SP. Supplementary Problems

SP1. Vectors and Vector Functions

SP1-A-1. Using the two expressions for the dot product of two vectors, prove the law of cosines, $|C|^2 = |A|^2 + |B|^2 - 2|A||B|\cos\theta$, using vector methods.

(Since the equality of the two expressions for the dot product was proved using the law of cosines, this shows that the cosine law and the dot product equality are equivalent theorems: either implies the other.)

SP1-A-2. Prove using vectors that the diagonals of a parallelogram are equal if and only if it is a rectangle.

SP1-A-3. Prove using vector methods that the four midpoints of a quadrilateral in 3-space lie in a plane and are the vertices of a parallelogram.

SP1-A-4. a) Let $\mathbf{i}'$ and $\mathbf{j}'$ be perpendicular unit vectors in the plane. Then any vector A can be expressed in terms of $\mathbf{i}'$ and $\mathbf{j}'$:

$$A = a\,\mathbf{i}' + b\,\mathbf{j}'\ .$$

Show how to determine the coefficients a and b in terms of A, $\mathbf{i}'$, and $\mathbf{j}'$, by using scalar products.

b) Suppose $\mathbf{i}' = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Find $\mathbf{j}'$, and then express the vector $A = 2\,\mathbf{i} + 3\,\mathbf{j}$ in the $\mathbf{i}'$-$\mathbf{j}'$ system by using the method of part (a).

c) Do part (b) another way by first expressing $\mathbf{i}, \mathbf{j}$ in terms of $\mathbf{i}', \mathbf{j}'$, and then substituting. (This method is substantially harder to carry out by hand in higher dimensions; one needs good computer software.)

SP1-A-5. Let A and B be vectors in space. Prove that $|A+B|^2 = |A|^2 + |B|^2$ if and only if A and B are orthogonal.

SP1-B-1. Let $A = \mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$. Find a vector C such that $A \cdot C = 1$ and $A \times C = -18\,\mathbf{i} + 9\,\mathbf{k}$.

SP1-B-2. Three vertices of a parallelogram $ABCD$ (labeled in clockwise order) are $A(0,1,-2)$, $B(1,3,-1)$, and $D(-1,4,0)$. Find the fourth vertex and the area of the parallelogram.

SP1-C-1. Invert the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix}$ and use your answer to solve $AX = \begin{pmatrix} 6 \\ 0 \\ -12 \end{pmatrix}$

SP1-D-1. a) Find the intersection of the planes $x - y + 2z = 10$ and $z - y = 2$.

b) Find the angle between the two planes.

SP1-D-2. a) Find in parametric form the equation of the line through the points $(1,2,-1)$ and $(2,1,3)$.

b) Where does this line intersect the plane $2x-3y+z+14=0$?

SP1-D-3. At noon, a snail starts at the center of an open clock face on a public building. It creeps at a steady rate along the minute hand, reaching the end of the hand at 1:00 PM. The minute hand is 1 meter long.

Write parametric equations for the position of the snail at time t (in hours), taking the center of the clock face as the origin in the xy-plane.

SP1-D-4. a) What part of a train is moving backwards when the train moves forwards?

b) A circular disc has inner radius a and outer radius b. Its inner circle rolls along the positive x-axis without slipping.

Find parametric equations for the motion of a point P on its outer edge, assuming P starts at the point $(0, a-b)$. Use as parameter the angle θ through which the disc has rolled.

c) Sketch the curve that P traces out.

d) Show that the parametric equations imply that P is moving backwards in an interval containing its lowest point.

SP1-E-1. Suppose a particle moves along a circle of radius a and center at the origin.

a) Express its position vector R in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

b) Without using your book, derive expressions for the velocity and acceleration vectors for the above motion, in terms of $\mathbf{u}_r, \mathbf{u}_\theta$.

c) Compare your answers in part (b) with the general expression for $\mathbf{v}$ and $\mathbf{a}$ in your book, and explain why certain terms are missing in your answer.

SP1-E-2. a) Suppose a particle moves along a fixed ray going out from the origin. Show its acceleration vector is $\mathbf{a} = r''\mathbf{u}_r$.

b) Conversely, show that if a particle moves so that $\mathbf{a} = r''\mathbf{u}_r + f(t)\mathbf{u}_\theta$, where $f(t)$ is any differentiable function of t, then it moves along a ray going out from the origin, and $f(t) = 0$. (Use the book's formulas called for in problem E1 above.)

