\underline{m=3}$: $-2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\underline{m=-1}$: $2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$

Initial condition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $2c_1 - 2c_2 = 1$, $c_1 + c_2 = 0$

Soln: $\vec{x} = \frac14\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} - \frac14\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$ $\therefore c_1 = \frac14$, $c_2 = -\frac14$

4D-1

26. Characteristic equation: $m^2 + 4$; $m = 2i$ eigenvalue

Corresponding eigenvector:

$\begin{cases} (1-2i)\alpha_1 - 5\alpha_2 = 0 \\ \alpha_1 + (-1-2i)\alpha_2 = 0 \end{cases}$ these are multiples of each other

Possible choices for eigenvector: $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$ or $\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$

The second choice gives as the soln $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}i\right)(\cos 2t + i\sin 2t)$

with real part $\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t$, imag part: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)$

$\therefore x = (c_1 + 2c_2)\cos 2t + (c_2 - 2c_1)\sin 2t$

$y = c_1\cos 2t + c_2\sin 2t$

The other choice of eigenvector leads to $x = 5a_1\cos 2t + 5a_2\sin 2t$, $y = (a_1 - 2a_2)\cos 2t + (2a_1 + a_2)\sin 2t$

(an equivalent solution).

4D-2

Characteristic equation is $m^2 - 6m + 25 = 0$

$\therefore m = 3 \pm 4i$, by quadratic formula

Using $3+4i$ as complex eigenvalue, corresponding eigenvector comes from equation $(3-m)\alpha_1 + 4\alpha_2 = 0$ $\therefore \vec{\alpha} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Corresponding solution is formed from real + imag. parts of $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}i\right)e^{3t}(\cos 4t + i\sin 4t)$, giving

$x = e^{3t}(c_1\cos 4t + c_2\sin 4t)$

$y = e^{3t}(c_1\sin 4t - c_2\cos 4t)$

4D-3

Char. equation is $(m-2)^2(m+1) = 0$

eigenvalue -1 gives eq'ns $3\alpha_1 + 3\alpha_2 + 3\alpha_3 = 0$, $-3\alpha_3 = 0$, $3\alpha_3 = 0$ $\vec{\alpha} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

rpt'd eigenvalue 2 gives eq'ns $\begin{cases} 3\alpha_2 + 3\alpha_3 = 0 \\ -3\alpha_2 - 3\alpha_3 = 0 \\ 0 = 0 \end{cases}$ which have $\underline{2}$ lin. indep't sol'ns, namely $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ — thus 2 is a complete eigenvalue

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{2t}$

or $x = c_1e^{-t} + c_2e^{+2t}$

$y = -c_1e^{-t} + c_3e^{+2t}$

$z = -c_3e^{+2t}$

4D-4

a) $A_1' = (A_2 - A_1) + (A_3 - A_1)$ $A_2 - A_1 = x_2 - x_1$, $A_1' = x_1'$

rate of change of salt in cell 1; rate of diffusion from $2 \to 1$; rate of diffusion from $3 \to 1$

$\therefore x_1' = x_2 - x_1 + x_3 - x_1 = -2x_1 + x_2 + x_3$

Similarly, $x_2' = x_1 - 2x_2 + x_3$

$x_3' = x_1 + x_2 - 2x_3$

b) Characteristic eq'n is $m^3 + 6m^2 + 9m = 0$

$= m(m+3)^2$

Eigenvalue 0 gives eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, normal mode is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

($e^{0t} = 1$, notice)

Eigenvalue -3 gives for eigenvector equations just $\alpha_1 + \alpha_2 + \alpha_3 = 0$ (all 3 eq'ns are same)

This is a complete eigenvalue: it has multiplicity 2 and 2 lin indep sol'ns: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

normal modes: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: all 3 cells have same amt of salt — stays that way

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$ — one cell is at starting concentration A_0 + stays that way; other two cells are equally above + below A_0 at start; salt flows from one to other until "at ∞" they all have A_0 salt in them.

$\boxed{4E-1}$ $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ Solving to get eigenvectors:

$\lambda^2 - 3\lambda - 10 = 0$ $\lambda = 5$ gives $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(\lambda - 5)(\lambda + 2) = 0$ $\lambda = -2$ gives $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

[eq'ns are: $-a_1 + 2a_2 = 0$ and $6a_1 + 2a_2 = 0$, respectively]

$\therefore$ coörd. change is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

[can multiply each column by a constant and it's still OK]

Check it decouples: $x = 2u + v$
$y = u - 3v$

$\therefore$ substituting into system:

$2u' + v' = 4(2u + v) + 2(u - 3v)$
$= 10 - 2v$

$u' - 3v' = 5u + 6v$, similarly

Multiply $\left\{ \begin{array}{l} \text{top eqn by 3 and add} \\ \text{bot. eqn by 2 and subtract} \end{array} \right.$

and you get $u' = 5u$
$v' = -2v$ decoupled!

$\boxed{4E-2}$ $\vec{x}' = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ use the eigenvectors given in 4D-4:

variable change matrix is:

$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$; $\vec{x} = E\vec{u}$ is the change of vars.

(cols are eigenvectors)

To check, use matrices: $\vec{u} = E^{-1}\vec{x}$ + subst. into system

$$\boxed{\vec{u}' = E^{-1} A E \vec{u}}$$

is the new system. Calculating:

$$\vec{u}' = \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 3 \end{bmatrix}}_{A \cdot E} \vec{u}$$

$$\vec{u}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \vec{u}$$

So system is decoupled: $u_1' = 0$
$u_2' = -3u_2$
$u_3' = -3u_3$

$\boxed{4F-1}$ $x'' + px' + qx = 0$

a) $x' = y$ $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$

b) $y' = -qx - py$

$\therefore$ Wronskian of two solutions $\vec{x}_1$ and $\vec{x}_2$ is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, or $\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$, since $y_i = x_i'$, which is the usual Wronskian of x_1 and x_2.

$\boxed{4F-2}$

a) Neither is a constant multiple of the other.

b) $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

c) Since $W = 0$ when $t = 0$, $\vec{x}_1$ and $\vec{x}_2$ cannot be solutions of $\vec{x}' = A(t)\vec{x}$, where the entries of $A(t)$ are continuous.

d)
To find $A(t)$ explicitly, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\boxed{\vec{x}' = A\vec{x}}$

Then since $\begin{bmatrix} t \\ 1 \end{bmatrix}$ is soln, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$ or: $1 = at + b$, $0 = ct + d$

Since $\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ is soln, $\begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ or: $2t = at^2 + b2t$, $2 = ct^2 + d2t$

These are 4 equations for a, b, c, d. Solving:
$a = 0$, $b = 1$, $c = -2/t^2$ $d = 2/t$ so <u>not contin. at $t=0$</u>

$\boxed{4F-3}$

a) $\begin{vmatrix} \alpha_1 e^{m_1 t} & \alpha_2 e^{m_2 t} \\ \beta_1 e^{m_1 t} & \beta_2 e^{m_2 t} \end{vmatrix} = e^{(m_1+m_2)t} \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$.

$\vec{\alpha}_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ $= 0 \Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0$

$\vec{\alpha}_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ $\Leftrightarrow \vec{\alpha}_1, \vec{\alpha}_2$ are lin. dep't

b) Suppose $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 = \vec{0}$ Multiply by A:

$c_1 A\vec{\alpha}_1 + c_2 A\vec{\alpha}_2 = A\vec{0}$

$\therefore c_1 m_1 \vec{\alpha}_1 + c_2 m_2 \vec{\alpha}_2 = \vec{0}$

Multiply top eq'n by m_1, subtract from 3^{rd} eq'n, get

$c_2(m_2 - m_1)\vec{\alpha}_2 = \vec{0}$

But $m_1 \neq m_2$, $\vec{\alpha}_2 \neq \vec{0}$ (since it's an eigenvector)

$\therefore$ $c_2 = 0$

$\therefore$ also $c_1 = 0$ (since $c_1\vec{\alpha}_1 = \vec{0}$ & $\vec{\alpha}_1 \neq \vec{0}$)

$\boxed{4F-4}$

If $\vec{x}'(0) = \vec{0}$, then since $\vec{x}' = A\vec{x}$,

it follows that $A\,\vec{x}(0) = \vec{0}$, also.

Since A is nonsingular, we can multiply by A^{-1}, + get

$$\vec{x}(0) = \vec{0}.$$

$\therefore$ by the uniqueness theorem, $\vec{x}(t) = \vec{0}$ for all t.

Hypotheses needed: A can be a function of t (with continuous entries); require only that at time $t=0$, $A(0)$ is nonsingular — then above reasoning still applies.

$\boxed{4G-1}$

a) Gen soln is: $\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$

$\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$: $\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$

or: $c_1 + c_2 = 0$, $c_1 + 2c_2 = 1$ $\quad\therefore c_2 = 1,\ c_1 = -1$

$\therefore \vec{x}_2 = -\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$: solves $\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$

b) $\vec{x}_1 = 2\begin{bmatrix}1\\1\end{bmatrix}e^{3t} - \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ solves $\vec{x}_1(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$\therefore$ soln to $\vec{x}(0) = \begin{bmatrix}a\\b\end{bmatrix}$ is: $\boxed{a\vec{x}_1 + b\vec{x}_2}$

(since $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$)

$$a\vec{x}_1 + b\vec{x}_2 = (2a-b)\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + (b-a)\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$$

$\boxed{4G-2}$

a) $\vec{x}' = \begin{bmatrix}5 & -1\\3 & 1\end{bmatrix}\vec{x}$ Eigenvalues: $\begin{vmatrix}5-m & -1\\3 & 1-m\end{vmatrix} = m^2 - 6m + 8 = 0$, $m = 4, 2$

$m=4$: $\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\1\end{bmatrix}$ eigenvector, $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ soln

$m=2$: $3\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\3\end{bmatrix}$ eigvector, $\begin{bmatrix}1\\3\end{bmatrix}e^{2t}$ solution.

Fund. matrix: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix} = F(t)$ $\quad F(0) = \begin{bmatrix}1 & 1\\3 & 1\end{bmatrix}$ $\quad F(0)^{-1} = -\frac{1}{2}\begin{bmatrix}1 & -1\\-3 & 1\end{bmatrix}$

Soln to IVP:

$$= F(t)F(0)^{-1}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$$

[see p. 12, (34)]

$$= \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-3/2\\7/2\end{bmatrix} = -\frac{3}{2}\begin{bmatrix}e^{2t}\\3e^{2t}\end{bmatrix} + \frac{7}{2}\begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

b) Normalized fund. mx: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{1}{2}e^{2t}-\frac{1}{2}e^{4t}\\-\frac{3}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{3}{2}e^{2t}-\frac{1}{2}e^{4t}\end{bmatrix}$

Multiply this on right by $\begin{bmatrix}2\\-1\end{bmatrix}$ to get same answer.

$\boxed{4H-1}$

$A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$, $A^2 = \begin{bmatrix}a^2 & 0\\0 & b^2\end{bmatrix}$, $\dots$ $A^n = \begin{bmatrix}a^n & 0\\0 & b^n\end{bmatrix}$ by rules for mx. mult.

$$\therefore e^{At} = I + At + A^2\frac{t^2}{2!} + \dots$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}at & 0\\0 & bt\end{bmatrix} + \begin{bmatrix}\frac{a^2t^2}{2!} & 0\\0 & \frac{b^2t^2}{2!}\end{bmatrix} + \dots$$

$$= \begin{bmatrix}1 + at + \frac{a^2t^2}{2!} + \dots & 0\\0 & 1 + bt + \frac{b^2t^2}{2!} + \dots\end{bmatrix} = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}$$

$$\vec{x} = e^{At}\vec{x}_0 = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1e^{at}\\k_2e^{bt}\end{bmatrix}$$

verify: $x = k_1e^{at}$, $y = k_2e^{bt}$ is soln of: $\begin{cases}x' = ax\\y' = by\end{cases}$ with $x(0) = k_1$, $y(0) = k_2$ obvious!

$\boxed{4H-2}$

$A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ $\quad A^2 = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, $A^3 = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, $A^4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

after this it repeats (since $A^4 = I$)

ie, $A^5 = A$, $A^6 = A^2$, etc.

$$e^{At} = \begin{bmatrix}1 - t^2/2! + t^4/4! \cdots & t - t^3/3! + t^5/5! - \cdots\\-t + t^3/3! \cdots & 1 - t^2/2! + t^4/4! \cdots\end{bmatrix}$$

$$= \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$$

$$\vec{x} = e^{At}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1\cos t + k_2\sin t\\-k_1\sin t + k_2\cos t\end{bmatrix}$$

This obviously satisfies the system (I.V.P.): $x' = y$, $y' = -x$, $x(0) = k_1$, $y(0) = k_2$.

$\boxed{4H-4}$

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \dots \quad (*)$$

In general, for matrices B, C, (square),

$$\frac{d}{dt}B(t)C(t) = \frac{dB}{dt}C + B\frac{dC}{dt}$$

$$\therefore \frac{d}{dt}A(t)A(t) = \frac{dA}{dt}\cdot A + A\cdot\frac{dA}{dt}$$

$$\neq 2A\frac{dA}{dt}$$ since above two matrices are not = !!

