6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 a) the vectors are all unit vectors, pointing radially outward.
b) the vector at P has its head on the y-axis, and is perpendicular to it

6A-2 $\frac{1}{2}(-x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k})$

6A-3 $\omega(-z\,\mathbf{j} + y\,\mathbf{k})$

6A-4 A vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is parallel to the plane $3x - 4y + z = 2$ if it is perpendicular to the normal vector to the plane, $3\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$: the condition on M, N, P therefore is $3M - 4N + P = 0$, or $P = 4N - 3M$.

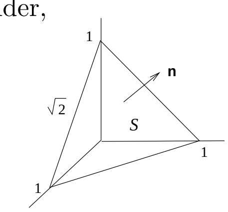
The most general such field is therefore $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + (4N - 3M)\,\mathbf{k}$, where M and N are functions of x, y, z.

6B. Surface Integrals and Flux

6B-1 We have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; therefore $\mathbf{F}\cdot\mathbf{n} = a$.

Flux through $S = \displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = a(\text{area of } S) = 4\pi\,a^3$.

6B-2 Since $\mathbf{k}$ is parallel to the surface, the field is everywhere tangent to the cylinder, hence the flux is 0.



6B-3 $\dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$ is a normal vector to the plane, so $\mathbf{F}\cdot\mathbf{n} = \dfrac{1}{\sqrt{3}}$.

Therefore, $\quad \text{flux} = \dfrac{\text{area of region}}{\sqrt{3}} = \dfrac{\frac{1}{2}(\text{base})(\text{height})}{\sqrt{3}} = \dfrac{\frac{1}{2}(\sqrt{2})(\frac{\sqrt{3}}{2}\sqrt{2})}{\sqrt{3}} = \dfrac{1}{2}.$

6B-4 $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; $\quad \mathbf{F}\cdot\mathbf{n} = \dfrac{y^2}{a}$. Calculating in spherical coordinates,

$$\text{flux} = \iint_S \frac{y^2}{a}\,dS = \frac{1}{a}\int_0^{\pi}\int_0^{\pi} a^4 \sin^3\phi\,\sin^2\theta\,d\phi\,d\theta = a^3\int_0^{\pi}\int_0^{\pi} \sin^3\phi\,\sin^2\theta\,d\phi d\theta.$$

Inner integral: $\sin^2\theta(-\cos\phi + \frac{1}{3}\cos^3\phi)\Big]_0^{\pi} = \frac{4}{3}\sin^2\theta$;

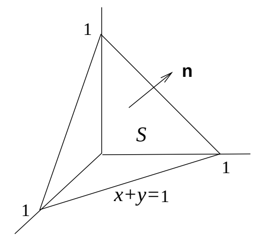
Outer integral: $\frac{4}{3}a^3(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\Big]_0^{\pi} = \frac{2}{3}\pi a^3$.

6B-5 $\mathbf{n} = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{z}{\sqrt{3}}$.

$$\text{flux } = \iint_S \frac{z}{\sqrt{3}}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{1}{\sqrt{3}}\iint_S (1-x-y)\frac{dx\,dy}{1/\sqrt{3}} = \int_0^1\int_0^{1-y}(1-x-y)\,dx\,dy.$$

Inner integral: $= x - \frac{1}{2}x^2 - xy\Big]_0^{1-y} = \frac{1}{2}(1-y)^2$.

Outer integral: $= \displaystyle\int_0^1 \frac{1}{2}(1-y)^2 dy = \frac{1}{2}\cdot -\frac{1}{3}\cdot(1-y)^3\Big]_0^1 = \frac{1}{6}$.



6B-6 $z = f(x,y) = x^2+y^2$ (a paraboloid). By (13) in Notes V9,

$$d\mathbf{S} = (-2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k})\,dx\,dy.$$

(This points generally "up", since the $\mathbf{k}$ component is positive.) Since $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_R (-2x^2 - 2y^2 + z)\,dxdy \qquad ,$$

where R is the interior of the unit circle in the xy-plane, i.e., the projection of S onto the xy-plane). Since $z = x^2+y^2$, the above integral

$$= -\iint_R (x^2+y^2)\,dxdy = -\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = -2\pi\cdot\frac{1}{4} = -\frac{\pi}{2}\ .$$

The answer is negative since the positive direction for flux is that of $\mathbf{n}$, which here points into the inside of the paraboloidal cup, whereas the flow $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is generally from the inside toward the outside of the cup, i.e., in the opposite direction.