SP1-E-3. Using the same formula in your book as in E1 and E2 above, show:

a) if the motion takes place on a circle of radius a centered at the origin, then the angular acceleration is $a\omega'$, and the radial acceleration is $-a\omega^2$, where ω is the angular velocity;

b) if the angular velocity is a non-zero constant and the force is central, then we have uniform motion on a circle centered at the origin;

c) if there is no radial component to the acceleration and r is a linear function of time, then we have uniform motion along a fixed ray.

SP2- Partial Differentiation

SP2-A-1. A frustum of a solid right circular cone is the piece between two horizontal planes perpendicular to the axis of the cone. (It looks like a solid lampshade.) If a and b are the radii of the two circular faces of the frustum, its volume is

$$V = \frac{\pi}{3}(a^2 + ab + b^2)h \ .$$

a) Give an approximate expression for ΔV in terms of $h, a, b, \Delta a, \Delta b, \Delta h$.

b) If $a = 1, b = 2, h = 2$, to which variable is V most sensitive?

SP2-B-1. Use the method of least squares to find the best line through the three points $(0,0), (1,2)$, and $(2,2)$. Sketch the points and the line. (Do the work from scratch, as a minimum problem; don't use a formula.)

SP2-B-2. a) Guess what line the method of least squares will give as the best fit to the three points $(-1,0), (0,1)$, and $(1,0)$. (Plot the points and eyeball them.)

b) Then calculate the least squares line. (Do it as a minimum problem; don't use a formula.)

SP2-C-1. Let $f(u)$ be differentiable, and let $g(x,y) = f\left(\dfrac{x+y}{xy}\right)$. Prove that $x^2 g_x = y^2 g_y$.

SP2-C-2. a) Find the directional derivative of $f(x,y) = \cos \pi xy + xy^2$ in the direction of $A = \mathbf{i} + \mathbf{j}$, at $P(-1,1)$.

b) Use the definition of directional derivative and your answer to part (a) to find approximately how much $f(x,y)$ changes as we move from P a distance .1 along A.

SP2-C-3. a) Define what is meant by a contour curve of $f(x,y)$.

b) Prove that at any point (a,b), the vector ∇f is perpendicular to the contour curve of f passing through (a,b).

SP2-C-4. a) Find the equation of the tangent plane to the surface $3x^2 - y^2 + 3z^2 = 0$ at (x_0, y_0, z_0).

b) Show that the tangent plane always makes an angle of 60^o with the xy-plane.

SP2-C-5. a) Find the equation of the tangent plane to the surface $e^{2xy} + 2x^3z^2 - \sin \pi yz = 1$ at (0,1,1).

b) Use your answer to approximate z by a linear function of x and y for $(x,y) = (0,1)$.

SP3. Double Integrals

SP3-A-1. Evaluate $\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dx\, dy$ by changing the order of integration.

SP3-A-2. Find the average distance from the origin to the points in the triangle with vertices at the origin, $(a, 0)$ and $(a, a/\sqrt{2})$.

SP3-B-1. Find the center of mass of the plane region lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

SP3-B-2. Find the average distance from the origin to the points inside the circles

(a) $x^2 + y^2 = a^2$; (b) $(x-1)^2 + y^2 = 1$.

SP4. Line integrals, Conservative Fields, Green's Theorem

SP4-A-1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 1 + z)\,\mathbf{j} + yz\,\mathbf{k}$, and C is the curve:

a) the line from $(1, 1, 1)$ to $(2, 1, -2)$

b) the semicircle running counterclockwise from $(1, 1, 0)$ to $(-1, -1, 0)$.

SP4-B-1. Let $\mathbf{F} = (y^2 + 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$. Find a function f such that $F = \nabla f$. Do this by calculating $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a curve C consisting of line segments running from the origin to (x_1, y_1), first in the $\mathbf{i}$ direction, then in the $\mathbf{j}$ direction.

SP4-B-2. Carry out the work of the previous problem for $\mathbf{F} = (2xy + z)\,\mathbf{i} + x^2\,\mathbf{j} + x\,\mathbf{k}$ (the curve C runs from the origin to (x_1, y_1, z_1) and is made up of three line segments).

SP4-B-3. Define "conservative vector field", and prove that ∇f is a conservative vector field.