$\therefore$ in general,

$$\frac{d}{dt}A^n(t) \neq nA^{n-1}\frac{dA}{dt},$$

and so you can't differentiate $(*)$ term-by-term to get Ae^{At}.

4H-7

a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \ldots$

similarly, $A^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$

$\therefore e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 2t & t \end{bmatrix} + \begin{bmatrix} t^2/2! & 0 \\ 4t^2/2! & t^2/2! \end{bmatrix} + \cdots$

$= \begin{bmatrix} 1+t+t^2/2!+\cdots & 0 \\ 2t+4t^2/2!+6t^3/3!+\cdots & 1+t+t^2/2!+\cdots \end{bmatrix}$

But lower-left corner

$= 2t\left(1+\frac{2t}{2!}+\frac{3t^2}{3!}+\frac{4t^3}{4!}+\cdots\right) = 2te^t$

$\therefore e^{At} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix}$ ⊛

b) $e^{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}t} = e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t} \cdot e^{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}t}$

$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \quad \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2t & 0 \end{bmatrix}\right)$

(see book ex. 1 p 451) (higher powers of mx. are 0)

$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} =$ ⊛

c) Find F by solving the system:

$\begin{aligned} x' &= x \\ y' &= 2x+y \end{aligned} \Rightarrow \begin{aligned} x &= c_1e^t \\ y'-y &= 2c_1e^t \end{aligned}$

solving 2^{nd} equation as a linear eqn:

$(ye^{-t})' = 2c_1$

$ye^{-t} = 2c_1t + c_2$

$y = c_1 \cdot 2te^t + c_2e^t$

$\therefore F = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix} \qquad F(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

so $\quad e^{At} = F \cdot F(0)^{-1} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix}$

4I-1

$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} -5 \\ -8 \end{bmatrix}t + \begin{bmatrix} 2 \\ -8 \end{bmatrix}$

① Solve the reduced equation $\vec{x}' = A\vec{x}$

· char. eqn is $m^2+m-6=0$ roots: $m=-3$, $m=2$

$(m+3)(m-2)=0$

$\underline{m=-3}$: $+4\alpha_1+\alpha_2=0$ soln: $\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-3t}$ $\qquad \underline{m=2}$: $-\alpha_1+\alpha_2=0$ soln: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

$\underline{\text{Fund mx}}$: $\begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = F. \qquad F^{-1} = \begin{bmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{bmatrix}\frac{1}{5e^{-t}}$

$v' = F^{-1}\begin{bmatrix} -5t+2 \\ -8t-8 \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}}{5}(-5t+2) - \frac{e^{3t}}{5}(-8t-8) \\ \frac{4}{5}e^{-2t}(-5t+2) + \frac{e^{-2t}}{5}(-8t-8) \end{bmatrix} = \begin{bmatrix} \frac{3e^{3t}}{5}t + 2e^{3t} \\ -\frac{28}{5}e^{-2t}t \end{bmatrix}$

$\therefore v = \begin{bmatrix} \frac{te^{3t}}{5} + \frac{3}{5}e^{3t} \\ +\frac{14}{5}e^{-2t}t + \frac{7}{5}e^{-2t} \end{bmatrix}$

$\vec{x}_p = F\vec{v} = \begin{bmatrix} \frac{t}{5} + \frac{3}{5} + \frac{14}{5}t + \frac{7}{5} \\ -\frac{4t}{5} - \frac{12}{5} + \frac{14}{5}t + \frac{7}{5} \end{bmatrix} = \begin{bmatrix} 3t+2 \\ 2t-1 \end{bmatrix}$ Ans.

4I-2

a) Using the work from above:

$v' = \frac{1}{5}\begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^t + 2e^{4t} \\ 4e^{-4t} - 2e^{-t} \end{bmatrix}$

$v = \frac{1}{5}\begin{bmatrix} e^t + \frac{e^{4t}}{2} \\ -e^{-4t} + 2e^{-t} \end{bmatrix} \qquad \vec{x} = Fv = \frac{1}{5}\begin{bmatrix} e^{-2t} + e^t/2 - e^{-2t} + 2e^t \\ -4e^{-2t} - 2e^t - e^{-2t} + 2e^t \end{bmatrix}$

$\therefore \vec{x}_p = \frac{1}{5}\begin{bmatrix} 5/2\, e^t \\ -5e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t/2 \\ -e^{-2t} \end{bmatrix}$

Add to $\vec{x}_p$ the $\vec{x}_h = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

$\vec{x}_p = \vec{c}e^{-2t} + \vec{d}e^t$ Substitute in to the equation:

$-2\vec{c}e^{-2t} + \vec{d}e^t = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c}e^{-2t} + \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{d}e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}e^t$

$\therefore -2\vec{c} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\vec{d} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{d} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Writing the left side of the 1^{st} system as $-2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{c}$, it becomes (I'm just being cute — you could just write it all out & hack away) on subtracting $\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c}$ from both sides

$\begin{bmatrix} -3 & -1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or: $\begin{aligned} -3c_1 - c_2 &= 1 \\ -4c_1 &= 0 \end{aligned} \quad \therefore c_1 = 0, \; c_2 = -1$

Similarly for the other system:

$\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \qquad \begin{aligned} -d_2 &= 0 \\ -4d_1 + 3d_2 &= -2 \end{aligned} \quad \therefore d_2 = 0, \; d_1 = 1/2$

Thus $\quad \vec{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}e^t = \begin{bmatrix} e^t/2 \\ -e^{-2t} \end{bmatrix}$ same as before, ta-da.

4I-4

Solve reduced equation first: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$

char eqn: $m^2 - 1 = 0$

$\underline{m=1}$: $\alpha_1 - \alpha_2 = 0$ $\qquad$ $\underline{m=-1}$: $3\alpha_1 - \alpha_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ soln. $\qquad$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ soln.

To find particular soln, since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ is a soln of reduced equation, we have to use as the trial soln not just $\vec{c}e^t$ but

$$\vec{x}_p = \vec{c}e^t + \vec{d}te^t$$

Substituting into the ODE's:

$$\vec{c}e^t + \vec{d}e^t + \vec{d}te^t = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}(\vec{c}e^t + \vec{d}te^t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

$$\therefore \quad \vec{c} + \vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{c} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{d}$$

Solving second system: (as done in prob (2b)) $\quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}k$

solving first system: $\quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -1-k \end{bmatrix}$

Subtract $3\times$ first row from second: $\quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -4+2k \end{bmatrix} \quad \therefore k = 2$

get: $-c_1 + c_2 = -1$ $\qquad$ so take $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (other $\vec{c}$ are possible)

soln: $\vec{x}_g = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

to this could be added

4I-5

$\vec{x}' = A\vec{x} + \vec{X}_0$. Try $\vec{x}_p = \vec{c}$. Substituting:

$A\vec{c} + \vec{X}_0 = 0. \quad \therefore \quad \vec{x}_p = -A^{-1}\vec{X}_0$ if A is nonsingular!

[If A is singular, you only get soln $\vec{x}_p = \vec{c}$ if $A\vec{c} = -\vec{x}_0$ is consistent. In general, if rank $A = n - r$, you use $\vec{x}_p = \vec{c}_0 + \vec{c}_1 t + \cdots + \vec{c}_{r-1}t^{r-1}$

$\boxed{5A-1}$ (a) Critical points occur where $x^2 - y^2 = 0$ and $x - xy = 0$

Now $x^2 - y^2 = 0 \Rightarrow x = \pm y$

Also $x - xy = 0 \Rightarrow x(1-y) = 0$

$\Rightarrow x = 0$ or $y = 1$

$\therefore$ $x = 0$ and $y = 0$

OR $y = 1$ and $x = 1$

OR $y = 1$ and $x = -1$

$\therefore$ $(0,0)$, $(1,1)$ and $(-1,1)$ are the critical points

(b) Critical points occur where $1 - x + y = 0$ and $y + 2x^2 = 0$

i.e. $y = x - 1$

Then $0 = x - 1 + 2x^2$

i.e. $x = \frac{1}{2}$ or $x = -1$

But $x = \frac{1}{2} \Rightarrow y = -\frac{1}{2}$

and $x = -1 \Rightarrow y = -2$

$\therefore$ $(\frac{1}{2}, -\frac{1}{2})$ and $(-1,-2)$ are the critical points.

$\boxed{5A-2}$ (a) Let $y = x'$

Then $y' = x'' = -\mu(x^2-1)x' - x$

The autonomous equations are then

$$\begin{cases} x' = y \\ y' = -\mu(x^2-1)y - x \end{cases}$$

Critical points occur at

$y = 0$

$-\mu(x^2-1)y - x = 0$ i.e. at $(0,0)$

(b) Let $y = x'$

Then $y' = x'' = x' - 1 + x^2$

The autonomous equations are then

$$\begin{cases} x' = y \\ y' = y - 1 + x^2 \end{cases}$$

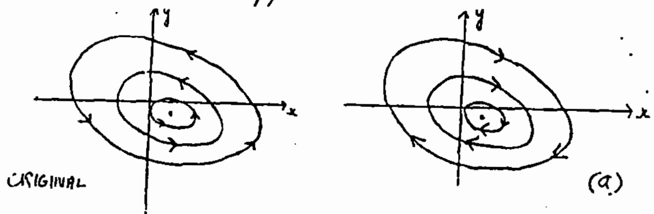
Critical points occur at

$y = 0$

$y - 1 + x^2 = 0$ $\therefore x^2 = 1$ $\therefore x = \pm 1$

So the critical points occur at $(1,0)$ and $(-1,0)$

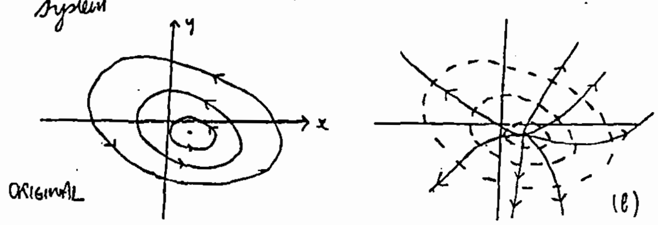
$\boxed{5A-3}$ (a) For this system the tangent vector $(-f(x,y), -g(x,y))$ to the trajectories is equal in magnitude but opposite in direction to the tangent vector $(f(x,y), g(x,y))$ to the original system. So the trajectories are the same but are traversed in the opposite direction



The critical points occur at

$$\left.\begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array}\right\}$$ i.e. the same for both systems

$\boxed{5A-3}$ (b) For this system the tangent vector $(g(x,y), -f(x,y))$ to the trajectories is perpendicular to the tangent vector $(f(x,y), g(x,y))$ to the original system. So (b) represents the orthogonal trajectories of the original system



The critical points of (b) occur at

$$\left.\begin{array}{l} g(x,y) = 0 \\ -f(x,y) = 0 \end{array}\right\}$$ i.e. the same as for the original system

$\boxed{5A-4}$ a) Let $u = t - t_0$, let $\bar{x}(t) = x_1(t-t_0)$.
Then $x_1(t-t_0) = x_1(u)$ as a function of u
$= \bar{x}(t)$ as a function of t

[As an example: if $x_1 = t^2$, then $x_1(u) = u^2$; and $\bar{x}(t) = t^2 - 2t_0 t + t_0^2$]

By hypothesis:

$$\frac{dx_1(t)}{dt} = f(x_1(t), y_1(t)) \qquad \frac{dy_1(t)}{dt} = g(x_1(t), y_1(t))$$

$\therefore$ changing letters formally:

$$\frac{dx_1(u)}{du} = f(x_1(u), y_1(u)) \qquad \frac{dy_1(u)}{du} = f(x_1(u), y_1(u)) \quad (*)$$

But $\frac{d\bar{x}(t)}{dt} = \frac{dx_1(u)}{du}\cdot\frac{du}{dt} = \frac{dx_1(u)}{du}$; similarly $\frac{d\bar{y}(t)}{dt} = \frac{dy_1(u)}{du}$

Therefore, from $(*)$ we get

$$\frac{d\bar{x}(t)}{dt} = f(\bar{x}(t), \bar{y}(t)) \qquad \frac{d\bar{y}(t)}{dt} = g(\bar{x}(t), \bar{y}(t)),$$

which shows that $\bar{x}(t)$, $\bar{y}(t)$ is also a solution.

$\begin{cases}\bar{x}(t) \\ \bar{y}(t)\end{cases} = \begin{cases}x_1(t-t_0) \\ y_1(t-t_0)\end{cases}$ represent the same motion as $\begin{cases}x_1(t) \\ y_1(t)\end{cases}$, but occurring t_0 time-units later.

That is, $\begin{cases}\bar{x}(t_1+t_0) \\ \bar{y}(t_1+t_0)\end{cases} = \begin{cases}x_1(t_1) \\ y_1(t_1)\end{cases}$ so wherever $\begin{cases}x_1 \\ y_1\end{cases}$ is at time t_1, $\begin{cases}\bar{x} \\ \bar{y}\end{cases}$ is there at time $t_1 + t_0$.

[This is the essential property of an autonomous system — the vector field does not change with time, so if we start at a given point t_0 seconds later, we follow the same trajectory path as before, but delayed by t_0 seconds.]