6B-8 On the cylindrical surface, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{a}$, $\mathbf{F}\cdot\mathbf{n} = \dfrac{y^2}{a}$.
In cylindrical coordinates, since $y = a\sin\theta$, this gives us $\mathbf{F}\cdot d\mathbf{S} = \mathbf{F}\cdot\mathbf{n}\,dS = a^2\sin^2\theta\,dz\,d\theta$.

$$\text{Flux } = \int_{-\pi/2}^{\pi/2}\int_0^k a^2\sin^2\theta\,dz\,d\theta = a^2h\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta = a^2h\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_{-\pi/2}^{\pi/2} = \frac{\pi}{2}a^2h\ .$$

6B-12 Since the distance from a point $(x,y,0)$ up to the hemispherical surface is z,

$$\text{average distance } = \frac{\iint_S z\,dS}{\iint_S dS}.$$

In spherical coordinates, $\displaystyle\iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} a\cos\phi\cdot a^2\sin\phi\,d\phi\,d\theta.$

Inner: $= a^3\displaystyle\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = a^3(\frac{\sin^2\phi}{2}\Big]_0^{\pi/2} = \frac{a^3}{2}$. Outer: $= \dfrac{a^3}{2}\displaystyle\int_0^{2\pi} d\theta = \pi a^3$.

Finally, $\displaystyle\iint_S dS =$ area of hemisphere $= 2\pi a^2$, so average distance $= \dfrac{\pi a^3}{2\pi a^2} = \dfrac{a}{2}$.

6C. Divergence Theorem

6C-1a div $\mathbf{F} = M_x + N_y + P_z = 2xy + x + x = 2x(y+1)$.

6C-2 Using the product and chain rules for the first, symmetry for the others,

$$(\rho^n x)_x = n\rho^{n-1}\,\frac{x}{\rho}\,x + \rho^n, \quad (\rho^n y)_y = n\rho^{n-1}\,\frac{y}{\rho}\,y + \rho^n, \quad (\rho^n z)_z = n\rho^{n-1}\,\frac{z}{\rho}\,z + \rho^n;$$

adding these three, we get $\quad \text{div}\,\mathbf{F} = n\rho^{n-1}\dfrac{x^2+y^2+z^2}{\rho} \;+\; 3\rho^n \;=\; \rho^n(n+3)$.

Therefore, div $\mathbf{F} = 0 \;\Leftrightarrow\; n = -3$.

6C-3 Evaluating the triple integral first, we have div $\mathbf{F} = 3$, therefore

$$\iiint_D \text{div}\,\mathbf{F}\,dV = 3(\text{vol.of } D) = 3\,\frac{2}{3}\pi a^3 = 2\pi a^3.$$

To evaluate the double integral over the closed surface $S_1 + S_2$, the normal vectors are:

$$\mathbf{n}_1 = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} \quad (\text{hemisphere } S_1), \qquad \mathbf{n}_2 = -\,\mathbf{k} \quad (\text{disc } S_2);$$

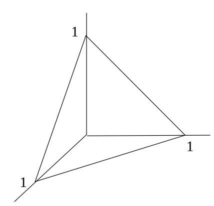
using these, the surface integral for the flux through S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \frac{x^2+y^2+z^2}{a}\,dS + \iint_{S_2} -z\,dS = \iint_{S_1} a\,dS,$$

since $x^2+y^2+z^2 = \rho^2 = a^2$ on S_1, and $z = 0$ on S_2. So the value of the surface integral is

$$a(\text{area of } S_1) = a(2\pi a^2) = 2\pi a^3,$$

which agrees with the triple integral above.



6C-5 The divergence theorem says $\quad \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV.$

Here div $\mathbf{F} = 1$, so that the right-hand integral is just the volume of the tetrahedron, which is $\frac{1}{3}$(base)(height)$= \frac{1}{3}(\frac{1}{2})(1) = \frac{1}{6}$.