SP4-B-4. a) Let $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$ be a continuously differentiable vector field. Prove that if $\mathbf{F}$ is a gradient field, then $M_y = N_x$.

b) Show the converse is false, by calculating $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$ and C is $x^2 + y^2 = 1$. (Why does the answer show the converse is false?)

SP4-B-5. If the $\mathbf{j}$-component of ∇f is proportional to y^3, find $f(1, 2, -1) - f(1, -2, -1)$.

SP4-B-6. If $f_y = 3xz^2 + x^2 y$ and $f_x = 3yz^2 + xy^2 + \cos(\pi x/2)$, find $f(1, 2, -1) - f(5, -1, -1)$.

SP4-C-6. By using Green's theorem, show that if the derivatives are continuous and $M_y = N_x$ in a simply-connected region of the plane (one with no holes), then $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a conservative field in this region.

SP4-D-1. Evaluate $\int \mathbf{F} \cdot \mathbf{n} ds$ where $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$ and C is $(t+1)\,\mathbf{i} + t^2\,\mathbf{j}$, $0 \le t \le 1$.

SP4-D-2. Find the flux of $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ outward across the triangle running from (0,0) to (1,0) to (0,1) and back to (0,0).

SP4-D-3. Let $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j})$. For what values of the integer n will the flux of $\mathbf{F}$ across a circle of radius a and center at the origin be independent of a?

SP4-D-4. Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$. Show that if div $\mathbf{F} = 0$, then the flux of $\mathbf{F}$ across all closed curves C is zero.

SP5. Triple Integrals

SP5-A-1. Find the volume of the region bounded by the elliptic paraboloids $z = x^2 + 4y^2$ and $z = 8 - x^2 - 4y^2$.

(Hint: make a change of variable $x = 2u$, calculate the volume in the uyz-system; how will this be related to the volume in the xyz-system?)

SP5-A-2. Set up an iterated integral in cylindrical coordinates giving the moment of inertia about the z-axis of the region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the paraboloid $2z = x^2 + y^2$.

SP5-A-3. Find the center of mass of the region bounded above by the plane $2x + 4y - z = 0$ and below by the paraboloid $z = x^2 + y^2$.

SP5-B-1. Find the average distance of the points in a sphere of radius a from

(a) its center (b) a fixed plane passing through its center.

SP5-B-2. Find the average distance from the origin to the points in the solid cone given in spherical coordinates by $0 \le \phi \le \pi/4, \quad 0 \le \rho\cos\phi \le 1$.

SP5-B-3. Find the gravitational attraction of a solid right circular cone (base radius is R, height h) on a unit mass at its vertex. Take the density $\delta = 1$. (Place the cone so as to make the calculation easiest.)

SP5-B-4. Evaluate $\iiint_V z^2 dV$, where V is the region bounded above by the unit sphere with center at the origin, and below by the unit sphere with center at $x = y = 0, z = 1$.

(Begin by writing the equations of both spheres in spherical coordinates.)

SP5-B-5. Let the density of a region in space be given by $\delta(x,y,z) = xyz$. Let V be the volume bounded above by the sphere of radius a with center at the origin, and below by the cone $3z = 2\sqrt{3}\sqrt{x^2+y^2}$. Find the average density of the region V.

SP6. Surface Integrals, Divergence Theorem, Stokes' Theorem

SP6-A-1. Let S denote the closed, positively oriented surface whose top is the cap of the sphere $x^2 + y^2 + z^2 = 2a^2$, $z \geq a$, and whose bottom is the flat circular disc at the height $z = a$. Find the flux of $\mathbf{F} = xz\,\mathbf{i} - yz\,\mathbf{k} + y^2\,\mathbf{k}$ outward through S

a) directly, by calculating the two surface integrals;

b) by applying the divergence theorem.

SP6-A-2 Evaluate $\int\int_S F \cdot \mathbf{n}\, dS$, where S is the sphere of radius a centered at the origin, and $F = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

a) directly, by evaluating the surface integral;

b) by applying the divergence theorem.

SP6-A-3 Suppose div $\mathbf{F} = 0$. Show that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ has the same value for any two similarly oriented and non-intersecting surfaces spanning the same simple closed curve C. Interpret the result physically in terms of fluid flow.

(Take two such surfaces, S and T; apply the divergence theorem to the region they surround.)