(b) Let $\begin{pmatrix}x_1(t)\\y_1(t)\end{pmatrix}$ $\begin{pmatrix}x_2(t)\\y_2(t)\end{pmatrix}$ be two trajectories which intersect at (a, b)

i.e. $\begin{pmatrix}x_1(t_0)\\y_1(t_0)\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x_2(t_1)\\y_2(t_1)\end{pmatrix}$ some t_0, t_1.

By part (a) $\begin{pmatrix}\bar{x}_2(t)\\\bar{y}_2(t)\end{pmatrix} \equiv \begin{pmatrix}x_2(t-t_0+t_1)\\y_2(t-t_0+t_1)\end{pmatrix}$

is also a solution to the ODE

But $\begin{pmatrix}\bar{x}_2(t_0)\\\bar{y}_2(t_0)\end{pmatrix} = \begin{pmatrix}x_2(t_1)\\y_2(t_1)\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

Thus by the uniqueness theorem

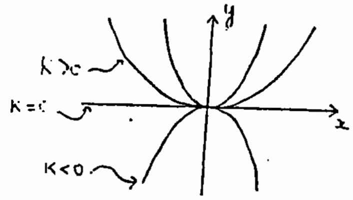
$$\begin{pmatrix}x_1(t)\\y_1(t)\end{pmatrix} = \begin{pmatrix}\bar{x}_2(t)\\\bar{y}_2(t)\end{pmatrix} = \begin{pmatrix}x_2(t-t_0+t_1)\\y_2(t-t_0+t_1)\end{pmatrix} \text{ for all } t$$

i.e. $\begin{pmatrix}x_1(t)\\y_1(t)\end{pmatrix}$ and $\begin{pmatrix}x_2(t)\\y_2(t)\end{pmatrix}$ are the same trajectory and differ at most by a change in parameter.

$\boxed{5B-1}$ (a) $\frac{y'}{x'} = \frac{dy}{dx} = \frac{-2y}{-x}$

$\therefore \frac{dy}{y} = 2\frac{dx}{x}$

$\therefore y = Kx^2$



(b) Let $\vec{x}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$ and $M = \begin{pmatrix}-1 & 0\\0 & -2\end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$. This has solution

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

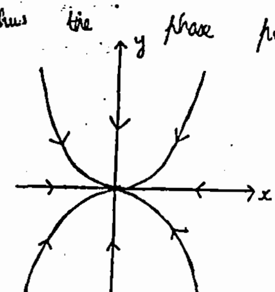
where λ_1 and λ_2 are the (distinct) eigenvalues of M with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

Here $\lambda_1 = -1$, $\lambda_2 = -2$

$\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

Then $\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}c_1 e^{-t}\\c_2 e^{-2t}\end{pmatrix}$ All trajectories $\rightarrow (0,0)$ as $t \to +\infty$

Thus the phase picture is:



The new trajectories are $\begin{cases}x = 0\\y = c_2 e^{-2t}\end{cases}$ ($c > 0, < 0, = 0$)
i.e. the positive and negative y-axis, and the trivial trajectory $\vec{x}(t) = 0$ (the origin)

c) As the picture shows, 3 trajectories are need to cover a typical solution curve from part (a): ↘, ↙, and • (the origin).

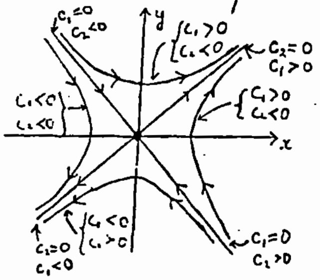
(d) This system may be obtained from the original by replacing t by $-t$. Thus we have the same trajectories but with the direction of the arrows reversed.

$\boxed{5B-2}$

a) $\frac{dy/dt}{dx/dt} = \frac{x}{y}$ $\therefore \frac{dy}{dx} = \frac{x}{y}$ soln: $y^2 - x^2 = c$ hyperbolas shown

b) $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

Thus the phase picture is.



The only "new" trajectory is the trivial one $\vec{x}(t) = 0$ (the origin)

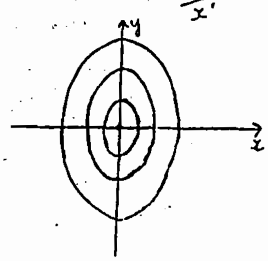
c) In general, each solution curve (part (b)) is covered by one trajectory. However, the two lines \ and / each require 3 trajectories to cover them.

(d) The system $\begin{cases} x' = -y \\ y' = -x \end{cases}$

has the same trajectories as the original system except the arrows are reversed.

$\boxed{5B-3}$

(a) $\frac{y'}{x'} = \frac{dy}{dx} = \frac{-2x}{y}$



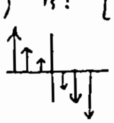
$\therefore y\,dy = -2x\,dx$

$\therefore \frac{y^2}{2} + x^2 = C$

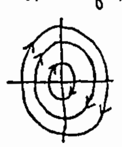
These are ellipses (for $C > 0$) and the point $(0,0)$ (for $C = 0$)

(b) For example, along the x-axis $(y=0)$, the tangent vectors ~~are~~ at $(x_0, 0)$ is: $\begin{cases} x' = 0 \\ y' = -2x_0 \end{cases}$, i.e., $(0, -2x_0)$

Thus the field is



So the direction of motion along the ellipses is clockwise.



$\boxed{5B-4}$

(a) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

Thus $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$

with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The system has a critical point at $(0,0)$ which is a saddle point

The general solution is

$$\vec{x}(t) = C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t}$$

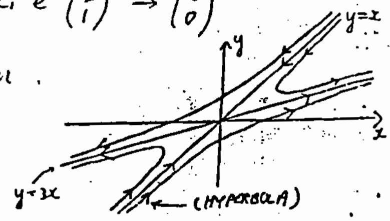
For $C_1 = 0$ and as $t \to \infty$

$$\vec{x}(t) = C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Also for $C_2 = 0$ and $t \to -\infty$

$$\vec{x}(t) = C_1 e^{t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the behaviour near the saddle point looks like



(b) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$.

M has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$

with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The system has an unstable node at $(0,0)$

The general solution is

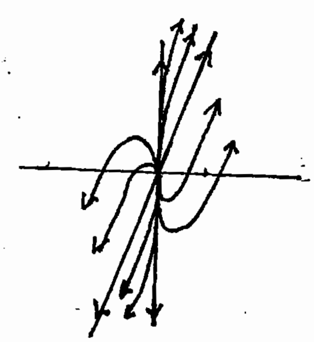
$$\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

So as $t \to -\infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus the behaviour near the node looks like:

For $t \approx -\infty$, $C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$ is dominant term, $\therefore$ solns are near the y-axis

For $t \approx \infty$, $C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$ dominates so solns are parallel to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$



5B-4

(c) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}$

Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = -4$, $\lambda_2 = -1$

with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

The system has an asympt. stable node at $(0,0)$

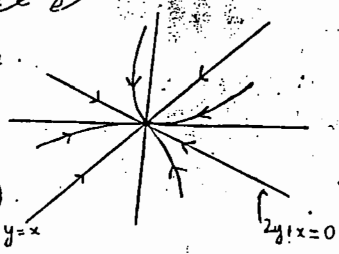
The general solution is

$$\vec{x}(t) = c_1 \vec{v}_1 e^{-4t} + c_2 \vec{v}_2 e^{-t}$$

As $t \to \infty$ all trajectories $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

~~$x(t) = c_1 e^{-4t} + 2c_2 e^{-t}$, $y(t) = c_1 e^{-4t} - c_2 e^{-t}$ — non-trivial curve!~~

The behaviour near the node looks like.



For $t \approx -\infty$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$ dominates so sol'ns are parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
For $t \approx \infty$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t}$ dominates, so sol'ns are $\underbrace{\text{close to}}_{\text{"like"}}$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.)

(d) Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $M = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

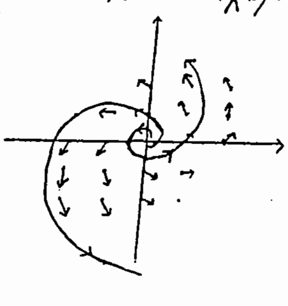
Then $\vec{x}'(t) = M\vec{x}(t)$

M has eigenvalues $\lambda_1 = 1 + i\sqrt{2}$, $\lambda_2 = 1 - i\sqrt{2}$

The system then has an unstable spiral around $(0,0)$.

When $y = 0$
$x' = x$
$\therefore x$ is increasing where the spiral cuts the +ve x-axis

As we e^t behaviour the spiral is outwards from the origin



e) $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \vec{x}$

Eigenvalues are $\pm i$ (pure imaginary), so the system is a stable center.

(The curves are ellipses, since $\frac{dy}{dx} = \frac{-2x-y}{x+y}$ which integrates easily after cross-multiplying to $2x^2 + 2xy + y^2 = c$.)

Direction of motion:

For example, at $(1,0)$, the vector field is $x' = 1$, $y' = -2$

. so motion is counterclockwise.

(a few other vectors are shown, inaccurately drawn...)

5B-5

(a) Let $y = x'$

Then, assuming $m \neq 0$,

$$y' = x'' = -\tfrac{c}{m} x' - \tfrac{k}{m} x$$

$\therefore$ The system is thus $\begin{cases} x' = y \\ y' = -\frac{k}{m} x - \frac{c}{m} y \end{cases}$

(b) The eigenvalues of $M = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$

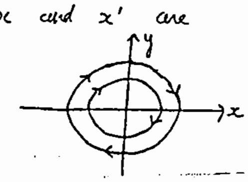
are $\lambda_\pm = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

(i) $c = 0 \Rightarrow \lambda_\pm = \pm i \sqrt{\frac{k}{m}}$

Thus there is a stable centre at $(0,0)$.

Physically we'd expect this as putting $c = 0$ ($m, k > 0$) in the ODE gives the SHM equation. Then x and x' are periodic with period $2\pi\sqrt{\frac{m}{k}}$

Thus we expect periodic trajectories in phase space



. Here $c^2 - 4km < 0$

(ii) $\sqrt{c^2 - 4km} = 2i\sqrt{km}\left(1 - \frac{c^2}{4km}\right)^{1/2}$.

or, neglecting c, $\approx 2i\sqrt{km}$

Then $\lambda_\pm = -\frac{c}{m} \pm i\sqrt{\frac{k}{m}}$ (eigenvalues)

The behaviour near $(0,0)$ is that of an asymptotically stable spiral. (since $-\frac{c}{m} < 0$)

The "radius" of the spiral decays as $t \to \infty$ like $e^{-\frac{c}{m}t}$ i.e. very slowly indeed!

Physically we have lightly damped harmonic motion e.g. a particle at the end of a spring oscillating in air. The motion is almost simple harmonic but the amplitude of oscillation decays slowly with time.

(iii) No!

When $c^2 - 4km \geq 0$, then as $k, m \geq 0$

we see $\sqrt{c^2 - 4km} \leq |c|$

Thus adding or subtracting $\sqrt{c^2 - 4km}$ to $-c$ cannot change its sign.

i.e. when the λ's are real, ~~either~~ they're both ~~positive or~~ ~~both~~ negative (since $c \geq 0$ always).

5C-1

$x' = x - y + xy$
$y' = 3x - 2y - xy$

linearization is (at $(0,0)$): $x' = x - y$, $y' = 3x - 2y$

$\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$ char eqn: $m^2 + m + 1 = 0$, $m = \frac{-1 \pm \sqrt{-3}}{2}$ $\therefore$ asymp. stable spiral

5C-2

$x' = x + 2x^2 - y^2$
$y' = x - 2y + x^3$

lin'zn: $x' = x$, $y' = x - 2y$ $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$

eigenvalues are $1, -2$ (since mx. is Δular) $\therefore$ unstable saddle

5C-3

$x' = 2x + y + xy^3$
$y' = x - 2y - xy$

lin'zn: $x' = 2x + y$, $y' = x - 2y$ $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$m^2 - 5 = 0$
$m = \pm\sqrt{5}$ unstable saddle

5C-4

$x' = 1 - y$
$y' = x^2 - y^2$

critical pts: $1 - y = 0$ $\therefore y = 1$; $x^2 - y^2 = 0$ $\therefore x = \pm 1$ $(1,1)$ and $(-1,1)$.

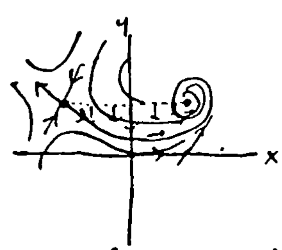
At $(1,1)$: since in general the Jac. matrix (of partial derivs) is $\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$, the lin'zn is $\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$m^2 + 2m + 2 = 0$
$m = -1 \pm \frac{\sqrt{-4}}{2} = -1 \pm i$ $\therefore$ asym. stable spiral.

At $(-1,1)$: lin'zn is (again using Jacobian): $\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$ $\therefore m^2 + 2m - 2 = 0$, $m = -1 \pm \sqrt{3} = -2.73, +.73$

$\therefore$ unstable saddle. Eigenvectors: $-m\alpha_1 - \alpha_2 = 0$ $\therefore \begin{bmatrix} 1 \\ -m \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ -.73 \end{bmatrix}, \begin{bmatrix} 1 \\ 2.73 \end{bmatrix}$

Using this info:



(Along dotted line, $y = 1$, a few dir. field vectors are drawn, using the original system: $x' = 0$, $y' = x^2 - 1$

A few other vectors are drawn in to help the sketch

5C-5

This one is work, but instructive: think of x, y as 2 population which mutually destroy each other: $x - x^2$, $3y - 2y^2$ represent their "natural" growth laws, the $-xy$ terms their mutual destruction. [Like two hostile tribes, non-cannibalistic].