6C-6 The divergence theorem says $\quad \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV.$

Here div $\mathbf{F} = 1$, so the right-hand integral is the volume of the solid cone, which has height 1 and base radius 1; its volume is $\frac{1}{3}$(base)(height)$= \pi/3$.

6C-7a Evaluating the triple integral first, over the cylindrical solid D, we have

$$\text{div}\,\mathbf{F} = 2x + x = 3x; \qquad \iiint_D 3x\,dV = 0,$$

since the solid is symmetric with respect to the yz-plane. (Physically, assuming the density is 1, the integral has the value $\bar{x}$(mass of D), where $\bar{x}$ is the x-coordinate of the center of mass; this must be in the yz plane since the solid is symmetric with respect to this plane.)

To evaluate the double integral, note that $\mathbf{F}$ has no $\mathbf{k}$-component, so there is no flux across the two disc-like ends of the solid. To find the flux across the cylindrical side,

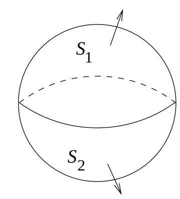
$$\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j}, \qquad \mathbf{F}\cdot\mathbf{n} = x^3 + xy^2 = x^3 + x(1-x^2) = x,$$

since the cylinder has radius 1 and equation $x^2+y^2=1$. Thus

$$\iint_S x\,dS = \int_0^{2\pi}\int_0^1 \cos\theta\,dz\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = 0.$$

6C-8 a) Reorient the lower hemisphere S_2 by reversing its normal vector; call the reoriented surface S_2'. Then $S = S_1 + S_2'$ is a closed surface, with the normal vector pointing outward everywhere, so by the divergence theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div }\mathbf{F}\,dV = 0,$$



since by hypothesis div $\mathbf{F}=0$. The above shows

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = -\iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S},$$

since reversing the orientation of a surface changes the sign of the flux through it.

b) The same statement holds if S_1 and S_2 are two oriented surfaces having the same boundary curve, but not intersecting anywhere else, and oriented so that S_1 and S_2' (i.e., S_2 with its orientation reversed) together make up a closed surface S with outward-pointing normal.

6C-10 If div $\mathbf{F}=0$, then for any closed surface S, we have by the divergence theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div }\mathbf{F}\,dV = 0.$$

Conversely: $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface $S \quad\Rightarrow\quad$ div $\mathbf{F}=0$.

For suppose there were a point P_0 at which $(\text{div }\mathbf{F})_0 \neq 0$ — say $(\text{div }\mathbf{F})_0 > 0$. Then by continuity, div $\mathbf{F}>0$ in a very small spherical ball D surrounding P_0, so that by the divergence theorem (S is the surface of the ball D),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div }\mathbf{F}\,dV \quad > \quad 0.$$

But this contradicts our hypothesis that $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface S.

6C-11 flux of $\mathbf{F} = \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{n} = \iiint_D \text{div }\mathbf{F}\,dV = \iiint_D 3\,dV = 3(\text{vol. of } D)$.

6D. Line Integrals in Space

6D-1 a) $C:\quad x=t,\ dx=dt;\quad y=t^2,\ dy=2t\,dt;\quad z=t^3,\ dz=3t^2\,dt;$

$$\int_C y\,dx + z\,dy - x\,dz \;=\; \int_0^1 (t^2)dt + t^3(2t\,dt) - t(3t^2\,dt)$$

$$= \int_0^1 (t^2+2t^4-3t^3)dt \;=\; \frac{t^3}{3}+\frac{2t^5}{5}-\frac{3t^4}{4}\Big]_0^1 \;=\; \frac{1}{3}+\frac{2}{5}-\frac{3}{4} \;=\; -\frac{1}{60}.$$

b) $C:\quad x=t,\ y=t,\ z=t;\qquad \displaystyle\int_C y\,dx + z\,dy - x\,dz \;=\; \int_0^1 t\,dt \;=\; \frac{1}{2}.$