SP6-A-4. a) Show that the gravitational field exerted by a mass at the origin is of the form

$$\mathbf{F}(x, y, z) = c(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})(x^2 + y^2 + z^2)^{3/2}\ .$$

b) Find the flux of $\mathbf{F}$ through a sphere of radius a centered at the origin.

c) Calculate div $\mathbf{F}$. What is its value at (0,0,0)?

d) Show the flux of $\mathbf{F}$ through a closed surface not containing the origin is zero.

e) Show that the flux of $\mathbf{F}$ through a closed surface S containing the origin is $4\pi c$. (Take a sufficiently small sphere S' centered at the origin, and apply the divergence theorem to the region between S and S'.)

SP6-A-5. a) Show that the flux of the position vector $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

b) Verify part (a), by calculating the surface integral, when S is a sphere of radius a, centered at the origin.

c) Verify part (a), by calculating the surface integral, when S is the cube centered at the origin and having edges of length 2 parallel to the three coordinate axes. (It is not necessary to calculate six surface integrals: use symmetry.)

d) Let $\mathbf{n}$ be the unit normal (pointing outwards) for a closed surface S. Show that it is impossible for the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ to be orthogonal to $\mathbf{n}$ at every point on the surface.

SP6-B-1. a) Calculate curl $y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

b) Prove that under suitable hypotheses about $f(x, y, z)$, we have curl $\nabla f = 0$.

Selected Solutions and Hints

SP1.

SP1-A1. $|C|^2 = C \cdot C = (A - B) \cdot (A - B)$. SP1-A2. Two diagonals are $A - B$ and $A + B$.

SP1-A4. a) $a = A \cdot \mathbf{i}'; b = A \cdot \mathbf{j}'$. b) $\mathbf{j}' = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$; $A = (5\,\mathbf{i}' + \mathbf{j}')/\sqrt{2}$

SP1-A5. Essentially same as SP1-A2.

SP1-B1. $2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$ SP1-B2. $(0, 6, 1), \sqrt{35}$

SP1-C1. $A^{-1} = [5/6, -2/3, -1/2; 1/6, -1/3, -1/2; -1/2, 1, 1/2]$; $X = [11; 7; -9]$ (column vector)

SP1-D1. a) $x = t, y = 6 - t, z = 8 - t$ b) $\pi/6$ SP1-D2. a) $x = 1 + t, y = 2 - t, z = -1 + 4t$ b) (0,3,-5)

SP2.

SP2-A1. b)b SP2-B1. $y = x + 1/3$ SP2-B2. $y = 1/3$ SP2-C2. a) $-1/\sqrt{2}$ b) $-1/10\sqrt{2}$

SP2-C3. b) Parametrize the curve by $x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and differentiate the function $f(x(t), y(t))$.

SP2-C5. a) $2x + \pi y + \pi z = 2\pi$ b) $z = 2 - (2/\pi)x - y$

SP3.

SP3-A1. $1 - \sin 1$ SP3-A2. $(2a/\sqrt{3})(1/3 + (\ln 3)/4)$ SP3-B1.. $\overline{x} = a(15\pi + 32)/6(8 + \pi)$, $\overline{y} = 0$

SP3-B2. a)$2a/3$ b) $32/9\pi$

SP4.

SP4-A1. a) $-\pi$ b) 4 c) $-1/12$

SP4-B1 a) $x^2 + xy^2$ b) $x^2y + xz$ SP4-B4. b) $\oint F \cdot d\mathbf{r} = 2\pi$, so field is not conservative.

SP4-B5. 0 B6. -42

SP4-C1. Hint: the double integral is like a center of mass integral – use this to simplify calculations (use symmetry, etc.)

SP4-D1. 9/4 SP4-D2. 1 SP4-D3. $n = -2$

SP5.

SP5-A1. 8π SP5-A2 $\int_0^{2\pi} \int_0^2 \int_{a^2/2}^{\sqrt{8-r^2}} r^3 dz dr d\theta$ SP5-A3. $(1, 2, 25/3)$

SP5-B1. a) $3a/4$ b) $3a/8$ SP5-B2. $\sqrt{2} - 1/2$ SP5-B3. $2\pi G h (1 - h(h^2 + R^2)^{-1/2})\,\mathbf{k}$

SP5-B4. $59\pi/480$

SP6.

SP6-B1. a) $-(\mathbf{i} + \mathbf{j} + \mathbf{k})$ SP6-B2. a) $a = 6, b = 4, c = 2$ b) $f = 3x^2yz + xy^3z^2 + 2y^2z^3$

SP6-C1. a) 0 b) $-\pi$ C3. 0

18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T.Shifrin and S. LeDuc