$x' = x - x^2 - xy$
$y' = 3y - 2y^2 - xy$

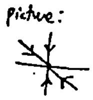
Critical points: $x(1 - x - y) = 0$
$y(3 - 2y - x) = 0$

From equation 1, either $x = 0$, or $1 - x - y = 0$.
If $x = 0$, eqn 2 says: $y = 0$ or $y = 3/2$
If $1 - x - y = 0$, eqn 2 says:
either $y = 0$ (in which case $1 - x = 0$, $x = 1$)
or $3 - 2y - x = 0$ (in which case we solve the 2 eqns: $1 - x - y = 0$, $3 - 2y - x = 0$ getting $y = 2$, $x = -1$

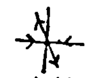
Summary: critical points are
$(0,0)$, $(0, 3/2)$, $(1,0)$, $(-1,2)$.

Now we determine their types: Jacobian matrix: $\begin{bmatrix} 1 - 2x - y & -x \\ -y & -x + 3 - 4y \end{bmatrix}$

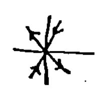
$(0,0)$: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$  unstable node.

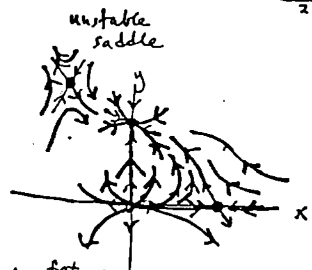
$(0, 3/2)$: $\begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$ eigenvs: $-1/2$, -3 vectors: $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ picture: 

asymp. stable node

$(1,0)$ $\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ eigenvs: -1, 2 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  unstable saddle

$(-1,2)$ $\begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$ $m^2 + 3m - 2 = 0$, $m = \frac{-3 \pm \sqrt{17}}{2}$; $m \approx 1/2$, $m \approx -7/2$ $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$





the fat lines are impressionistic pieces of solution curves. Note there is no mutual coexistence! The tribe y always wins, (unless there is none of it to start with), essentially because of its stronger growth rate.

NL

5D-1

a) Putting right-side of equations in (2) $= 0$ gives (assume $x \neq 0, y \neq 0$)

$$-\frac{x}{y} = 1 - x^2 - y^2 = \frac{y}{x} \qquad \therefore -x^2 = y^2$$

$$\text{so } x^2 + y^2 = 0 \quad \therefore \begin{matrix} x=0 \\ y=0 \end{matrix}$$

(contradiction)

b) $(\cos t, \sin t)$ satisfies the system (just substitute); trajectory is the unit circle.

c) Equation (3) shows that if $R > 1$, the direction field points in towards the unit ⊙, and (along circle of radius R) if $R < 1$, it points out towards the unit circle. Thus every solution curve is always getting closer to the unit ⊙.

5D-2

a) Bendixson criterion:

$$\operatorname{div}(f,g) = (1+3x^2) + (1+3y^2) > 0$$

$\therefore$ no limit cycle in xy-plane

b) System has no critical points, since $x^2+y^2=0 \Rightarrow x=0,\ y=0$, and this does not make $1+x-y=0$.

$\therefore$ no limit cycles.

c) System has no critical points if $x < -1$, $\therefore$ no limit cycles in this region.

[To see this: $x^2 - y^2 = 0 \Rightarrow y = \pm x$

$2x + x^2 + y^2 = 0$ and $y = \pm x$ $\Rightarrow 2x + 2x^2 = 0 \quad \therefore x = 0, -1$

thus critical pts. are $(0,0), (-1,1), (-1,-1)$.]

d) Bendixson's criterion:

$$\operatorname{div}(f,g) = a + 2bx - 2cy + 2cy - 2bx = a$$

$\therefore$ no limit cycles in xy-plane if $a \neq 0$.

5D-3

The system (7) is

$$x' = y$$
$$y' = -v(x) - u(x)y$$

a) By Bendixson's criterion,

$\operatorname{div}(f,g) = 0 - u(x) < 0$ for all x, y if $u(x) > 0$.

$\therefore$ no periodic solution.

b) $v(x) > 0 \Rightarrow$ system has no critical point [at a critical point, $y = 0$, $\therefore v(x) = 0$]

$\therefore$ no periodic solution.

5D-5 (like 5D-1)

$\boxed{5E\text{-}1}$ a) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{bmatrix}1 & -4\\ 2 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0).$$

Char. eqn: $\lambda^2 + 7 = 0$

$(0,0)$ is a center.

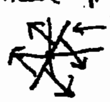
For non-lin. system, $(0,0)$ could be a center; or, unstable or asymptotically stable spiral.

b) linearization is

$$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}3 & -1\\ -6 & +2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{at } (0,0)$$

char. eqn: $\lambda^2 - 5\lambda = 0$, $\lambda = 0, 5$

$\therefore$ $(0,0)$ is not isolated – it is one of a line of critical points, all unstable:



For non-linear system, picture could stay like this; or turn into an unstable node or saddle.

$\boxed{5E\text{-}2}$ a) $x' = y$, $y' = x(1-x)$ $\quad J = \begin{bmatrix}0 & 1\\ 1-2x & 0\end{bmatrix}$

Crit. pts: $(0,0)$, $(1,0)$

At $\underline{(0,0)}$, $J = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\lambda^2 - 1 = 0$

$\lambda = 1$, $\vec{\alpha} = \begin{pmatrix}1\\1\end{pmatrix}$; $\lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\-1\end{pmatrix}$.

This is an unstable saddle.

At $(1,0)$, $J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $\lambda = \pm i$

This is a center, clockwise motion.

For non-linear system, three possibilities:

(1,0) center $\quad$ asymp. stable spiral $\quad$ unstable spiral

$\boxed{5E\text{-}2}$ b) $x' = x^2 - x + y$, $y' = -yx^2 - y$

Crit. pts: $\begin{cases} x^2 - x - y = 0 \\ -y(x^2+1) = 0 \end{cases}$ $\quad \therefore y = 0$, $x = 0, 1$

Two crit. pts: $(0,0)$, $(1,0)$.

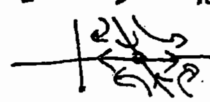
$$J = \begin{bmatrix}2x-1 & 1\\ -2xy & -x^2-1\end{bmatrix}$$

At $(0,0)$: $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$ $\quad \lambda = -1$, $\vec{\alpha} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad$ repeated incomplete eigenvalue asy. stable node

Picture:

At $(1,0)$: $\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}$ $\quad \lambda_1 = 1$, $\vec{\alpha}_1 = \begin{pmatrix}1\\0\end{pmatrix}$; $\lambda_2 = -2$, $\vec{\alpha}_2 = \begin{pmatrix}1\\-3\end{pmatrix}$

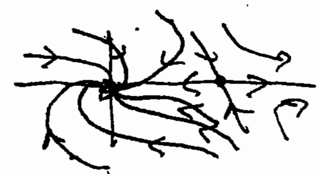
Picture:



unstable saddle.

For non-linear system, two possibilities

(0,0) becomes an asym. stable node



(0,0) becomes an asym. stable spiral

$\boxed{5E\text{-}3}$ The new system is

$$x' = \tfrac{5}{4}ax - pxy,$$
$$y' = -by + qxy$$

whose critical pt is $\left(\frac{b}{q}, \frac{5a/4}{p}\right)$.

Crit. pt. for the orig. system is: $\left(\frac{b}{q}, \frac{a}{p}\right)$.

So the effect is to leave the flower population the same, but to increase the borer population by 25%.

Section 6 Solutions

$\boxed{6A-1}$ All of these use the ratio test:
if $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$, $\sum_0^\infty b_n$ converges if $L<1$, diverges if $L>1$.

a) $\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \left(\frac{n+1}{n}\right)|x| \to |x|$ as $n\to\infty$

$\therefore$ converges if $|x|<1$, so $R=1$

b) $\left|\frac{x^{2(n+1)}}{(n+1)2^{n+1}}\cdot\frac{n\cdot 2^n}{x^{2n}}\right| = \frac{n}{(n+1)2}\cdot|x|^2$

$\xrightarrow[\text{as } n\to\infty]{} \frac{1}{2}|x|^2$, and $\frac{|x|^2}{2}<1$ if $|x|<\sqrt{2}$

$\therefore$ converges if $|x|<\sqrt{2}$, so $R=\sqrt{2}$

c) $\left|\frac{(n+1)!\,x^{n+1}}{n!\;\;x^n}\right| = (n+1)|x| \to \infty$ as $n\to\infty$ (if $x\neq 0$).

$\therefore$ converges only when $x=0$; $R=0$.

d) $\left|\frac{[2(n+1)]!}{(n+1)!^2}\cdot x^{n+1}\cdot\frac{(n!)^2}{(2n)!\,x^n}\right|$

$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\cdot|x| \to 4|x|$ as $n\to\infty$

$\therefore$ converges if $4|x|<1$, i.e., $|x|<\frac{1}{4}$, so $R=1/4$

$\boxed{6A-2}$ a) $\frac{1}{1-x} = \sum_0^\infty x^n$

$\therefore \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_1^\infty n x^{n-1}$

$= \sum_0^\infty (n+1)x^n$

(replacing n by $n+1$).

b) $e^x = \sum_0^\infty \frac{x^n}{n!}$, $\therefore e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n}}{n!}$

$x e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{n!}$

$\boxed{6A-2}$ c) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \sum_0^\infty (-1)^n x^{2n}$

Integrating:

$\tan^{-1}x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1} + \cancelto{0}{c}$

($c=0$: substitute $x=0$ on both sides to see that $c=0$)

d) $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_0^\infty (-1)^n x^n$

Integrating:

$\ln(1+x) = \sum_0^\infty \frac{(-1)^n x^{n+1}}{n+1} + \cancelto{0}{c}$

(see that $c=0$ by substituting $x=0$ on both sides)

[series could also be written $\sum_1^\infty (-1)^{n-1}\frac{x^n}{n}$
(putting n for $n+1$)]

$\boxed{6A-3}$ a) $y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$

$y' = \sum_0^\infty \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_0^\infty \frac{x^{2n}}{(2n)!}$

$y'' = \sum_1^\infty \frac{2n\,x^{2n-1}}{(2n)!} = \sum_1^\infty \frac{x^{2n-1}}{(2n-1)!}$

the 0 term disappears

$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$ (changing $n\to n+1$)

This shows $y''=y$, or $y''-y=0$.

b) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots$

$e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}$

$\therefore \frac{e^x-e^{-x}}{2} = \frac{2x}{2}+\frac{2x^3}{2\,3!}+\frac{2}{2}\frac{x^5}{5!}+\ldots$

$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$.

4a) $\sum_0^\infty x^{3n+2} = x^2\sum_0^\infty x^{3n}$

$= x^2\cdot\frac{1}{1-x^3}$.

(since $\sum_0^\infty x^{3n} = \sum_0^\infty (x^3)^n = \frac{1}{1-(x^3)}$).

$\boxed{6A-4}$ b) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Integrate both sides:

$$\sum_0^\infty \frac{x^{n+1}}{n+1} = -\ln(1-x) + \underset{\substack{(\text{substitute}\\ x=0)}}{c=0}$$

$$\therefore \sum_0^\infty \frac{x^n}{n+1} = -\frac{\ln(1-x)}{x}.$$

4c) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Differentiating, $\sum_1^\infty n x^{n-1} = \frac{1}{(1-x)^2}$

$$\therefore \sum_1^\infty n x^n = \frac{x}{(1-x)^2}$$

note 1 or 0 (makes no difference)

$\boxed{6B-1}$

a) Since $y(0)=1$,

$$y = 1 + a_1 x + a_2 x^2 + a_3 x^3$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$y^2 = (1 + a_1 x + a_2 x^2 + \dots)(1 + a_1 x + a_2 x^2 + \dots)$$
$$= 1 + 2a_1 x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \dots$$

(this is far enough to get a_3.)

$y' = x + y^2$ says that

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = 1 + (2a_1 + 1)x + (2a_2 + a_1^2)x^2 + \dots$$

$\therefore$ equating coefficients of like powers of x gives us:

$$a_1 = 1,\ 2a_2 = 2a_1 + 1 = 3,\ \therefore a_2 = 3/2$$
$$3a_3 = 2a_2 + a_1^2 = 4,\ \therefore a_3 = 4/3$$

So: $\boxed{y = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots}$

b) Using Taylor's formula: $y(0)=1$

$$y' = x + y^2 \qquad \therefore y'(0) = 0 + 1^2 = 1$$
$$\therefore y'' = 1 + y' \cdot 2y \qquad y''(0) = 1 + 1\cdot(2\cdot 1) = 3$$
$$y''' = y'' \cdot 2y + y' \cdot 2y' \qquad y'''(0) = 3\cdot 2 + 1\cdot 2 = 8$$

By Taylor's formula,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$
$$\therefore y = 1 + x + \frac{3}{2}x^2 + \frac{8}{6}x^3 + \dots$$

just as in part (a).