c) $C = C_1 + C_2 + C_3$; $\quad C_1 : y = z = 0$; $\quad C_2 : x = 1,\ z = 0$; $\quad C_3 : x = 1,\ y = 1$

$$\int_C y\,dx + z\,dy - x\,dz \;=\; \int_{C_1} 0 + \int_{C_2} 0 + \int_0^1 -dz \;=\; -1.$$

d) $C : x = \cos t,\ y = \sin t,\ z = t$; $\quad \int_C zx\,dx + zy\,dy + x\,dz$

$$= \int_0^{2\pi} t\cos t(-\sin t\,dt) + t\sin t(\cos t\,dt) + \cos t\,dt \;=\; \int_0^{2\pi} \cos t\,dt \;=\; 0.$$

6D-2 The field $\mathbf{F}$ is always pointed radially outward; if C lies on a sphere centered at the origin, its unit tangent $\mathbf{t}$ is always tangent to the sphere, therefore perpendicular to the radius; this means $\mathbf{F}\cdot\mathbf{t} = 0$ at every point of C. Thus $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = 0$.

6D-4 a) $\mathbf{F} = \nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

b) (i) Directly, letting C be the helix: $x = \cos t,\ y = \sin t,\ z = t$, from $t = 0$ to $t = 2n\pi$,

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2\cos t(-\sin t)dt + 2\sin t(\cos t)dt + 2t\,dt = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (ii) Choose the vertical path $x = 1,\ y = 0,\ z = t$; then

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (iii) By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,0,2n\pi) - f(1,0,0) = 91^2 + (2n\pi)^2) - 1^2 = (2n\pi)^2$$

6D-5 By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin(xyz)\Big|_Q - \sin(xyz)\Big|_P,$$

where C is any path joining P to Q. The maximum value of this difference is $1 - (-1) = 2$, since $\sin(xyz)$ ranges between -1 and 1.

For example, any path C connecting $P : (1, 1, -\pi/2)$ to $Q : (1, 1, \pi/2)$ will give this maximum value of 2 for $\int_C \mathbf{F}\cdot d\mathbf{r}$.

6E. Gradient Fields in Space

6E-1 a) Since $M = x^2,\ N = y^2,\ P = z^2$ are continuously differentiable, the differential is exact because $N_z = P_y = 0,\quad M_z = P_x = 0,\quad M_y = N_x = 0$; $f(x, y, z) = (x^3 + y^3 + z^3)/3$.

b) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = 2xy, \quad M_z = P_x = y^2, \quad M_y = N_x = 2yz; \quad f(x, y, z) = xy^2\ .$$

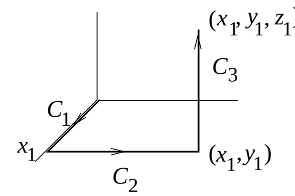
c) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = x, \quad M_z = P_x = y, \quad M_y = N_x = 6x^2 + z;\ f(x, y, z) = 2x^3y + xyz.$$

6E-2 $\quad \operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & yz & xyz^2 \end{vmatrix} = (xz^2 - y)\,\mathbf{i} - yz^2\,\mathbf{j} - x^2\,\mathbf{k}.$

6E-3 a) It is easily checked that curl $\mathbf{F} = 0$.

b) (i) using method I:



$$f(x_1, y_1, z_1) = \int_{(0,0,0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{x_1} x\,dx + \int_0^{y_1} y\,dy + \int_0^{z_1} z\,dz = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + \frac{1}{2}z_2^2.$$

Therefore $f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + c$.

(ii) Using method II: We seek $f(x, y, z)$ such that $f_x = 2xy + z$, $f_y = x^2$, $f_z = x$.

$f_x = 2xy + z \quad \Rightarrow \quad f = x^2y + xz + g(y, z)$.
$f_y = x^2 + g_y = x^2 \quad \Rightarrow \quad g_y = 0 \quad \Rightarrow \quad g = h(z)$
$f_z = x + h'(z) = x \quad \Rightarrow \quad h' = 0 \quad \Rightarrow \quad h = c$

Therefore $f(x, y, z) = x^2y + xz + c$.

