4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1

a) All vectors in the field are identical; continuously differentiable everywhere.
b) The vector at P has its tail at P and head at the origin; field is cont. diff. everywhere.
c) All vectors have unit length and point radially outwards; cont. diff. except at $(0,0)$.
d) Vector at P has unit length, and the clockwise direction perpendicular to OP.

4A-2 a) $a\,\mathbf{i} + b\,\mathbf{j}$ b) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^2}$ c) $f'(r)\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r}$

4A-3 a) $\mathbf{i} + 2\mathbf{j}$ b) $-r(x\,\mathbf{i} + y\,\mathbf{j})$ c) $\dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{r^3}$ d) $f(x,y)(\mathbf{i} + \mathbf{j})$

4A-4 $k \cdot \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2}$

4B. Line Integrals in the Plane

4B-1

a) On C_1: $y = 0,\ dy = 0$; therefore $\displaystyle\int_{C_1} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}.$

On C_2: $y = 1 - x^2,\ dy = -2x\,dx$; $\displaystyle\int_{C_2} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} (2x^2 - 1)\,dx - 4x^2\,dx$

$$= \int_{-1}^{1} (-2x^2 - 1)\,dx = -\left[\frac{2}{3}x^3 + x\right]_{-1}^{1} = -\frac{4}{3} - 2 = -\frac{10}{3}.$$

b) C: use the parametrization $x = \cos t,\ y = \sin t$; then $dx = -\sin t\,dt,\ dy = \cos t\,dt$

$$\int_C xy\,dx - x^2\,dy = \int_{\pi/2}^{0} -\sin^2 t\cos t\,dt - \cos^2 t\cos t\,dt = -\int_{\pi/2}^{0} \cos t\,dt = -\sin t\Big]_{\pi/2}^{0} = 1.$$

c) $C = C_1 + C_2 + C_3$; $C_1 : x = dx = 0$; $C_2 :\ y = 1 - x$; $C_3 : y = dy = 0$

$$\int_C y\,dx - x\,dy = \int_{C_1} 0 + \int_0^1 (1-x)dx - x(-dx) + \int_{C_3} 0 = \int_0^1 dx = 1.$$

d) $C : x = 2\cos t,\ y = \sin t$; $dx = -2\sin t\,dt$ $\displaystyle\int_C y\,dx = \int_0^{2\pi} -2\sin^2 t\,dt = -2\pi.$

e) $C :\ x = t^2,\ y = t^3$; $dx = 2t\,dt,\ dy = 3t^2\,dt$

$$\int_C 6y\,dx + x\,dy = \int_1^2 6t^3(2t\,dt) + t^2(3t^2\,dt) = \int_1^2 (15t^4)\,dt = 3t^5\Big]_1^2 = 3 \cdot 31.$$

f) $\displaystyle\int_C (x+y)dx + xy\,dy = \int_{C_1} 0 + \int_0^1 (x+2)dx = \frac{x^2}{2} + 2x\Big]_0^1 = \frac{5}{2}.$

4B-2 a) The field $\mathbf{F}$ points radially outward, the unit tangent $\mathbf{t}$ to the circle is always perpendicular to the radius; therefore $\mathbf{F} \cdot \mathbf{t} = 0$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\,ds = 0$

b) The field $\mathbf{F}$ is always tangent to the circle of radius a, in the clockwise direction, and of magnitude a. Therefore $\mathbf{F} = -a\mathbf{t}$, so that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\,ds = -\int_C a\,ds = -2\pi a^2$.

4B-3 a) maximum if C is in the direction of the field: $C = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

b) minimum if C is in the opposite direction to the field: $C = -\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

c) zero if C is perpendicular to the field: $C = \pm\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$

d) max $= \sqrt{2}$, min $= -\sqrt{2}$: by (a) and (b), for the max or min $\mathbf{F}$ and C have respectively the same or opposite constant direction, so $\int_C \mathbf{F}\cdot d\mathbf{r} = \pm|\mathbf{F}|\cdot|C| = \pm\sqrt{2}$.

4C. Gradient Fields and Exact Differentials

4C-1 a) $\mathbf{F} = \nabla f = 3x^2y\,\mathbf{i} + (x^3+3y^2)\,\mathbf{j}$

b) (i) Using y as parameter, C_1 is: $x = y^2$, $y = y$; thus $dx = 2y\,dy$, and

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} 3(y^2)^2y\cdot 2y\,dy + [(y^2)^3+3y^2]\,dy = \int_{-1}^{1}(7y^6+3y^2)\,dy = (y^7+y^3)\Big]_{-1}^{1} = 4.$$

b) (ii) Using y as parameter, C_2 is: $x = 1$, $y = y$; thus $dx = 0$, and

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}(1+3y^2)\,dy = (y+y^3)\Big]_{-1}^{1} = 4.$$

b) (iii) By the Fundamental Theorem of Calculus for line integrals,

$$\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A).$$

Here $A = (1,-1)$ and $B = (1,1)$, so that $\displaystyle\int_C \nabla f\cdot d\mathbf{r} = (1+1) - (-1-1) = 4$.

4C-2 a) $\mathbf{F} = \nabla f = (xye^{xy} + e^{xy})\,\mathbf{i} + (x^2e^{xy})\,\mathbf{j}$.

b) (i) Using x as parameter, C is: $x = x$, $y = 1/x$, so $dy = -dx/x^2$, and so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (e+e)\,dx + (x^2e)(-dx/x^2) = (2ex - ex)\Big]_1^0 = -e.$$

b) (ii) Using the F.T.C. for line integrals, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(0,\infty) - f(1,1) = 0 - e = -e$.

4C-3 a) $\mathbf{F} = \nabla f = (\cos x\cos y)\,\mathbf{i} - (\sin x\sin y)\,\mathbf{j}$.

b) Since $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is path-independent, for any C connecting $A:(x_0,y_0)$ to $B:(x_1,y_1)$, we have by the F.T.C. for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin x_1\cos y_1 - \sin x_0\cos y_0$$

This difference on the right-hand side is maximized if $\sin x_1\cos y_1$ is maximized, and $\sin x_0\cos y_0$ is minimized. Since $|\sin x\cos y| = |\sin x||\cos y| \le 1$, the difference on the right hand side has a maximum of 2, attained when $\sin x_1\cos y_1 = 1$ and $\sin x_0\cos y_0 = -1$.

(For example, a C running from $(-\pi/2, 0)$ to $(\pi/2, 0)$ gives this maximum value.)

4C-5 a) $\mathbf{F}$ is a gradient field only if $M_y = N_x$, that is, if $2y = ay$, so $a = 2$.

By inspection, the potential function is $f(x,y) = xy^2 + x^2 + c$; you can check that $\mathbf{F} = \nabla f$.

b) The equation $M_y = N_x$ becomes $e^{x+y}(x+a) = xe^{x+y} + e^{x+y}$, which $= e^{x+y}(x+1)$. Therefore $a = 1$.

To find the potential function $f(x,y)$, using Method 2 we have

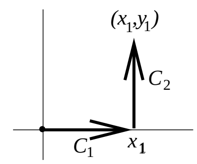
$f_x = e^y e^x (x+1) \;\Rightarrow\; f(x,y) = e^y x e^x + g(y)$.

Differentiating, and comparing the result with N, we find

$f_y = e^y x e^x + g'(y) = xe^{x+y}$; therefore $g'(y) = 0$, so $g(y) = c$ and $f(x,y) = x\,e^{x+y} + c$.

4C-6 a) $ydx - xdy$ is not exact, since $M_y = 1$ but $N_x = -1$.

b) $y(2x+y)\,dx + x(2y+x)\,dy$ is exact, since $M_y = 2x + 2y = N_x$.



Using Method 1 to find the potential function $f(x,y)$, we calculate the line integral over the standard broken line path shown, $C = C_1 + C_2$.

$$f(x_1, y_1) = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(x_1,y_1)} y(2x+y)\,dx + x(2y+x)\,dy.$$

On C_1 we have $y = 0$ and $dy = 0$, so $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$.

On C_2, we have $x = x_1$ and $dx = 0$, so $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{y_1} x_1(2y + x_1)\,dx = x_1 y_1^2 + x_1^2 y_1$.

Therefore, $f(x,y) = x^2 y + xy^2$; to get all possible functions, add $+c$.

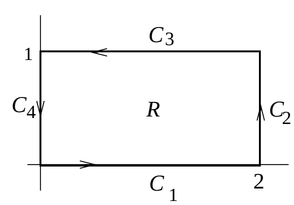
4D. Green's Theorem

4D-1 a) Evaluating the line integral first, we have $C:\; x = \cos t,\; y = \sin t$, so

$$\oint_C 2y\,dx + x\,dy = \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 3\sin^2 t)\,dt = t - 3\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg]_0^{2\pi} = -\pi.$$

For the double integral over the circular region R inside the C, we have

$$\iint_R (N_x - M_y)\,dA = \iint_R (1-2)\,dA = -\text{ area of } R = -\pi.$$



b) Evaluating the line integral, over the indicated path $C = C_1 + C_2 + C_3 + C_4$,

$$\oint_C x^2 dx + x^2 dy = \int_0^2 x^2 dx + \int_0^1 4\,dy + \int_2^0 x^2 dx + \int_1^0 0\,dy \;=\; 4,$$

since the first and third integrals cancel, and the fourth is 0.

For the double integral over the rectangle R,

$$\iint_R 2x\,dA = \int_0^2 \int_0^1 2x\,dydx = x^2\Big]_0^2 = 4.$$

c) Evaluating the line integral over $C = C_1 + C_2$, we have

$$C_1: \ x = x,\ y = x^2; \quad \int_{C_1} xy\,dx + y^2 dy = \int_0^1 x\cdot x^2 dx + x^4 \cdot 2x\,dx = \frac{x^4}{4} + \frac{x^6}{3}\Big]_0^1 = \frac{7}{12}$$

$$C_2: \ x = x,\ y = x; \quad \int_{C_2} xy\,dx + y^2 dy = \int_1^0 (x^2 dx + x^2 dx) = \frac{2}{3}x^3\Big]_1^0 = -\frac{2}{3}.$$

Therefore, $\displaystyle\oint_C xy\,dx + y^2 dy = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}$.

Evaluating the double integral over the interior R of C, we have

$$\iint_R -x\,dA = \int_0^1 \int_{x^2}^x -x\,dydx;$$

evaluating: Inner: $\displaystyle -xy\Big]_{y=x^2}^{y=x} = -x^2 + x^3$; Outer: $\displaystyle -\frac{x^3}{3} + \frac{x^4}{4}\Big]_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$.

4D-2 By Green's theorem, $\displaystyle\oint_C 4x^3 y\,dx + x^4\,dy = \iint (4x^3 - 4x^3)\,dA = 0$.

This is true for every closed curve C in the plane, since M and N have continuous derivatives for all x, y.

4D-3 We use the symmetric form for the integrand since the parametrization of the curve does not favor x or y; this leads to the easiest calculation.

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt = \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt$$

Using $\sin^2 t\cos^2 t = \frac{1}{4}(\sin 2t)^2 = \frac{1}{4}\cdot\frac{1}{2}(1 - \cos 4t)$, the above $\displaystyle = \frac{3}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)_0^{2\pi} = \frac{3\pi}{8}$.

4D-4 By Green's theorem, $\displaystyle\oint_C -y^3 dx + x^3 dy = \iint_R (3x^2 + 3y^2)\,dA > 0$, since the integrand is always positive outside the origin.

4D-5 Let C be a square, and R its interior. Using Green's theorem,

$$\oint_C xy^2 dx + (x^2 y + 2x)\,dy = \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = 2(\text{area of } R).$$

4E. Two-dimensional Flux

4E-1 The vector $\mathbf{F}$ is the velocity vector for a rotating disc; it is at each point tangent to the circle centered at the origin and passing through that point.

a) Since $\mathbf{F}$ is tangent to the circle, $\mathbf{F}\cdot\mathbf{n} = 0$ at every point on the circle, so the flux is 0.

b) $\mathbf{F} = x\,\mathbf{j}$ at the point $(x, 0)$ on the line. So if $x_0 > 0$, the flux at x_0 has the same magnitude as the flux at $-x_0$ but the opposite sign, so the net flux over the line is 0.

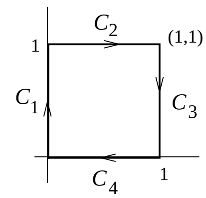
c) $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = x\,\mathbf{j}\cdot -\mathbf{j} = -x$. Thus $\displaystyle\int \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{1}{2}$.

4E-2 All the vectors of $\mathbf{F}$ have length $\sqrt{2}$ and point northeast. So the flux across a line segment C of length 1 will be

a) maximal, if C points northwest;
b) minimal, if C point southeast;
c) zero, if C points northeast or southwest;
d) -1, if C has the direction and magnitude of $\mathbf{i}$ or $-\mathbf{j}$; the corresponding normal vectors are then respectively $-\mathbf{j}$ and $-\mathbf{i}$, by convention, so that $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i}+\mathbf{j})\cdot -\mathbf{j} = -1.$ or $(\mathbf{i}+\mathbf{j})\cdot -\mathbf{i} = -1.$
e) respectively $\sqrt{2}$ and $-\sqrt{2}$, since the angle θ between $\mathbf{F}$ and n is respectively 0 and π, so that respectively $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}|\cos\theta = \pm\sqrt{2}$.

4E-3 $\displaystyle\int_C M\,dy - N\,dx = \int_C x^2 dy - xy\,dx = \int_0^1 (t+1)^2 2t\,dt - (t+1)t^2\,dt$

$$= \int_0^1 (t^3+3t^2+2t)\,dt = \frac{t^4}{4} + t^3 + t^2\Big]_0^1 = \frac{9}{4}.$$



4E-4 Taking the curve $C = C_1 + C_2 + C_3 + C_4$ as shown,

$$\int_C x\,dy - y\,dx = \int_{C_1} 0 + \int_0^1 -dx + \int_1^0 dy + \int_{C_4} 0 = -2.$$

4E-5 Since $\mathbf{F}$ and $\mathbf{n}$ both point radially outwards, $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = a^m$, at every point of the circle C of radius a centered at the origin.

a) The flux across C is $a^m \cdot 2\pi a = 2\pi a^{m+1}$.
b) The flux will be independent of a if $m = -1$.

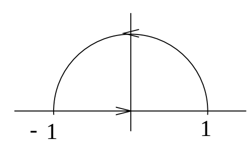
4F. Green's Theorem in Normal Form

4F-1 a) both are 0 b) div $\mathbf{F} = 2x + 2y$; curl $\mathbf{F} = 0$ c) div $\mathbf{F} = x + y$; curl $\mathbf{F} = y - x$

4F-2 a) div $\mathbf{F} = (-\omega y)_x + (\omega x)_y = 0$; curl $\mathbf{F} = (\omega x)_x - (-\omega y)_y = 2\omega$.

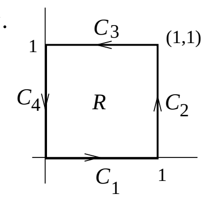
b) Since $\mathbf{F}$ is the velocity field of a fluid rotating with constant angular velocity (like a rigid disc), there are no sources or sinks: fluid is not being added to or subtracted from the flow at any point.

c) A paddlewheel placed at the origin will clearly spin with the same angular velocity ω as the rotating fluid, so by Notes V4,(11), the curl should be 2ω at the origin. (It is much less clear that the curl is 2ω at all other points as well.)



4F-3 The line integral for flux is $\displaystyle\int_C x\,dy - y\,dx$; its value is 0 on any segment of the x-axis since $y = dy = 0$; on the upper half of the unit semicircle (oriented counterclockwise), $\mathbf{F}\cdot\mathbf{n} = 1$, so the flux is the length of the semicircle: π.

Letting R be the region inside C, $\displaystyle\iint_R \text{div}\,\mathbf{F}\,dA = \iint_R 2\,dA = 2(\pi/2) = \pi.$



4F-4 For the flux integral $\displaystyle\oint_C x^2 dy - xy\,dx$ over $C = C_1 + C_2 + C_3 + C_4$,

we get for the four sides respectively $\displaystyle\int_{C_1} 0 + \int_0^1 dy + \int_1^0 -x\,dx + \int_{C_4} 0 = \frac{3}{2}.$

For the double integral, $\displaystyle\iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R 3x\,dA = \int_0^1\int_0^1 3x\,dydx = \frac{3}{2}x^2\Big]_0^1 = \frac{3}{2}.$

4F-5 $r = (x^2+y^2)^{1/2} \;\Rightarrow\; r_x = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \dfrac{x}{r}$; by symmetry, $r_y = \dfrac{y}{r}$.

To calculate div $\mathbf{F}$, we have $M = r^n x$ and $N = r^n y$; therefore by the chain rule, and the above values for r_x and r_y, we have

$$M_x = r^n + nr^{n-1}x\cdot\frac{x}{r} = r^n + n\,r^{n-2}x^2; \quad \text{similarly (or by symmetry),}$$

$$N_y = r^n + nr^{n-1}y\cdot\frac{y}{r} = r^n + n\,r^{n-2}y^2, \quad \text{so that}$$

$$\operatorname{div}\mathbf{F} = M_x + N_y = 2r^n + nr^{n-2}(x^2+y^2) = r^n(2+n), \text{ which } = 0 \text{ if } n = -2.$$

To calculate curl $\mathbf{F}$, we have by the chain rule

$$N_x = nr^{n-1}\cdot\frac{x}{r}\cdot y; \quad M_y = nr^{n-1}\cdot\frac{y}{r}\cdot x, \quad \text{so that curl } \mathbf{F} = N_x - M_y = 0, \text{ for all } n.$$

4G. Simply-connected Regions

4G-1 Hypotheses: the region R is simply connected, $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ has continuous derivatives in R, and curl $\mathbf{F} = 0$ in R.

Conclusion: $\mathbf{F}$ is a gradient field in R (or, $M\,dx + N\,dy$ is an exact differential).

a) curl $\mathbf{F} = 2y - 2y = 0$, and R is the whole xy-plane. Therefore $\mathbf{F} = \nabla f$ in the plane.

b) curl $\mathbf{F} = -y\sin x - x\sin y \neq 0$, so the differential is not exact.

c) curl $\mathbf{F} = 0$, but R is the exterior of the unit circle, which is not simply-connected; criterion fails.

d) curl $\mathbf{F} = 0$, and R is the interior of the unit circle, which is simply-connected, so the differential is exact.

e) curl $\mathbf{F} = 0$ and R is the first quadrant, which is simply-connected, so $\mathbf{F}$ is a gradient field.

4G-2 a) $f(x,y) = xy^2 + 2x$ b) $f(x,y) = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}$

c) Using Method 1, we take the origin as the starting point and use the straight line to (x_1, y_1) as the path C. In polar coordinates, $x_1 = r_1\cos\theta_1,\ y_1 = r_1\sin\theta_1$; we use r as the parameter, so the path is $C: x = r\cos\theta_1,\ y = r\sin\theta_1,\ 0 \le r \le r_1$. Then

$$f(x_1,y_1) = \int_C \frac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \int_0^{r_1}\frac{r\cos^2\theta_1 + r\sin^2\theta_1}{\sqrt{1-r^2}}\,dr$$

$$= \int_0^{r_1}\frac{r}{\sqrt{1-r^2}}\,dr = -\sqrt{1-r^2}\Big]_0^{r_1} = -\sqrt{1-{r_1}^2} + 1.$$

Therefore, $$\frac{x\,dx + y\,dy}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$$

Another approach: $x\,dx + y\,dy = \frac{1}{2}d(r^2)$; therefore $\dfrac{x\,dx+y\,dy}{\sqrt{1-r^2}} = \dfrac{1}{2}\dfrac{d(r^2)}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2})$.
(Think of r^2 as a new variable u, and integrate.)

4G-3 By Example 3 in Notes V5, we know that $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\dfrac{1}{r}\right)$.

Therefore, $\displaystyle\int_{(1,1)}^{(3,4)} = -\frac{1}{r}\bigg]_{\sqrt{2}}^{5} = \frac{1}{\sqrt{2}} - \frac{1}{5}.$

4G-4 By Green's theorem $\displaystyle\oint_C xy\,dx + x^2\,dy = \iint_R x\,dA.$

For any plane region of density 1, we have $\iint_R x\,dA = \bar{x}\cdot$(area of R), where $\bar{x}$ is the x-component of its center of mass. Since our region is symmetric with respect to the y-axis, its center of mass is on the y-axis, hence $\bar{x} = 0$ and so $\iint_R x\,dA = 0$.

4G-5

a) yes
b) no (a circle surrounding the line segment lies in R, but its interior does not)
c) yes (no finite curve could surround the entire positive x-axis)
d) no (the region does not consist of one connected piece)
e) yes if $\theta_0 < 2\pi$; no if $\theta_0 \geq 2\pi$, since then R is the plane with $(0,0)$ removed
f) no (a circle between the two boundary circles lies in R, but its interior does not)
g) yes

4G-6

a) continuously differentiable for $x, y > 0$; thus R is the first quadrant without the two axes, which is simply-connected.

b) continuous differentiable if $r < 1$; thus R is the interior of the unit circle, and is simply-connected.

c) continuously differentiable if $r > 1$; thus R is the exterior of the unit circle, and is not simply-connected.

d) continuously differentiable if $r \neq 0$; thus R is the plane with the origin removed, and is not simply-connected.

e) continuously differentiable if $r \neq 0$; same as (d).

4H. Multiply-connected Regions

4H-1 a) 0; 0 b) 2; 4π c) -1; -2π d) -2; -4π

4H-2 In each case, the winding number about each of the points is given, then the value of the line integral of **F** around the curve.

a) $(1,-1,1)$; $2 - \sqrt{2} + \sqrt{3}$
b) $(-1,0,1)$; $-2 + \sqrt{3}$
c) $(-1,0,0)$; -2
d) $(-1,-2,1)$; $-2 - 2\sqrt{2} + \sqrt{3}$

18.02 Notes and Exercises by A. Mattuck with the assistance of T.Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.