$\boxed{6B-2}$

a) $y = \sum_0^\infty a_n x^n$

$$y' = \sum_{0\,(\text{or }1)}^\infty n a_n x^{n-1} \rightsquigarrow \sum_0^\infty (n+1)a_{n+1} x^n$$

$y' - y = x$ says that

$$(n+1)a_{n+1} - a_n = 0 \quad \text{if } n \neq 1$$
$$= 1 \quad \text{if } n = 1,$$

that is, (since $y(0)=0$):

$$a_0 = 0, \quad a_{n+1} = \frac{a_n}{n+1} \quad \text{if } n \neq 1$$
$$\text{and} \quad 2a_2 - a_1 = 1.$$

This gives:

$$a_0 = 0,\ a_1 = 0,\ a_2 = 1/2,\ a_3 = \frac{1}{3}\cdot\frac{1}{2},$$
$$a_4 = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2},\ \text{etc.}\dots$$

so $y = \sum_{n=2}^\infty \frac{x^n}{n!} = e^x - 1 - x$

b) $y = \sum_0^\infty a_n x^n \xrightarrow{\cdot x} -xy = \sum_0^\infty a_n x^{n+1}$

$$y' = \sum_{1\,(\text{or }0)}^\infty n a_n x^{n-1} \xrightarrow{n \to n+2} \sum_{-1} (n+2)a_{n+2} x^{n+1}$$

note

$y' = -xy \Rightarrow$ ~~a_n~~

$$(n+2)a_{n+2} = -a_n \qquad n = 0,1,2,\dots$$
$$a_1 = 0 \qquad (\text{corresponding to } n=-1)$$
$$a_0 = 1 \qquad (\text{since } y(0)=1)$$
$$\therefore a_{n+2} = \frac{-a_n}{n+2} \qquad n = 0,1,2,\dots$$

so

$$a_0 = 1,\ a_2 = \frac{-1}{2},\ a_4 = \frac{1}{4}\cdot\frac{1}{2},\ a_6 = \frac{-1}{6\cdot 4\cdot 2}$$
$$a_1 = a_3 = a_5 = \dots = 0.$$

So $y = \sum_0^\infty \frac{x^{2n}(-1)^n}{2^n \cdot n!} = e^{-x^2/2}$

6B-2

c) $y = \sum_0^\infty a_n x^n$

$y' = \sum_{0\ (\text{or } 1)}^\infty n a_n x^{n-1} \rightsquigarrow \sum_0^\infty (n+1) a_{n+1} x^n$

$x y' = \sum_0^\infty n a_n x^n$

$\therefore (1-x)y' - y = 0 \implies$ (equating the coeff of x^n to 0)

$$(n+1)a_{n+1} - n a_n - a_n = 0$$

or $a_{n+1} = \frac{(n+1)a_n}{n+1} = a_n$

$y(0) = 1 \implies a_0 = 1$

$$\therefore\ y = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

6C-1

a) $\sum_1^\infty a_n x^{n+3} \underset{n\to n-3}{=} \sum_4 a_{n-3} x^n$

this starts with x^4, so this must also

b) $\sum_0^\infty n(n-1)a_n x^{n-2} \underset{n\to n+2}{=} \sum_0^\infty (n+2)(n+1)a_{n+2} x^n$

starts with x^0, so this must also

c) $\sum_1^\infty (n+1)a_n x^{n-1} \underset{n\to n+1}{=} \sum_0^\infty (n+2)a_{n+1} x^n$

starts with x^0, so this must also

6C-2

$y = \sum_0^\infty a_n x^n \underset{\cdot 4}{\rightsquigarrow} 4y = \sum_0^\infty 4 a_n x^n$

$y'' = \sum_0^\infty a_n \cdot n(n-1) x^{n-2} \underset{n\to n+2}{\rightsquigarrow} \sum_0^\infty a_{n+2}(n+2)(n+1) x^n$

$y'' - 4y = 0 \implies a_{n+2}(n+2)(n+1) - 4a_n = 0$

or $\boxed{a_{n+2} = \frac{4\, a_n}{(n+2)(n+1)}}$ Recursion formula

$\therefore\ a_2 = \frac{4a_0}{2\cdot 1}$, $a_4 = \frac{4}{4\cdot 3}\cdot\frac{4}{2\cdot 1} a_0 = \frac{4^2}{4!} a_0$

$a_3 = \frac{4a_1}{3\cdot 2}$ $\quad a_5 = \frac{4}{5\cdot 4}\cdot\frac{4}{3\cdot 2}\cdot a_1 = \frac{4^2 a_1}{5!}$

continued above

6C-2

(continued)

get one series by taking $a_0 = 1, a_1 = 0$:

$y_0 = 1 + \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 + \frac{4^3}{6!}x^6 + \dots$

other series: take $a_0 = 0$, $a_1 = 1$

$y_1 = x + \frac{4x^3}{3!} + \frac{4^2x^5}{5!} + \dots$

In summation notation:

$y_0 = \sum_0^\infty \frac{4^n x^{2n}}{n!}$, $\quad y_1 = \sum_0^\infty \frac{4^n x^{2n+1}}{(2n+1)!}$

(can also write numerator as $(2x)^{2n}$)

6C-3

Not solved.

6C-4

$\boxed{y'' - 2xy' + ky = 0}$; $\boxed{k = 2m}$.

$y = \sum_0^\infty a_n x^n \underset{\cdot 2m}{\rightsquigarrow} \sum_0^\infty 2m a_n x^n$

$y' = \sum_0^\infty n a_n x^{n-1} \underset{\cdot -2x}{\rightsquigarrow} \sum_0^\infty -2n a_n x^n$

$y'' = \sum_0^\infty n(n-1)a_n x^{n-2} \underset{n\to n+2}{\rightsquigarrow} \sum_0 (n+2)(n+1)a_{n+2} x^n$

Since $y'' - 2xy' + ky = 0$, this gives

$(n+2)(n+1)a_{n+2} - 2na_n + 2ma_n = 0$

or $\boxed{a_{n+2} = \frac{2(n-m)}{(n+2)(n+1)} a_n}$

If $n = m$, then $a_{m+2} = 0$, etc.

So: if m is odd,
take $a_0 = 0$, $a_1 = 1$; then
all $a_0 = a_2 = a_4 = \dots = 0$
and all $a_{m+2} = a_{m+4} = 0 = .$
so $y_1 = a_1 x + a_3 x^3 + \dots + a_m x^m$

If m is even, take $a_1 = 0$.
then similarly, (so $a_3 = 0, a_5 = 0 ..$)
$y_0 = a_0 + a_1 x^2 + \dots + a_m x^m$

$\boxed{6C\text{-}5}$

$$\boxed{y'' = xy}$$

$$y = \sum_0^\infty a_n x^n \quad \underset{\cdot x}{\overset{xy:}{\rightsquigarrow}} \quad \sum_0^\infty a_n x^{n+1} = \sum_1^\infty a_{n-1} x^n$$

$$y'' = \sum_0^\infty n(n-1) a_n x^{n-2} \quad \underset{n \to n+2}{\rightsquigarrow} \quad \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

Equating coeff's of like powers of x (since $y'' = xy$) gives

$$(n+2)(n+1) a_{n+2} = a_{n-1} \quad (n \geq 1) \quad \rightsquigarrow \quad \therefore \boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}, \quad n \geq 1.$$

$$= 0 \quad (n = 0) \qquad\qquad a_2 = 0$$

Recursion formula

$\therefore a_0, a_1$ are arbitrary, $2a_2 = 0$ (so $a_2 = 0$),

($\therefore a_5 = a_8 = a_{11} = \ldots = 0$ by the recursion formula)

and other terms are: $a_3 = \dfrac{a_0}{3\cdot 2}$, $a_6 = \dfrac{a_0}{6\cdot 5\cdot 3\cdot 2} \ldots$

$a_4 = \dfrac{a_1}{4\cdot 3}$, $a_7 = \dfrac{a_1}{7\cdot 6\cdot 4\cdot 3} \ldots$

Taking $a_0 = 1$, $a_1 = 0$ gives $y_0 = 1 + \dfrac{x^3}{3\cdot 2} + \dfrac{x^6}{6\cdot 5\cdot 3\cdot 2} + \cdots + \dfrac{x^{3n}}{3n\cdot(3n-1)(3n-3)\ldots 3\cdot 2} + \ldots$

taking $a_0 = 0$, $a_1 = 1$ gives $y_1 = x + \dfrac{x^4}{4\cdot 3} + \dfrac{x^7}{7\cdot 6\cdot 4\cdot 3} + \cdots + \dfrac{x^{3n+1}}{(3n+1)\cdot 3n\cdot(3n-2)\ldots 4\cdot 3} + \ldots$

$\boxed{6C\text{-}6}$

$$\boxed{y'' - x^2y'' - 2xy' + 6y = 0}$$

$$y = \sum_0^\infty a_n x^n \rightsquigarrow 6y = \sum_0^\infty 6a_n x^n$$

$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow -2xy' = \sum_0^\infty -2n a_n x^n$$

$$y'' = \sum_0^\infty n(n-1) a_n x^{n-2} \rightsquigarrow y'' = \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

$$\rightsquigarrow -x^2y'' = -\sum_0^\infty n(n-1) a_n x^n$$

Equating coeff of x^n to 0 gives:

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 6a_n = 0$$

or $a_{n+2} = a_n \dfrac{[n(n-1) + 2n - 6]}{(n+2)(n+1)}$

or $\boxed{a_{n+2} = \dfrac{(n+3)(n-2)}{(n+2)(n+1)} a_n}$

RECURSION FORMULA.

This gives solutions

$y_0 = 1 - 3x^2 \quad (a_0 = 1, a_1 = 0 = a_3 = a_5 = \ldots)$

$y_1 = x - \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 - \cdots \quad \Leftarrow$

Radius of convergence for y_1 is determined by ratio test: $\left|\dfrac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \dfrac{(n+3)(n-2)}{(n+2)(n+1)}|x|^2 \underset{\text{as } n\to\infty}{\longrightarrow} x^2$, $|x|^2 < 1$ if $|x| < 1$

$\therefore R = 1$. This is expected, since in standard form, ODE is $y'' - \frac{2x}{1-x^2}y' + \frac{6}{1-x^2}y = 0$, and coefficients become infinite at $|x| = 1$.

$\boxed{6C\text{-}7}$

$$y = \sum_0^\infty a_n x^n \rightsquigarrow xy = \sum_1^\infty a_{n-1} x^n$$

$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow 2y' = 2\sum_0^\infty (n+1) a_{n+1} x^n$$

$$y'' = \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$$

$\therefore y'' + 2y' + (x-1)y = 0$ leads to the recursion:

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_{n-1} - a_n = 0$$

leading to two sol'ns

$y_0 = 1 + x^2/2 - x^3/2 + \ldots \quad \left(\begin{smallmatrix} a_0 = 1 \\ a_1 = 0 \end{smallmatrix}\right)$

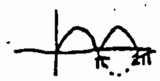
$y_1 = x - x^2 + 5/6\, x^3 + \ldots \quad (a_0 = 0, a_1 = 1)$

FOURIER SERIES

7A-1

a) For $\sin kt$, $\cos kt$ the frequency is k, and $\boxed{(\text{frequency})(\text{period}) = 2\pi}$.

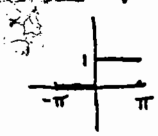
$\therefore \quad \frac{\pi}{3} \cdot P = 2\pi, \qquad P = \boxed{6}$

b)  Period is $\boxed{\pi}$: $|\sin(t+\pi)| = |-\sin t| = |\sin t|$

c) $\cos 3t$ has period $= \frac{2\pi}{3}$ (see problem 4)
$\cos^2 3t$ has period $\frac{1}{2}\cdot\frac{2\pi}{3}$ (as in prob. 9):

$$\left(\cos 3\left(t+\tfrac{\pi}{3}\right)\right)^2 = (\cos(3t+\pi))^2 = (-\cos(3t))^2 = (\cos(3t))^2$$

7A-2 a)



$$a_n = \frac{1}{\pi}\int_0^\pi \cos nt\,dt = \frac{\sin nt}{\pi n}\Big]_0^\pi = 0$$

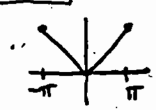
$$\left(a_0 = \frac{1}{\pi}\int_0^\pi dt = 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nt\,dt = \frac{-\cos nt}{n\pi}\Big]_0^\pi = \frac{-(-1)^n-(-1)}{n\pi}$$

$$= \frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd}\end{cases}$$

$$\therefore\ f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

7A-2 b



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\,dt = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2}$$

$$= \boxed{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\cos nt}_{\text{even function}}\,dt = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$$

$$= \frac{2}{\pi}\left[t\,\frac{\sin nt}{n} - \int \frac{\sin nt}{n}\,dt\right]_0^\pi$$

$$= \frac{2}{\pi}\left(0 + \left[\frac{\cos nt}{n^2}\right]_0^\pi\right) = \frac{2}{\pi}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ -4/\pi n^2, & n \text{ odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\sin nt}_{\text{odd function}}\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \dots\right)$$

7A-3

$$\int_{-\pi}^{\pi}\cos mt\cos nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(m+n)t + \cos(m-n)t\right)dt$$

$$\begin{cases} = \frac{1}{2}\left[\frac{\sin(m+n)t}{m+n} + \frac{\sin(m-n)t}{m-n}\right]_{-\pi}^{\pi} = 0 \text{ if } m \neq n \\ = \frac{1}{2}\left[\frac{\sin 2mt}{2m} + t\right]_{-\pi}^{\pi} = \frac{\pi-(-\pi)}{2} = \pi, \text{ if } m = n\end{cases}$$

7A-4

a) $$\int_P^{a+P} f(t)\,dt = \int_0^a f(u+P)\,du = \int_0^a f(u)\,du$$

$u = t-P$, so $t = u+P$ (since $f(u+P) = f(u)$)

Then: (b)

$$\int_a^{a+P} f(t)\,dt = \int_a^P f(t)\,dt + \int_P^{a+P} f(t)\,dt$$

$$= \int_a^P f(t)\,dt + \int_0^a f(t)\,dt \quad \text{by the first part}$$

$$= \int_0^P f(t)\,dt\,.$$

7B-1. a) $a_0 = 2\int_0^1 (1-t)\,dt = 2t - t^2\Big]_0^1 = 1$

$a_n = 2\int_0^1 (1-t)\cos n\pi t\,dt$ Integ. by parts:

$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} - \int (-1)\frac{\sin n\pi t}{n\pi}\,dt\right]_0^1$

$= 2\left[(1-t)\frac{\sin n\pi t}{n\pi} + -\frac{\cos n\pi t}{(n\pi)^2}\right]_0^1$

$= \frac{-2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ 4/n^2\pi^2, & n \text{ odd}\end{cases}$

$f(t) \sim \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \ldots\right)$

Fourier cosine series (picture below)

b) $b_n = 2\int_0^1 (1-t)\sin n\pi t\,dt$ Integ. by parts.