(iii) If $f_x = yz$, $f_y = xz$, $f_z = xy$, then by inspection, $f(x, y, z) = xyz + c$.

6E-4 Let $F = f - g$. Since ∇ is a linear operator, $\nabla F = \nabla f - \nabla g = \mathbf{0}$

We now show: $\nabla F = \mathbf{0} \Rightarrow F = c$.

Fix a point $P_0 : (x_0, y_0, z_0)$. Then by the Fundamental Theorem for line integrals,

$$F(P) - F(P_0) = \int_{P_0}^{P} \nabla F \cdot d\mathbf{r} = 0.$$

Therefore $F(P) = F(P_0)$ for all P, i.e., $F(x, y, z) = F(x_0, y_0, z_0) = c$.

6E-5 $\mathbf{F}$ is a gradient field only if these equations are satisfied:

$$N_z = P_y : \ 2xz + ay = bxz + 2y \qquad M_z = P_x : \ 2yz = byz \qquad M_y = N_x : \ z^2 = z^2.$$

Thus the conditions are: $a = 2$, $b = 2$.

Using these values of a and b we employ Method 2 to find the potential function f:

$f_x = yz^2 \quad \Rightarrow \quad f = xyz^2 + g(y, z)$;
$f_y = xz^2 + g_y = xz^2 + 2yz \quad \Rightarrow \quad g_y = 2yz \quad \Rightarrow \quad g = y^2z + h(z)$
$f_z = 2xyz + y^2 + h'(z) = 2xyz + y^2 \quad \Rightarrow \quad h = c$;

therefore, $f(x, y, z) = xyz^2 + y^2z + c$.

6E-6 a) $Mdx + Ndy + Pdz$ is an exact differential if there exists some function $f(x, y, z)$ for which $df = Mdx + Ndy + Pdz$; that, is, for which $f_x = M$, $f_y = N$, $f_z = P$.

b) The given differential is exact if the following equations are satisfied:

$N_z = P_y : \quad (a/2)x^2 + 6xy^2z + 3byz^2 = 3x^2 + 3cxy^2z + 12yz^2$;
$M_z = P_x : \quad axy + 2y^3z = 6xy + cy^3z$
$M_y = N_x : \quad axz + 3y^2z^2 = axz + 3y^2z^2$.

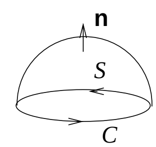
Solving these, we find that the differential is exact if $a = 6$, $b = 4$, $c = 2$.

c) We find $f(x, y, z)$ using method 2:

$$f_x = 6xyz + y^3z^2 \quad \Rightarrow \quad f = 3x^2yz + xy^3z^2 + g(y, z);$$
$$f_y = 3x^2z + 3xy^2z^2 + g_y = 3x^2z + 3xy^2z^2 + 4yz^3 \quad \Rightarrow \quad g_y = 4yz^3 \quad \Rightarrow \quad g = 2y^2z^3 + h(z)$$
$$f_z = 3x^2y + 2xy^3z + 6y^2z^2 + h'(z) = 3x^2y + 2xy^3z + 6y^2z^2 \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h = c.$$

Therefore, $\quad f(x, y, z) = 3x^2yz + xy^3z^2 + 2y^2z^3 + c.$

6F. Stokes' Theorem



6F-1 a) For the line integral, $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C xdx + ydy + zdz = 0,$
since the differential is exact.

For the surface integral, $\quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \mathbf{0}$, and therefore $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0.$

b) Line integral: $\displaystyle\oint_C ydx + zdy + xdz = \oint_C ydx,$ since $z = 0$ and $dz = 0$ on C.

Using $x = \cos t, \quad y = \sin t, \quad \displaystyle\int_0^{2\pi} -\sin^2 t\, dt = -\int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt = -\pi.$

Surface integral: curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}\,; \qquad \mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\iint_S \nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = -\iint_S (x + y + z)\, dS.$$

To evaluate, we use $x = r\cos\theta, \quad y = r\sin\theta, \quad z = \rho\cos\phi. \quad r = \rho\sin\phi, \quad dS = \rho^2 \sin\phi\, d\phi d\theta;$
note that $\rho = 1$ on S. The integral then becomes

$$-\int_0^{2\pi}\int_0^{\pi/2} [\sin\phi(\cos\theta + \sin\theta) + \cos\phi]\sin\phi\, d\phi\, d\theta$$

Inner: $\displaystyle -\left[(\cos\theta + \sin\theta)(\frac{\phi}{2} - \frac{\sin 2\phi}{4}) + \frac{1}{2}\sin^2\phi\right]_0^{\pi/2} \quad = \quad -\left[(\cos\theta + \sin\theta)\frac{\pi}{4} + \frac{1}{2}\right]\,;$

Outer: $\displaystyle\int_0^{2\pi}\left(-\frac{1}{2} - (\cos\theta + \sin\theta)\frac{\pi}{4}\right) d\theta = -\pi.$

6F-2 The surface S is: $\ z = -x - y$, so that $f(x, y) = -x - y$.

$$\mathbf{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dx\, dy \;=\; \langle 1, 1, 1\rangle\, dx\, dy.$$

(Note the signs: $\mathbf{n}$ points upwards, and therefore should have a positive k-component.)

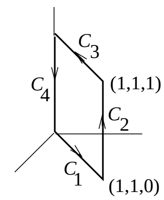
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Therefore $\displaystyle\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_{S'} 3\, dA = -3\pi$, where S' is the projection of S, i.e., the interior of the unit circle in the xy-plane.

As for the line integral, we have $C: \quad x = \cos t, \quad y = \sin t \quad z = -\cos t - \sin t,$ so that

$$\oint_C ydx + zdy + xdz = \int_0^{2\pi} \left[-\sin^2 t - (\cos^2 t + \sin t \cos t) + \cos t(\sin t - \cos t) \right] dt$$
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t - \cos^2 t)\, dt = \int_0^{2\pi} \left[-1 - \frac{1}{2}(1 + \cos 2t) \right] dt = -\frac{3}{2} \cdot 2\pi = -3\pi.$$

6F-3 Line integral: $\oint_C yz\, dx + xz\, dy + xy\, dz$ over the path $C = C_1 + \ldots + C_4$:



$\int_{C_1} = 0$, since $z = dz = 0$ on C_1;

$\int_{C_2} = \int_0^1 1 \cdot 1\, dz = 1$, since $x = 1,\ y = 1,\ dx = 0,\ dy = 0$ on C_2;

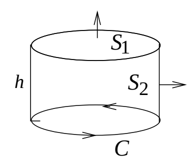
$\int_{C_3} ydx + xdy = \int_1^0 xdx + xdx = -1$, since $y = x,\ z = 1,\ dz = 0$ on C_3;

$\int_{C_4} = 0$, since $x = 0,\ y = 0$ on C_4.

Adding up, we get $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0$. For the surface integral,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = \mathbf{i}\,(x - x) - \mathbf{j}\,(y - y) + \mathbf{k}\,(z - z) = \mathbf{0}; \text{ thus } \iint \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

6F-5 Let S_1 be the top of the cylinder (oriented so $\mathbf{n} = \mathbf{k}$), and S_2 the side.



a) We have $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & x^2 \end{vmatrix} = -2x\,\mathbf{j} + 2\,\mathbf{k}$.

For the top: $\iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} 2\, dS = 2(\text{area of } S_1) = 2\pi a^2$.

For the side: we have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{a}$, and $dS = dz \cdot a\, d\theta$, so that

$$\iint_{S_2} \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^h \frac{-2xy}{a} a\, dz\, d\theta = \int_0^{2\pi} -2h(a\cos\theta)(a\sin\theta)\, d\theta = -ha^2 \sin^2\theta \Big]_0^{2\pi} = 0.$$

Adding, $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} + \iint_{S_2} = 2\pi a^2$.

b) Let C be the circular boundary of S, parameterized by $x = a\cos\theta,\ y = a\sin\theta,\ z = 0$. Then using Stokes' theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C -y\, dx + x\, dy + x^2\, dz = \int_0^{2\pi} (a^2 \sin^2\theta + a^2 \cos^2\theta)\, d\theta = 2\pi a^2.$$

6G. Topological Questions

6G-1 a) yes b) no c) yes d) no; yes; no; yes; no; yes

6G-2 Recall that $\rho_x = x/\rho$, etc. Then, using the chain rule,

$$\text{curl } \mathbf{F} = (n\rho^{n-1} z\, \frac{y}{\rho} - n\rho^{n-1} y\, \frac{z}{\rho})\,\mathbf{i} + (n\rho^{n-1} z\, \frac{x}{\rho} - n\rho^{n-1} x\, \frac{z}{\rho})\,\mathbf{j} + (n\rho^{n-1} y\, \frac{x}{\rho} - n\rho^{n-1} x\, \frac{y}{\rho})\,\mathbf{k}\,.$$

Therefore curl $\mathbf{F} = \mathbf{0}$. To find the potential function, we let P_0 be any convenient starting point, and integrate along some path to $P_1 : (x_1, y_1, z_1)$. Then, if $n \neq -2$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{P_0}^{P_1} \rho^n (x\,dx + y\,dy + z\,dz) = \int_{P_0}^{P_1} \rho^n \,\frac{1}{2}\, d(\rho^2)$$

$$= \int_{P_0}^{P_1} \rho^{n+1} d\rho = \frac{\rho^{n+2}}{n+2}\bigg]_{P_0}^{P_1} = \frac{\rho_1^{n+2}}{n+2} - \frac{\rho_0^{n+2}}{n+2} = \frac{\rho_1^{n+2}}{n+2} + c, \text{ since } P_0 \text{ is fixed.}$$

Therefore, we get $\mathbf{F} = \nabla \dfrac{\rho^{n+2}}{n+2}$, if $n \neq -2$.

If $n = -2$, the line integral becomes $\displaystyle\int_{P_0}^{P_1} \frac{d\rho}{\rho} = \ln \rho_1 + c$, so that $\mathbf{F} = \nabla(\ln \rho)$.

6H. Applications and Further Exercises

6H-1 Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$. By the definition of curl $\mathbf{F}$, we have

$$\nabla \times \mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}\,,$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$

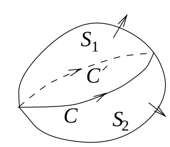
If all the mixed partials exist and are continuous, then $P_{xy} = P_{yx}$, etc. and the right-hand side of the above equation is zero: div (curl $\mathbf{F}$) = 0.

6H-2 a) Using the divergence theorem, and the previous problem, (D is the interior of S),

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iiint_D \text{div curl } \mathbf{F}\, dV = \iiint_D 0\, dV = 0.$$

b) Draw a closed curve C on S that divides it into two pieces S_1 and S_2 both having C as boundary. Orient C compatibly with S_1, then the curve C' obtained by reversing the orientation of C will be oriented compatibly with S_2. Using Stokes' theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{C'} \mathbf{F} \cdot d\mathbf{r} = 0,$$



since the integral on C' is the negative of the integral on C.

Or more simply, consider the limiting case where C has been shrunk to a point; even as a point, it can still be considered to be the boundary of S. Since it has zero length, the line integral around it is zero, and therefore Stokes' theorem gives

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

6H-10 Let C be an oriented closed curve, and S a compatibly-oriented surface having C as its boundary. Using Stokes' theorem and the Maxwell equation, we get respectively

$$\iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{B}\cdot d\mathbf{r} \qquad \text{and} \qquad \iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \iint_S \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S} = \frac{1}{c}\frac{d}{dt}\iint_S E\cdot d\mathbf{S}.$$

Since the two left sides are the same, we get $\displaystyle\oint_C \mathbf{B} \cdot d\mathbf{r} = \frac{1}{c}\frac{d}{dt}\iint_S \mathbf{E} \cdot d\mathbf{S}.$

In words: for the magnetic field $\mathbf{B}$ produced by a moving electric field $\mathbf{E}(t)$, the magneto-motive force around a closed loop C is, up to a constant factor depending on the units, the time-rate at which the electric flux through C is changing.

18.02 Notes and Exercises by A. Mattuck and Bjorn Poonen with the assistance of T.Shifrin and S. LeDuc