$= 2\left[(1-t)\left(\frac{-\cos n\pi t}{n\pi}\right) - \int (-1)\left(\frac{-\cos n\pi t}{n\pi}\right)\right]_0^1$

(this part is 0)

$= 2\left[0 + \frac{1}{n\pi}\right]$

$\therefore f(t) \sim \frac{2}{\pi}\left[\sin\pi t + \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} + \ldots\right]$

Fourier sine series (picture below)

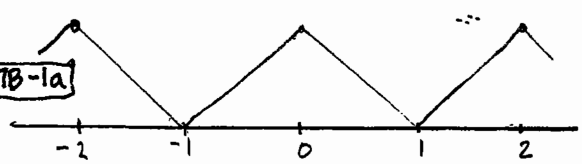
7B-3 a) $\int_{-a}^0 f(t)\,dt = \int_a^0 f(-u)(-du) = \int_0^a f(u)\,du$

f even (let $t = -u$) ($f(-u) = f(u)$)

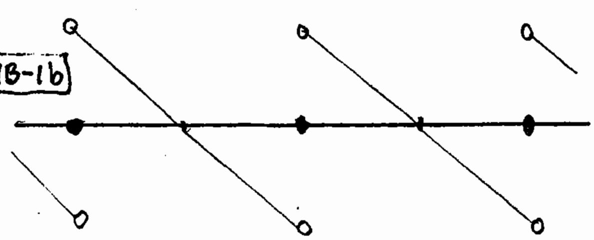
b) $\int_{-a}^0 f(t)\,dt = \int_a^0 -f(u)(-du) = -\int_0^a f(u)\,du$

f odd $t = -u$, $f(-u) = -f(u)$

7B-1a

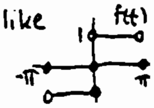


7B-1b



7B-2a $x'' + 2x = 1$, $x(0) = x(\pi) = 0$

) First expand 1 in a Fourier sine series. This means the (odd) periodic extension f(t) looks like



. We can then get a F. sine series for x(t), + it will fit the bdry. conditions.

By (21), 8.1,

$f(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$ (*)

) Look for a series $x(t) = \sum b_n \sin nt$ (this satisfies $x(0) = x(\pi) = 0$).

$x'' = \sum -b_n \cdot n^2 \sin nt$

$+\ 2x = \sum 2b_n \sin nt$ Adding

$f(x) = \sum b_n(2-n^2)\sin nt$

$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$

$\therefore b_n = 0$, n even

$b_n = \frac{4}{\pi}\cdot\frac{1}{2-n^2}\cdot\frac{1}{n}$, if n is odd

$= \frac{-4}{n(n^2-2)\pi}$, n odd.

$\therefore x(t) = \frac{-4}{\pi}\sum_{n\,\text{odd}} \frac{\sin nt}{n(n^2-2)}$, $0 \le t \le \pi$

7B-2b

$x'' + 2x = t$, $x'(0) = x'(\pi) = 0$

a) Expand t in a Fourier cosine series; (we will then get a F. cosine series for x(t), + it will satisfy the 2 endpoint conditions).

Get $a_n = \frac{2}{\pi}\int_0^\pi t\cos nt\,dt$ Integ by parts

$= \frac{2}{\pi}\left[t\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^\pi = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}$

$a_n = \begin{cases} = \frac{-4}{n^2\pi} & \text{if } n \text{ odd} \\ = 0 & \text{if } n \text{ even.}\end{cases}$ $a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$

$\therefore t \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \ldots\right)$

b) $x = \frac{A_0}{2} + \sum A_n \cos nt$ ($\times 2$)

$x'' = -\sum n^2 A_n \cos nt$ Adding,

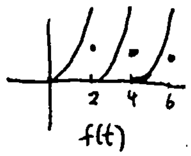
$t = A_0 + \sum A_n(2-n^2)\cos nt$

$\therefore A_0 = \frac{\pi}{2}$, $A_n = 0$ if n even $A_n = -4$

$A_n = -\frac{4}{\pi}\cdot\frac{1}{n^2(2-n^2)}$ if n odd

7B-4

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_1^\infty \frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_1^\infty \frac{\sin n\pi t}{n}$$



$$f'(t) \stackrel{?}{=} -\frac{4}{\pi^2}\sum \frac{\sin n\pi t}{n} - \frac{4}{\pi}\sum_1^\infty \cos n\pi t$$

This series doesn't converge (the cosine terms don't add up – for example, when $t=0$). So it certainly can't converge to $f'(t)$

7C-1

Preliminary remarks

$$mx'' + kx = F(t)$$

The natural frequency of the spring-mass system is $\omega_0 = \sqrt{k/m}$

The typical term of the Fourier expansion of $F(t)$ is $\cos\frac{n\pi}{L}t$, $\sin\frac{n\pi}{L}t$; thus we get pure resonance if and only if the Fourier series has a $\cos\frac{n\pi}{L}t$ or $\sin\frac{n\pi}{L}t$ term where $\boxed{\frac{n\pi}{L} = \omega_0}$

a) $\omega_0 = \sqrt{5}$ for spring-mass system

$L = 1$

Fourier series is $\sum b_n \sin n\pi t$

$n\pi \neq \sqrt{5}$ $\therefore$ no resonance

b) $\omega_0 = 2\pi$ $L=1$

Fourier series is $\sum b_n \sin n\pi t$, and $n\pi = 2\pi$ if $n=2$

Example 1, 8.4 shows that this term actually occurs in the Fourier series for $2t$ (just change scale). $\therefore$ get resonance

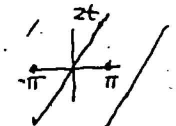
c) $\omega_0 = 3$ Fourier series is a sine series ($F(t)$ is odd):

$F(t) = \sum b_n \sin nt$ all odd n occur (see Problem 8.3/11, or ex. 1, 8.1)

$\therefore n=3$ occurs, & we get resonance.

7C-2

Fourier series for $F(t)$



will be same (up to factor 2) as the Fourier sine series in Example 1, **8.3** $(L=\pi)$

$$F(t) = 4\left(\sin t - \tfrac{1}{2}\sin 2t + \tfrac{1}{3}\sin 3t - \dots\right)$$

$$\begin{aligned} x &= \sum B_n \sin nt & \lfloor \times 3 \\ x'' &= \sum -B_n \cdot n^2 \sin nt & \text{Adding:} \\ \hline F(t) &= \sum B_n(3-n^2)\sin nt \end{aligned}$$

$$\therefore B_n = (-1)^{n+1}\cdot\frac{4}{n}\cdot\frac{1}{(3-n^2)} = \frac{(-1)^n\cdot 4}{n(n^2-3)}$$

7C-3a

The natural frequency of the undamped spring is $\omega_0 = \sqrt{18/2} = 3$

This frequency occurs in the Fourier series for $F(t)$ (see problem 3). Thus the $n=3$ term should dominate. (The actual series is

$$x_{sp}(t) \approx .25\sin(t-.0063) - .20\sin(2t-.02) + 4.44\sin(3t-1.5708) - .07\sin(4t-3.1130)\dots$$

(steady periodic soln – no transients)

7C-3b

The natural frequency of the undamped spring is $\sqrt{30/3} = \sqrt{10}$

Expanding the force in a Fourier series, since $L=1$ (half-period), & $F(t)$ is odd, it will be $F(t) = \sum b_n \sin n\pi t$

It's virtually certain all terms will occur (since $F(t)$ looks so messy) – (check soln to 8.4/5 in back of book)

$\therefore$ since $\sqrt{10} \approx \pi$, $b_1 \sin \pi t$ should be the dominant term in the $x_{sp}(t)$ series (this checks with answer given in back of book)

[NOTE: Edwards & Penney 4th edn: 8.4 (16), p. 590 has a sign error in denominators – cf. (13), which is correct.]

18.03 Solutions

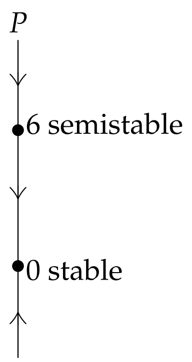
8: Extra Problems

8A. Bifurcation Diagrams

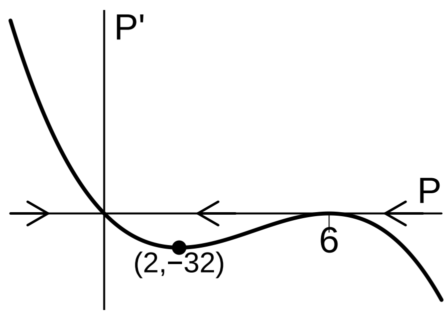
8A-1.

a) Critical points: $f(P) = 0 \Rightarrow P = 0, 6$. See below for phase line. The integral curves are not shown. Make sure you know how to sketch them as functions of P vs. t.

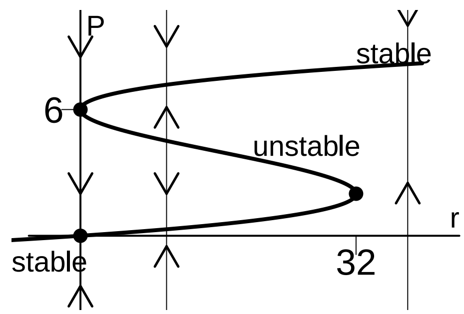
b) The picture below shows the graph of $P' = f(P)$ (i.e. when $r = 0$).
A positive r will raise the graph. As soon as $r > 32$ the graph will have only one zero and that zero will be above 6. **Solution.** $r = 32$.



Phaseline ($r = 0$)



P' vs. P (r=0)



Bifurcation

c) See above diagram. The curve of critical points is given by solving

$$P' = -P^3 + 12P^2 - 36P + r = 0 \Rightarrow r = P^3 - 12P^2 + 36P,$$

which is a sideways cubic. The phase-line for $r = 0$ is determined by the middle plot. The phase line for the other values of r then follow by continuity, i.e. the rP-plane is divided into two pieces by the curve, and arrows in the same piece have to point the same way.

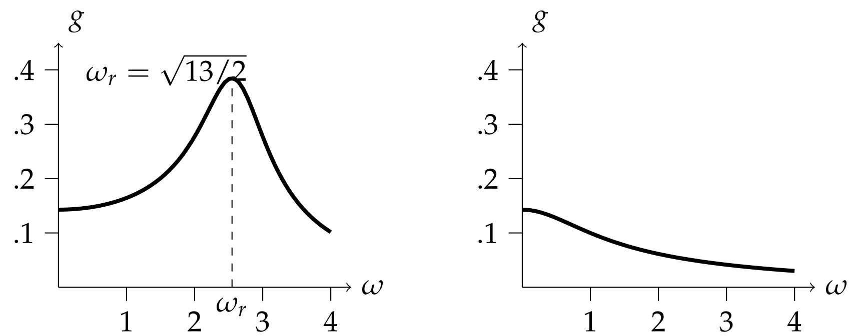
8B. Frequency Response

8B-1.

a) Characteristic polynomial: $p(s) = r^2 + r + 7$
Complexified ODE: $\tilde{x}'' + \tilde{x} + 7\tilde{x} = F_0 e^{i\omega t}$.
Particular solution (from Exp. Input Theorem): $\tilde{x}_p = F_0 e^{i\omega t}/p(i\omega) = F_0 e^{i\omega t}/(7 - \omega^2 + i\omega)$
Complex and real gain: $\tilde{g}(\omega) = 1/(7 - \omega^2 + i\omega)$, $g(\omega) = 1/|p(i\omega)| = 1/\sqrt{(7-\omega^2)^2 + \omega^2}$.
For graphing we analyze the term under the square root: $f(\omega) = (7 - \omega^2)^2 + \omega^2$.
Critical points: $f'(\omega) = -4\omega(7 - \omega^2) + 2\omega = 0 \Rightarrow \omega = 0$ or $\omega = \sqrt{13/2}$.
Evaluate at the critical points: $g(0) = 1/7$, $g(\sqrt{13/2}) = .385$
Find regions of increase and decrease by checking values of $f'(\omega)$:
On $[0, \sqrt{13/2}]$: $f(\omega) < 0 \Rightarrow f$ is decreasing $\Rightarrow g$ is increasing.
On $[\sqrt{13/2}, \infty]$: $f(\omega) > 0 \Rightarrow f$ is increasing $\Rightarrow g$ is decreasing.

The graph is given below.
This system has a (practical) resonant frequency = $\omega_r = \sqrt{13/2}$.

b) Characteristic polynomial: $p(s) = r^2 + 8r + 7$
Complex and real gain: $\tilde{g}(\omega) = 1/p(i\omega) = 1/(7 - \omega^2 + i8\omega)$, $g(\omega) = 1/|p(i\omega)| = 1/\sqrt{(7-\omega^2)^2 + 64\omega^2}$.
For graphing we analyze the term under the square root: $f(\omega) = (7-\omega^2)^2 + 64\omega^2$.
Critical points: $f'(\omega) = -4\omega(7-\omega^2) + 128\omega = 0 \Rightarrow \omega = 0$.
Since there are no positive critical points the graph is strictly decreasing.
Graph below.



Graphs for 8B-1a and 8B-1b.

8C. Pole Diagrams

8C-1.

a) All poles have negative real part: a, b, c, h.

b) All poles have nonzero imaginary part: b, d, e, f, h.

c) All poles are real: a, g.

d) Poles are real or complex conjugate pairs: a, b, c, g, h.

e) b, because the pole farthest to the right in b, is more negative than the one in c.

f) This is just the number of poles: a) 2, b) 2, c) 4, d) 2, e) 2, f) 4, g) 3, h) 2.

g) a) Making up a scale, the poles are -1 and -3 $\Rightarrow P(s) = (s+1)(s+3) \Rightarrow P(D) = D^2 + 4D + 3$.
b) Possible poles are $-3 \pm 2i \Rightarrow P(s) = (s+3-2i)(s+3+2i) \Rightarrow P(D) = D^2 + 6D + 13$.
c) Possible poles are $-1, -3, -2 \pm 2i \Rightarrow P(s) = (s+1)(s+3)(s+2-2i)(s+2+2i)$
$\Rightarrow P(D) = (D+1)(D+3)(D^2+4D+8) = D^4 + 8D^3 + 27D^2 + 44D + 24$.

h) System (h). The amplitude of the reponse is $1/|P(i\omega)|$. In the pole diagram $i\omega$ is on the imaginary axis. The poles represent values of s where $1/P(s)$ is infinite. The poles in system (h) are closer to the imaginary axis than those in system (b), so the biggest $1/|P(i\omega)|$ is bigger in (h) than (b).

9. 18.03 Linear Algebra Exercises Solutions

9A. Matrix Multiplication, Rank, Echelon Form

9A-1. (i) No. The pivots have to occur in descending rows.
(ii) Yes. There's only one pivotal column, and it's as required.
(iii) Yes. There's only one pivotal column, and it's as required.
(iv) No. The pivots have to occur in consecutive rows.
(v) Yes.

9A-2 (i) $\begin{bmatrix}4\end{bmatrix} \sim \begin{bmatrix}1\end{bmatrix}$.
(ii) This is reduced echelon already.
(iii) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$.
(iv) $\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -\frac{1}{2} & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & -\frac{3}{2} & 1\\ 0 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{2}{3}\\ 0 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & \frac{1}{3}\\ 0 & 1 & \frac{2}{3}\\ 0 & 0 & \frac{8}{3}\end{bmatrix} \sim$
$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$. (We'll soon see that this is not a surprise.)

9A-3 There are many answers. For example, $\begin{bmatrix}1\end{bmatrix}$ or $\begin{bmatrix}-1\end{bmatrix}$ work fine. Or $[\cos\theta \quad \sin\theta]$; etc.

9A-4 Here's several answers: $\begin{bmatrix}1\\ -1\\ 0\\ 0\end{bmatrix}$, $\begin{bmatrix}1\\ 1\\ 1\\ -3\end{bmatrix}$, $\begin{bmatrix}1\\ 2\\ -2\\ -1\end{bmatrix}$
The interesting thing to say is that the answer is any vector in the nullspace of $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$. The simplest solution is $\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$.

9A-5 (a) The obvious answer to this question is $\mathbf{v} = \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}$; for any matrix A with four columns, $A\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}$ is the third column of A.

But there are other answers: Remember, the general solution is any particular solution plus the general solution to the homogeneous problem. The reduced echelon form of

A may be obtained by subracting the last row from the first row: $R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

$R\mathbf{v} = \mathbf{0}$ has solutions which are multiples of $\begin{bmatrix} -3 \\ -2 \\ -1 \\ 1 \end{bmatrix}$. So for any t, $A \begin{bmatrix} 1-3t \\ 1-2t \\ 1-t \\ 1+t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) For any matrix with three rows, the product $[\,0 \quad 0 \quad 1\,]A$ is its third row. In this case, the rows are linearly independent—the only row vector $\mathbf{u}$ such that $\mathbf{u}A = \mathbf{0}$ is $\mathbf{u} = \mathbf{0}$—so there are no other solutions.

9A-6 (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$. Rank 1, as always for row $\times$ column with both nonzero.

(b) $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = [2]$ Rank 1 just because it's nonzero!

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$ Rank 2: the columns are linearly independent (as are the rows).

9B. Column Space, Null Space, Independence, Basis, Dimension

9B-1 First of all, any four vectors in $\mathbb{R}^3$ are linearly dependent. The question here is how. The first thing to do is to make the 3×4 matrix with these vectors as columns. Now linear relations among the columns are given by the null space of this matrix, so we want to find a nonzero element in this null space. To find one, row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The null space does't change under these row operations, and the null space is the space of linear relations among the columns. In the reduced echelon form it's clear that the first three columns are linearly independent. (This is clear from the original matrix, too, because the first three columns form an upper triangular matrix with nonzero entries down the diagonal.) The first three variables are pivotal, and the last is free. Set the last one equal to 1 and solve for the first two to get a basis vector for the null space:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Check: the sum of the first, second, and fourth columns is 4 times the third.

9B-2 (a) Reduction steps: row exchange the top two rows, to get a pivot at upper right; then subtract twice the new top row from the bottom row.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last transformation uses the pivot in the second row to eliminate other entries in the second column. This is now reduced echelon.

Stop and observe: The third column is twice the second minus the first. That's true in the original matrix as well! The fourth column is the 3 times the second minus twice the first. That's also true in the original matrix!

Those linear relations can be expressed as matrix multiplications. The first two variables are pivot variables and the last two are free variables. Setting $x_3 = 1$ and $x_4 = 0$ and then the other way around gives the two vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$, and they form a basis for the null space of A. Any linearly independent pair of linear combinations of them is another basis for the null space.

Similarly for A^T:

$$A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two variables are again pivotal, and the third is free. Set it equal to 1 and solve for the pivotal variables to find the basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Any nonzero multiple of this vector is another basis for the null space.

(b) We need a particular solution to $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. You could form the augmented matrix, by adjoining $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a fifth column, and row reduce the result. Maybe it's easier to just look and see what's there. For example the difference of the first two columns is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: so $\mathbf{x_p} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ works fine. Then the general solution is

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Then we need a particular solution to $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Again, the difference of the first two columns works; so the general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9B-3 (a) This is easy to do directly: all such vectors are multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, so that by itself forms a basis of this subspace. But we can do this as directed as well. One way to express the conditions is to say that $x_1 = x_2$, and $x_2 = x_3$, and $x_3 = x_4$. These three equations can be represented by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The fourth variable is free, and if we set it equal to 1 we find that the other three are 1 as well.

(b) A matrix representation of this relation is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = 0$. This is already in reduced echelon form! The first variable is pivotal, the last three are free. We find $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Now there are two equations, represented by the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$. The first two variables are pivotal and the last two are free. A basis: $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Looking back at the original equations, this makes sense, doesn't it?

9B-4 (a) If $c \neq 0$, there are pivots on three rows and the rank is 3. So $c = 0$. The rank will still be 3 unless the third row is a linear combination of the first two. The first column can't occur (because of the nonzero entries on the left), so we must have $d = 2$

(b) The answer is the same, since dim(Null space)+dim(Column space)=width.

9C. Determinants and Inverses

9C-1 (a) $R(\alpha)R(\beta) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$

$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R(\alpha+\beta).$

Geometrically, $R(\theta)$ rotates vectors in the plane by θ radians counterclockwise.

(b) $\det R(\theta) = (\cos\theta)^2 + (\sin\theta)^2 = 1$. $R(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(-\theta)$.

9C-2 (a) $\det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$; $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(b) $\det \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = 1$ (by cross-hatch, or because the determinant of an upper-triangular matrix is the product of the diagonal entries). To find the inverse, row reduce

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

so the inverse is $\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2$ by crosshatch. So we expect a 2 in the denominator of the inverse.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

so the inverse is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

The original matrix was symmetric. Is it an accident that the inverse is also symmetric?

(d) $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

9D. Eigenvalues and Eigenvectors

9D-1 (a) The eigenvalues of upper or lower triangular matrices are the diagonal entries: so for A we get 1 and 2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is clearly an eigenvector for value 1, as is any multiple. For $\lambda = 2$, we want a vector in the null space of $A - \lambda I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple will do nicely.

For B, the eigenvalues are 3 and 4. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is clearly an eigenvector for $\lambda = 4$, as is any multiple. For $\lambda = 3$, we want a vector in the null space of $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any multiple will do nicely.

(b) $AB = \begin{bmatrix} 4 & 4 \\ 2 & 8 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - (\mathrm{tr}A)\lambda + \det A = \lambda^2 - 12\lambda + 24 = (\lambda - 6)^2 - (36 - 24)$ has roots $\lambda_{1,2} = 6 \pm \sqrt{12}$.

(c) If $A\mathbf{x} = \lambda\mathbf{x}$, then $(cA)\mathbf{x} = cA\mathbf{x} = c\lambda\mathbf{x}$, so $c\lambda$ is an eigenvalue of cA. If $c = 0$ then $cA = 0$ and its only eigenvalue is 0; otherwise, this argument is reversible, so the eigenvalues of cA are exactly c times the eigenvalues of A.

(d) $A + B = \begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - 10\lambda + 23 = (\lambda - 5)^2 - (25 - 23)$ and so $\lambda_{1,2} = 5 \pm \sqrt{2}$.

9D-2 (a) $\lambda^2 - 2\lambda + 1 = (1 - \lambda)^2$.

(b) $(1 - \lambda)^3$.

(c) $\det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = (-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda) = -\lambda^3 + 3\lambda + 2$.

(d) $(1 - \lambda)(2 - \lambda)(3 - \lambda)(4 - \lambda)$.

9D-3 The eigenvalue equation is $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mathbf{v} = \lambda\mathbf{v}$. Let's write $\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$, where $\mathbf{x}$ has m components and $\mathbf{y}$ has n components. Then the eigenvalue equation is equivalent to the two equation $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{y} = \lambda\mathbf{y}$, for the same λ. λ is an eigenvalue if there is *nonzero* vector satisfying this equation. This means that *either* $\mathbf{x}$ *or* $\mathbf{y}$ must be nonzero. So one possiblity is that λ is an eigenvalue of A, $\mathbf{x}$ is a nonzero eigenvector for A and this eigenvalue, and $\mathbf{y} = \mathbf{0}$. Another possibility is that λ is an eigenvalue of B, $\mathbf{x} = \mathbf{0}$, and $\mathbf{y}$ is a nonzero eigenvector for B with this eigenvalue. Conclusion: The eigenvalues of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n$.

9D-4 The eigenvalue equation is $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$. This expands to $A\mathbf{y} = \lambda\mathbf{x}$ and $B\mathbf{x} = \lambda\mathbf{y}$. We want either $\mathbf{x}$ or $\mathbf{y}$ to be nonzero. We compute $AB\mathbf{x} = A(\lambda\mathbf{y}) = \lambda A\mathbf{y} = \lambda^2\mathbf{x}$ and $BA\mathbf{y} = B\lambda\mathbf{x}) = \lambda B\mathbf{x} = \lambda^2\mathbf{y}$. So if $\mathbf{x} \neq \mathbf{0}$, then λ^2 is an eigenvalue of AB, and if $\mathbf{y} \neq \mathbf{0}$ then λ^2 is an eigenvalue of BA. Enough said.

9E. Two Dimensional Linear Dynamics

9E-1 (a) Write $\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$, so $\dot{\mathbf{u}} = A\mathbf{u}$ with $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. (Attention to signs!)

(b) A is singular (columns are proportional) so one eigenvalue is 0. The other must be -2 because the trace is -2.

(c) 0 has eigenvectors given by the multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so one normal mode solution is the constant solution $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple. This is the steady state, with the same number of people in both rooms. To find an eigenvector for the eigenvalue -2, form $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; so we have $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple. The corresponding normal mode is $\mathbf{u_2} = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and the general solution is $\mathbf{u} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. With $\mathbf{u}(0) = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$, this gives the equations $a + b = 30, a - b = 10$, or $a = 20, b = 10$: so $\mathbf{u} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The first term is the steady state; the second is a transient, and it decays rapidly to zero.

When $t = 1$ we have $v(1) = 20 + 10e^{-2} \sim 21.36$ and $w(1) = 20 - 10e^{-2} \sim 18.64$.

(d) When $t = \infty$ (so to speak), the transient has died away and the rooms have equalized at 20 each.

9E-2 (a) This is the companion matrix for the harmonic oscillator, as explained in LA.7, with $\omega = 1$. The basic solutions are $\mathbf{u_1}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\mathbf{u_2}(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, and the general solution is a linear combination of them. A good way to think of this is to remember that the second entry is the derivative of the first, and the first can be any sinusoidal function of angular frequency 1: so let's write it as $\mathbf{u} = \begin{bmatrix} A\cos(t-\phi) \\ -A\sin(t-\phi) \end{bmatrix} = A\begin{bmatrix} \cos(t-\phi) \\ \sin(t-\phi) \end{bmatrix}$.

For these to lie on the unit circle we should take $A = 1$. That's it: $\mathbf{u} = \begin{bmatrix} \cos(t-\phi) \\ \sin(t-\phi) \end{bmatrix}$ for any ϕ. ϕ is both phase lag and time lag, since $\omega = 1$. The solution goes through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t = \phi$.

(b) Now the characteristic polynomial is $p_A(\lambda) = \lambda^2 - 1$, so there are two real eigenvalues, 1 and -1. [The phase portrait is a saddle.] Eigenvectors for 1 are killed by $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and multiples. The matrix is symmetric, so the eigenvectors for -1 are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and multiples. So the normal modes are $\mathbf{u_1} = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u_2} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution is $\mathbf{u} = ae^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + be^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

In order for $\mathbf{u}(t_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for some t_0, you must have $ae^{t_0} = 1$ and $be^{-t_0} = 0$, because the two eigenvectors are linearly independent. So $b = 0$, and a can be any positive number (because you can then solve for t_0): $ae^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $a > 0$. These are "ray" solutions.

9F. Normal Modes

9F-1 The characteristic polynomial is $p(s) = s^4 - c$. Its roots are the fourth roots of c. Write r for the real positive fourth root of $|c|$. If $c > 0$, the fourth roots of c are $\pm r$ and $\pm ir$. There are four exponential solutions, but only the ones with imaginary exponent give rise to sinusoidal solutions. They are the sinusoidal functions of angular frequency r. If $c < 0$, the fourth roots of c are $r\frac{\pm 1 \pm i}{\sqrt{2}}$. None of these is purely imaginary, so there are no (nonzero) sinusoidal solutions. When $c = 0$ the equation is $\frac{d^4x}{dt^4} = 0$. The solutions are the polynomials of degree at most 3. The constant solutions are sinusoidal.

9G. Diagonalization, Orthogonal Matrices

9G-1 The null space is (width)-(rank)=9 dimensional, so there are 9 linearly independent eigenvectors for the eigenvalue 0. There's just one more eigenvalue, which can't be zero because the sum of the eigenvalues is the trace, 5. It has to be 5.

9G-2 (a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ has eigenvalues 1 and 3. 1 has nonzero eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (or any nonzero multiple; by inspection or computation). For 3, $A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ so we have nonzero eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or any nonzero multiple). With these choices, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ has eigenvectors 0 (because it's singular) and 4 (because its trace is 4; or by computation). An eigenvector for 0 is a vector in the null space, so e.g. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple will do. For 4, $A - 4I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ so we have nonzero eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (or any nonzero multiple). With these choices, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

(b) $A^3 = S\Lambda^3 S^{-1}$; to be explicit, $\Lambda^3 = \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$.

$A^{-1} = S\Lambda^{-1}S^{-1}$; to be explicit, $\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$.

9G-3 [The (i, j) entry in $A^T A$ is the dot product of the ith and jth columns of A. The columns form an orthonormal set exactly when this matrix of dot products is the identity matrix.]

$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3)$, so eigenvalues -1 and -3 and $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$. Eigenvectors for 1 are killed by $A - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any nonzero multiple. We could find an eigenvector for 3 similarly, or just remember that eigenvectors for distinct eigenvalues of a symmetric matrix are orthogonal, and write down $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Each of these vectors has length $\sqrt{2}$, so an orthogonal S is given by $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(There are seven other correct answers; one could list the eigenvalues in the opposite order, and change the signs of the eigenvectors.)

9H. Decoupling

9H-1 The rabbit matrix is $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. We found eigenvalues $2, 5$ and eigenvector matrix $S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, with inverse $S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. The decoupling coordinates are the entries in $\mathbf{y}$ such that $S\mathbf{y} = \mathbf{x}$, or $\mathbf{y} = S^{-1}\mathbf{x} = \frac{1}{3}\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$: $y_1 = 2x_1 - x_2$ (for eigenvalue 2) and $y_2 = x_1 + x_2$ (for eigenvalue 5), or any nonzero multiples of these, do the trick. $y_1(t) = e^{2t}y_1(0)$, $y_2(t) = e^{5t}y_2(0)$. $y_2(t)$ is the sum of the populations, and it grows exponentially with rate 5, just as if there was no hedge.

9I. Matrix Exponential

9I-1 By time invariance the answer will be the same for any t_0; for example we could take $t_0 = 0$; so $\begin{bmatrix} x(t_0+1) \\ y(t_0+1) \end{bmatrix} = e^A \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix}$. Then we need to compute the exponential matrix. We recalled the diagonalization of the rabbit matrix above, so $e^{At} = Se^{\Lambda t}S^{-1}$, and $e^A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

9I-2 $p_A(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$ has roots $-1 \pm i$. Eigenvectors for $\lambda = -1+i$ are killed by $A - (-1+i)I = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$; for example $\begin{bmatrix} -1 \\ 1-i \end{bmatrix}$. The exponential solutions are $e^{(-1+i)t}\begin{bmatrix} -1 \\ 1-i \end{bmatrix}$ and its complex conjugate, so we get real solutions as real and imaginary parts, which we put into the columns in a fundamental matrix:

$$\Phi(t) = e^{-t}\begin{bmatrix} -\cos t & -\sin t \\ \cos t + \sin t & -\cos t + \sin t \end{bmatrix}$$

$\Phi(0) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}$$

9J. Inhomogeneous Solutions

9J-1

(a) As always, substitution of $\mathbf{u_p} = e^{2t}\mathbf{v}$ gives the exponential response formula

$$\mathbf{v} = \begin{bmatrix} -4 & -5 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 10 \\ -7 \end{bmatrix} \Rightarrow \mathbf{u_p} = -\frac{1}{5}e^{2t}\begin{bmatrix} 10 \\ -7 \end{bmatrix}.$$

(b) Complex replacement and the trial solution $\mathbf{z_p} = e^{2it}\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix} 2i-1 & 0 \\ -1 & 2i-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{-2-6i} \begin{bmatrix} 2i-2 \\ 1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -4-8i \\ -1+3i \end{bmatrix}$$

So

$$\begin{aligned} \mathbf{z_p} &= \frac{1}{20}(\cos(2t) + i\sin(2t)) \begin{bmatrix} -4-8i \\ -1+3i \end{bmatrix} \\ &= \frac{1}{20}\left(\begin{bmatrix} -4\cos(2t) + 8\sin(2t) \\ -\cos(2t) - 3\sin(2t) \end{bmatrix} + i \begin{bmatrix} -4\sin(2t) - 8\cos(2t) \\ -\sin(2t) + 3\cos(2t) \end{bmatrix} \right) \end{aligned}$$

Therefore

$$\mathbf{u_p} = \mathrm{Re}\,(\mathbf{z_p}) = \frac{1}{20} \begin{bmatrix} -4\cos(2t) + 8\sin(2t) \\ -\cos(2t) - 3\sin(2t) \end{bmatrix}$$

9J-2 We want to compute a particular solution to $\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} -e^t \\ 0 \end{bmatrix}$, using $\mathbf{x}(t) = \Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} dt$.

$\Phi(t) = Se^{\Lambda t}$, so $\Phi(t)^{-1} = e^{-\Lambda t}S^{-1}$. $S^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} = -\frac{1}{3}e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\Phi(t)^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} = e^{-\Lambda t}S^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-5t} \end{bmatrix} e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 2e^{-t} \\ e^{-4t} \end{bmatrix}$.

$\int \Phi(t)^{-1} \begin{bmatrix} -e^t \\ 0 \end{bmatrix} dt = \frac{1}{3} \begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix}$.

$\mathbf{x}(t) = \Phi(t)\frac{1}{3} \begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix} = \frac{1}{3}S \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ \frac{1}{4}e^{-4t} \end{bmatrix} = \frac{1}{12}e^t S \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \frac{e^t}{12} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \frac{e^t}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

We still need to get the initial condition right. We need a solution to the homogeneous equation with inital value $\frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Luckily, this is an eigenvector, for value 5, so the relevant solution is $\frac{e^{5t}}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the solution we seek is $\frac{e^t}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{e^{5t}}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

10. 18.03 PDE Exercises Solutions

10A. Heat Equation; Separation of Variables

10A-1 (i) Trying a solution $u(x,t) = v(x)w(t)$ leads to separated solutions $u_k(x,t) = v_k(x)w_k(t)$ where $v_k(x) = \sin(\pi kx)$, and $w_k(t) = e^{-2\pi^2k^2t}$, and $k = 0, 1, 2, \ldots$.

(ii) $u(x,0) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(\pi kx)$

(iii) $u(x,t) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(\pi kx)e^{-\pi^2k^2t}$

(iv) $u(\frac{1}{2}, 1) \approx .00032$

10A-2 (i) Separated solutions $u_k(x,t) = v_k(x)w_k(t)$ where $v_k(x) = \sin(\pi kx)$, and $w_k(t) = e^{-2\pi^2k^2t}$ (Note factor of 2 from (10A-2.1)) and $k = 0, 1, 2, \ldots$

(ii) $u(x,0) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi kx)}{k}$

(iii) $u(x,t) = \frac{4}{\pi}\sum_{k \text{ odd}} \frac{\sin(\pi kx)}{k} e^{-2\pi^2k^2t}$

10A-3 (a) $u_{st}(x,t) = U(x) = 1 - x$

(b) Since the heat equation is linear $\tilde{u}$ satisfies the PDE (10A-3.1). At the boundary ($x = 0$ and $x = 1$) we have $\tilde{u}(0,t) = u(0,1) - u_{st}(0,t) = 1 - 1 = 0$. Likewise $\tilde{u}(1,t) = 0$. That is, $\tilde{u}$ is a solution to the heat equation with homogeneous boundary conditions in 10A-1. The initial condition is $\tilde{u}(x,0) = x$. We found the coefficients for this in 10A-1.

$$\tilde{u}(x,t) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(\pi kx)e^{-2\pi^2k^2t}.$$

(c)

$$u(x,t) = U(x) + \tilde{u}(x,t) = 1 - x + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(\pi kx)e^{-2\pi^2k^2t}.$$

(d) $u(x,t) - U(x) = \tilde{u}(x,t)$ The term in the sum for $\tilde{u}$ that decays the slowest is when $k = 1$. Therefore we need $\frac{2}{\pi}e^{-2\pi^2T} = .01U(1/2) = .005$. Solving we get $T = .246$

10B. Wave Equation

10B-1 (a) Separating variables, we look for a solutions of the form $u(x,t) = v(x)w(t)$, which leads to $v''(x) = \lambda v(x)$ with $v(0) = v(\pi/2) = 0$, and hence

$$v_k(x) = \sin(2kx)$$

Consequently, $\ddot{w}_k = -(2k)^2c^2w_k$, whic implies

$$w_k(t) = A\cos(2ckt) + B\sin(2ckt)$$

The normal modes are

$$u_k(x,t) = \sin(2kx)(A\cos(2ckt) + B\sin(2ckt)),$$

where A and B must be specified by an initial position and velocity of the string.

(b) The main note (from the mode u_1) has frequency $\dfrac{2c}{2\pi} = \dfrac{c}{\pi}$. You will also hear the higher harmonics at the frequencies $\dfrac{ck}{\pi}$, $k = 2, 3, \ldots$. (The sound waves induced by the vibrating string depend on the frequency in t of the modes.)

(c) Longer strings have lower frequencies, lower notes, and shorter strings have higher frequencies, higher notes. If the length of the string is L, then the equations $v''(x) = \lambda v(x)$, $v(0) = v(L) = 0$ lead to solutions $v_k(x) = \sin(k\pi x/L)$. (In part (a), $L = \pi/2$.) The associated angular frequencies in the t variable are $kc\pi/L$, so the larger L, the smaller $kc\pi/L$ and the lower the note. Thus c is inversely proportional to the length of the string.

(d) When you tighten the string, the notes get higher, and the frequency you hear is increased. (Tightening the string increases the tension in the string and increases the spring constant, which corresponds to c. The frequencies of the sounds are directly proportional to c.)

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises