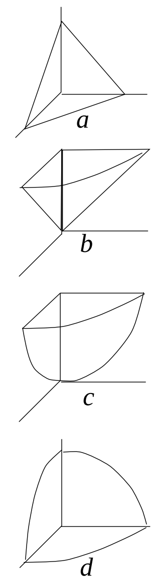
# 2. Partial Differentiation

## 2A. Functions and Partial Derivatives



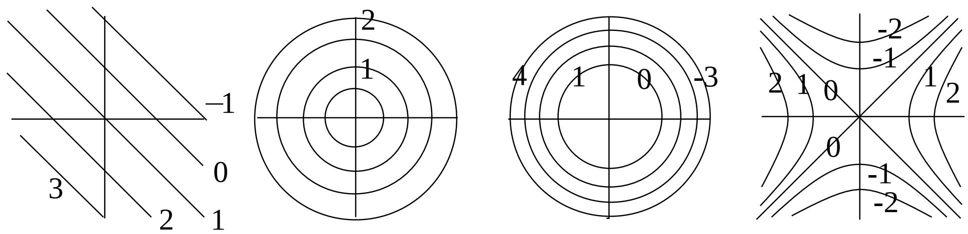

**2A-1** In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:

b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;

c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;

d) on the right, two level curves are labeled; the unlabeled ones are $-1$ and $-2$; the origin is the level curve 1;

The crude sketches of the graph in the first octant are at the right.




*a* *b* *c, d* *e*

**2A-2** a) $f_x = 3x^2y - 3y^2, \quad f_y = x^3 - 6xy + 4y$ b) $z_x = \dfrac{1}{y}, \quad z_y = -\dfrac{x}{y^2}$

c) $f_x = 3\cos(3x+2y), \quad f_y = 2\cos(3x+2y)$

d) $f_x = 2xye^{x^2y}, \quad f_y = x^2e^{x^2y}$ e) $z_x = \ln(2x+y) + \dfrac{2x}{2x+y}, \quad z_y = \dfrac{x}{2x+y}$

f) $f_x = 2xz, \quad f_y = -2z^3, \quad f_z = x^2 - 6yz^2$

**2A-3** a) both sides are $mnx^{m-1}y^{n-1}$

b) $f_x = \dfrac{y}{(x+y)^2}, \quad f_{xy} = (f_x)_y = \dfrac{x-y}{(x+y)^3}; \quad f_y = \dfrac{-x}{(x+y)^2}, \quad f_{yx} = \dfrac{-(y-x)}{(x+y)^3}.$

c) $f_x = -2x\sin(x^2+y), \quad f_{xy} = (f_x)_y = -2x\cos(x^2+y);$
$f_y = -\sin(x^2+y), \quad f_{yx} = -\cos(x^2+y)\cdot 2x.$

d) both sides are $f'(x)g'(y)$.

**2A-4** $(f_x)_y = ax+6y, \quad (f_y)_x = 2x+6y$; therefore $f_{xy} = f_{yx} \Leftrightarrow a = 2$. By inspection, one sees that if $a = 2, \quad f(x,y) = x^2y + 3xy^2$ is a function with the given $f_x$ and $f_y$.

**2A-5**

a) $w_x = ae^{ax}\sin ay, \quad w_{xx} = a^2e^{ax}\sin ay;$
$w_y = e^{ax}a\cos ay, \quad w_{yy} = e^{ax}a^2(-\sin ay);$ therefore $w_{yy} = -w_{xx}$.

b) We have $w_x = \dfrac{2x}{x^2+y^2}, \quad w_{xx} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$. If we interchange $x$ and $y$, the function $w = \ln(x^2+y^2)$ remains the same, while $w_{xx}$ gets turned into $w_{yy}$; since the interchange just changes the sign of the right hand side, it follows that $w_{yy} = -w_{xx}$.

## 2B. Tangent Plane; Linear Approximation

**2B-1** a) $z_x = y^2, \quad z_y = 2xy$; therefore at (1,1,1), we get $z_x = 1, \quad z_y = 2$, so that the tangent plane is $z = 1 + (x-1) + 2(y-1)$, or $z = x + 2y - 2$.

b) $w_x = -y^2/x^2$, $w_y = 2y/x$; therefore at (1,2,4), we get $w_x = -4$, $w_y = 4$, so that the tangent plane is $w = 4 - 4(x-1) + 4(y-2)$, or $w = -4x + 4y$.

**2B-2** a) $z_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{z}$; by symmetry (interchanging $x$ and $y$), $z_y = \dfrac{y}{z}$; then the tangent plane is $z = z_0 + \dfrac{x_0}{z_0}(x - x_0) + \dfrac{y_0}{z_0}(y - y_0)$, or $z = \dfrac{x_0}{z_0}x + \dfrac{y_0}{z_0}y$, since $x_0^2 + y_0^2 = z_0^2$.

b) The line is $x = x_0 t$, $y = y_0 t$, $z = z_0 t$; substituting into the equations of the cone and the tangent plane, both are satisfied for all values of $t$; this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

**2B-3** Letting $x, y, z$ be respectively the lengths of the two legs and the hypotenuse, we have $z = \sqrt{x^2+y^2}$; thus the calculation of partial derivatives is the same as in **2B-2**, and we get $\Delta z \approx \dfrac{3}{5}\Delta x + \dfrac{4}{5}\Delta y$. Taking $\Delta x = \Delta y = .01$, we get $\Delta z \approx \dfrac{7}{5}(.01) = .014$.

**2B-4** From the formula, we get $R = \dfrac{R_1 R_2}{R_1 + R_2}$. From this we calculate

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1+R_2}\right)^2, \text{ and by symmetry, } \frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1+R_2}\right)^2.$$

Substituting $R_1 = 1$, $R_2 = 2$ the approximation formula then gives $\Delta R = \dfrac{4}{9}\Delta R_1 + \dfrac{1}{9}\Delta R_2$.

By hypothesis, $|\Delta R_i| \le .1$, for $i = 1, 2$, so that $|\Delta R| \le \dfrac{4}{9}(.1) + \dfrac{1}{9}(.1) = \dfrac{5}{9}(.1) \approx .06$; thus

$$R = \frac{2}{3} = .67 \ \pm .06.$$

**2B-5** a) We have $f(x,y) = (x+y+2)^2$, $f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$. Therefore

at $(0,0)$, $f_x(0,0) = f_y(0,0) = 4$, $f(0,0) = 4$; linearization is $4 + 4x + 4y$;

at $(1,2)$, $f_x(1,2) = f_y(1,2) = 10$, $f(1,2) = 25$;
linearization is $10(x-1) + 10(y-2) + 25$, or $10x + 10y - 5$.

b) $f = e^x \cos y$; $f_x = e^x \cos y$; $f_y = -e^x \sin y$ .

linearization at $(0,0)$: $1 + x$; linearization at $(0, \pi/2)$: $-(y - \pi/2)$

**2B-6** We have $V = \pi r^2 h$, $\dfrac{\partial V}{\partial r} = 2\pi r h$, $\dfrac{\partial V}{\partial h} = \pi r^2$; $\Delta V \approx \left(\dfrac{\partial V}{\partial r}\right)_0 \Delta r + \left(\dfrac{\partial V}{\partial h}\right)_0 \Delta h$.
Evaluating the partials at $r = 2$, $h = 3$, we get

$$\Delta V \approx 12\pi \Delta r + 4\pi \Delta h.$$

Assuming the same accuracy $|\Delta r| \le \epsilon$, $|\Delta h| \le \epsilon$ for both measurements, we get

$$|\Delta V| \le 12\pi\,\epsilon + 4\pi\,\epsilon = 16\pi\,\epsilon, \quad \text{which is } < .1 \text{ if } \epsilon < \frac{1}{160\pi} < .002 .$$

**2B-7** We have $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Therefore at $(3,4)$, $r = 5$, and $\Delta r \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. If $|\Delta x|$ and $|\Delta y|$ are both $\le .01$, then

$$|\Delta r| \le \tfrac{3}{5}|\Delta x| + \tfrac{4}{5}|\Delta y| = \tfrac{7}{5}(.01) = .014 \text{ (or .02)}.$$

Similarly, $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$; $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$, so at the point $(3,4)$,

$$|\Delta\theta| \;\le\; |\tfrac{-4}{25}\Delta x| + |\tfrac{3}{25}\Delta y| \;\le\; \tfrac{7}{25}(.01) \;=\; .0028 \text{ (or } .003).$$

Since at $(3,4)$ we have $|r_y| > |r_x|$, $r$ is more sensitive there to changes in $y$; by analogous reasoning, $\theta$ is more sensitive there to $x$.

**2B-9** a) $w = x^2(y+1)$; $w_x = 2x(y+1) = 2$ at $(1,0)$, and $w_y = x^2 = 1$ at $(1,0)$; therefore $w$ is more sensitive to changes in $x$ around this point.

b) To first order approximation, $\Delta w \approx 2\Delta x + \Delta y$, using the above values of the partial derivatives.

If we want $\Delta w = 0$, then by the above, $2\Delta x + \Delta y = 0$, or $\Delta y/\Delta x = -2$ .

## 2C. Differentials; Approximations

**2C-1** a) $dw = \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$ b) $dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2dz$

c) $dz = \dfrac{2y\,dx - 2x\,dy}{(x+y)^2}$ d) $dw = \dfrac{t\,du - u\,dt}{t\sqrt{t^2-u^2}}$

**2C-2** The volume is $V = xyz$; so $dV = yz\,dx + xz\,dy + xy\,dz$. For $x = 5$, $y = 10$, $z = 20$,

$$\Delta V \approx dV = 200\,dx + 100\,dy + 50\,dz,$$

from which we see that $|\Delta V| \le 350(.1)$; therefore $V = 1000 \pm 35$.

**2C-3** a) $A = \frac{1}{2}ab\sin\theta$. Therefore, $dA = \frac{1}{2}(b\sin\theta\,da + a\sin\theta\,db + ab\cos\theta\,d\theta)$.

b) $dA = \frac{1}{2}(2\cdot\frac{1}{2}\,da + 1\cdot\frac{1}{2}\,db + 1\cdot 2\cdot\frac{1}{2}\sqrt{3}\,d\theta) = \frac{1}{2}(da + \frac{1}{2}\,db + \sqrt{3}\,d\theta)$;
therefore most sensitive to $\theta$, least senstitive to $b$, since $d\theta$ and $db$ have respectively the largest and smallest coefficients.

c) $dA = \frac{1}{2}(.02 + .01 + 1.73(.02) \approx \frac{1}{2}(.065) \approx .03$

**2C-4** a) $P = \dfrac{kT}{V}$; therefore $dP = \dfrac{k}{V}\,dT - \dfrac{kT}{V^2}\,dV$

b) $V\,dP + P\,dV = k\,dT$; therefore $dP = \dfrac{k\,dT - P\,dV}{V}$.

c) Substituting $P = kT/V$ into (b) turns it into (a).

**2C-5** a) $-\dfrac{dw}{w^2} = -\dfrac{dt}{t^2} - \dfrac{du}{u^2} - \dfrac{dv}{v^2}$; therefore $dw = w^2\left(\dfrac{dt}{t^2} + \dfrac{du}{u^2} + \dfrac{dv}{v^2}\right)$.

b) $2u\,du + 4v\,dv + 6w\,dw = 0$; therefore $dw = -\dfrac{u\,du + 2v\,dv}{3w}$.

## 2D. Gradient; Directional Derivative

**2D-1** a) $\nabla f = 3x^2\,\mathbf{i} + 6y^2\,\mathbf{j}$; $(\nabla f)_P = 3\,\mathbf{i} + 6\,\mathbf{j}$; $\left.\dfrac{df}{ds}\right|_{\mathbf{u}} = (3\,\mathbf{i} + 6\,\mathbf{j})\cdot\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} = -\dfrac{3\sqrt{2}}{2}$

b) $\nabla w = \dfrac{y}{z}\,\mathbf{i} + \dfrac{x}{z}\,\mathbf{j} - \dfrac{xy}{z^2}\,\mathbf{k}$; $(\nabla w)_P = -\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$; $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (\nabla w)_P\cdot\dfrac{\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}}{3} = -\dfrac{1}{3}$

c) $\nabla z = (\sin y - y\sin x)\,\mathbf{i} + (x\cos y + \cos x)\,\mathbf{j}$; $(\nabla z)_P = \mathbf{i} + \mathbf{j}$;

$$\left.\frac{dz}{ds}\right|_{\mathbf{u}} = (\,\mathbf{i} + \mathbf{j}\,)\cdot\frac{-3\,\mathbf{i} + 4\,\mathbf{j}}{5} = \frac{1}{5}$$

d) $\nabla w = \dfrac{2\,\mathbf{i} + 3\,\mathbf{j}}{2t+3u}$; $\quad (\nabla w)_P = 2\,\mathbf{i} + 3\,\mathbf{j}$; $\quad \left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \dfrac{4i - 3\,\mathbf{j}}{5} = -\dfrac{1}{5}$

e) $\nabla f = 2(u+2v+3w)(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$; $\quad (\nabla f)_P = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$

$$\left.\frac{df}{ds}\right|_{\mathbf{u}} = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,) \cdot \frac{-2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = -\frac{4}{3}$$

**2D-2** a) $\nabla w = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{4x-3y}$; $\quad (\nabla w)_P = 4\,\mathbf{i} - 3\,\mathbf{j}$

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (4\,\mathbf{i} - 3\,\mathbf{j}\,) \cdot \mathbf{u}$ has maximum 5, in the direction $\mathbf{u} = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{5}$,
and minimum $-5$ in the opposite direction.

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{3\,\mathbf{i} + 4\,\mathbf{j}}{5}$.

b) $\nabla w = \langle y+z, x+z, x+y\rangle$; $\quad (\nabla w)_P = \langle 1, 3, 0\rangle$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{10}$, direction $\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$; $\quad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{10}$, direction $-\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\mathbf{u} = \pm\dfrac{-3\,\mathbf{i} + \mathbf{j} + c\,\mathbf{k}}{\sqrt{10+c^2}}$ (for all $c$)

c) $\nabla w = 2\sin(t-u)\cos(t-u)(\,\mathbf{i} - \mathbf{j}\,) = \sin 2(t-u)(\,\mathbf{i} - \mathbf{j}\,)$; $\quad (\nabla w)_P = \mathbf{i} - \mathbf{j}$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{2}$, direction $\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$; $\quad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{2}$, direction $-\dfrac{-\,\mathbf{i} + \mathbf{j}}{\sqrt{2}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

**2D-3** a) $\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$; $\quad (\nabla f)_P = \langle 4, 12, 36\rangle$; $\quad$ normal at $P$: $\langle 1, 3, 9\rangle$;
tangent plane at $P$: $x + 3y + 9z = 18$

b) $\nabla f = \langle 2x, 8y, 18z\rangle$; $\quad$ normal at $P$: $\langle 1, 4, 9\rangle$, $\quad$ tangent plane: $x + 4y + 9z = 14$.

c) $(\nabla w)_P = \langle 2x_0, 2y_0, -2z_0\rangle$; $\quad$ tangent plane: $x_0(x-x_0) + y_0(y-y_0) - z_0(z-z_0) = 0$,
or $x_0x + y_0y - z_0z = 0$, since $x_0^2 + y_0^2 - z_0^2 = 0$.

**2D-4** a) $\nabla T = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{x^2+y^2}$; $\quad (\nabla T)_P = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5}$;

$T$ is increasing at $P$ most rapidly in the direction of $(\nabla T)_P$, which is $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$.

b) $|\nabla T| = \dfrac{2}{\sqrt{5}}$ = rate of increase in direction $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$. Call the distance to go $\Delta s$, then

$$\frac{2}{\sqrt{5}}\,\Delta s = .20 \quad \Rightarrow \quad \Delta s = \frac{.2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22.$$

c) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5} \cdot \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \dfrac{6}{5\sqrt{2}}$;

$$\frac{6}{5\sqrt{2}}\,\Delta s = .12 \quad \Rightarrow \quad \Delta s = \frac{5\sqrt{2}}{6}\,(.12) \;\approx\; (.10)(\sqrt{2}) \;\approx\; .14$$

d) In the directions orthogonal to the gradient: $\pm\dfrac{2\,\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

**2D-5** a) isotherms = the level surfaces $x^2 + 2y^2 + 2z^2 = c$, which are ellipsoids.

b) $\nabla T = \langle 2x, 4y, 4z \rangle$; $(\nabla T)_P = \langle 2, 4, 4 \rangle$; $|(\nabla T)_P| = 6$;
for most rapid decrease, use direction of $-(\nabla T)_P$ : $-\frac{1}{3}\langle 1, 2, 2 \rangle$

c) let $\Delta s$ be distance to go; then $-6(\Delta s) = -1.2$; $\Delta s \approx .2$

d) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \langle 2, 4, 4 \rangle \cdot \dfrac{\langle 1, -2, 2 \rangle}{3} = \dfrac{2}{3}$; $\dfrac{2}{3}\Delta s \approx .10 \ \Rightarrow\ \Delta s \approx .15.$

**2D-6** $\nabla uv = \langle (uv)_x, (uv)_y \rangle = \langle uv_x + vu_x, uv_y + vu_y \rangle = \langle uv_x, uv_y \rangle + \langle vu_x + vu_y \rangle = u\nabla v + v\nabla u$

$$\nabla(uv) = u\nabla v + v\nabla u \quad\Rightarrow\quad \nabla(uv)\cdot\mathbf{u} = u\nabla v\cdot\mathbf{u} + v\nabla u\cdot\mathbf{u} \quad\Rightarrow\quad \left.\frac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\frac{dv}{ds}\right|_{\mathbf{u}} + v\left.\frac{du}{ds}\right|_{\mathbf{u}}.$$

**2D-7** At $P$, let $\nabla w = a\,\mathbf{i} + b\,\mathbf{j}$. Then

$$a\,\mathbf{i} + b\,\mathbf{j} \cdot \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = 2 \quad\Rightarrow\quad a + b = 2\sqrt{2}$$
$$a\,\mathbf{i} + b\,\mathbf{j} \cdot \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} = 1 \quad\Rightarrow\quad a - b = \sqrt{2}$$

Adding and subtracting the equations on the right, we get $a = \frac{3}{2}\sqrt{2}$, $b = \frac{1}{2}\sqrt{2}$.

**2D-8** We have $P(0,0,0) = 32$; we wish to decrease it to 31.1 by traveling the shortest distance from the origin $\mathbf{0}$; for this we should travel in the direction of $-(\nabla P)_{\mathbf{0}}$.

$$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z \rangle; \quad (\nabla P)_{\mathbf{0}} = \langle 2, 1, 2 \rangle. \qquad |(\nabla P)_{\mathbf{0}}| = 3.$$

Since $(-3)\cdot(\Delta s) = -.9 \quad\Rightarrow\quad \Delta s = .3$, we should travel a distance .3 in the direction of $-(\nabla P)_{\mathbf{0}}$. Since $|-\langle 2,1,2 \rangle| = 3$, the distance .3 will be $\frac{1}{10}$ of the distance from $(0,0,0)$ to $(-2,-1,-2)$, which will bring us to $(-.2, -.1, -.2)$.

**2D-9** In these, we use $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} \approx \dfrac{\Delta w}{\Delta s}$: we travel in the direction $\mathbf{u}$ from a given point $P$ to the nearest level curve $C$; then $\Delta s$ is the distance traveled (estimate it by using the unit distance), and $\Delta w$ is the corresponding change in $w$ (estimate it by using the labels on the level curves).

a) The *direction* of $\nabla f$ is perpendicular to the level curve at $A$, in the increasing sense (the "uphill" direction). The *magnitude* of $\nabla f$ is the directional derivative in that direction: from the picture, $\dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$.

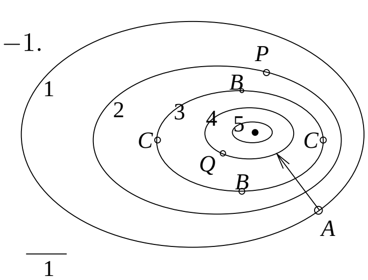
b), c) $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}}$, $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}}$, so $B$ will be where $\mathbf{i}$ is tangent to the level curve and $C$ where $\mathbf{j}$ is tangent to the level curve.

d) At P, $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/3} = -.6$; $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{1} = -1.$

e) If $\mathbf{u}$ is the direction of $\mathbf{i} + \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$

f) If $\mathbf{u}$ is the direction of $\mathbf{i} - \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_{u} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/4} = -.8$

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.



## 2E. Chain Rule

**2E-1**

a) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5$

(ii) $w = xyz = t^6$; $\quad \dfrac{dw}{dt} = 6t^5$

b) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4\sin t\cos t$

(ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$; $\quad \dfrac{dw}{dt} = -2\sin 2t$

c) (i) $\dfrac{dw}{dt} = \dfrac{2u}{u^2+v^2}(-2\sin t) + \dfrac{2v}{u^2+v^2}(2\cos t) = -\cos t\sin t + \sin t\cos t = 0$

(ii) $w = \ln(u^2+v^2) = \ln(4\cos^2 t + 4\sin^2 t) = \ln 4$; $\quad \dfrac{dw}{dt} = 0.$

**2E-2** a) The value $t = 0$ corresponds to the point $(x(0), y(0)) = (1, 0) = P$.

$$\left.\frac{dw}{dt}\right|_0 = \left.\frac{\partial w}{\partial x}\right|_P \left.\frac{dx}{dt}\right|_0 + \left.\frac{\partial w}{\partial y}\right|_P \left.\frac{dy}{dt}\right|_0 = -2\,\sin t + 3\,\cos t\Big]_0 = 3.$$

b) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t.$

$\dfrac{dw}{dt} = 0$ when $2t = \dfrac{\pi}{2} + n\pi$, therefore when $t = \dfrac{\pi}{4} + \dfrac{n\pi}{2}$.

c) $t = 1$ corresponds to the point $(x(1), y(1), z(1)) = (1, 1, 1)$.

$$\left.\frac{df}{dt}\right|_1 = 1 \cdot \left.\frac{dx}{dt}\right|_1 - 1 \cdot \left.\frac{dy}{dt}\right|_1 + 2 \cdot \left.\frac{dz}{dt}\right|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5.$$

d) $\dfrac{df}{dt} = 3x^2y\dfrac{dx}{dt} + (x^3+z)\dfrac{dy}{dt} + y\dfrac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4.$

**2E-3** a) Let $w = uv$, where $u = u(t),\ v = v(t)$; $\quad \dfrac{dw}{dt} = \dfrac{\partial w}{\partial u}\dfrac{du}{dt} + \dfrac{\partial w}{\partial v}\dfrac{dv}{dt} = v\,\dfrac{du}{dt} + u\,\dfrac{dv}{dt}.$

b) $\dfrac{d(uvw)}{dt} = vw\dfrac{du}{dt} + uw\dfrac{dv}{dt} + uv\dfrac{dw}{dt}$; $\quad e^{2t}\sin t + 2te^{2t}\sin t + te^{2t}\cos t$

**2E-4** The values $u = 1,\ v = 1$ correspond to the point $x = 0,\ y = 1$. At this point,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = 2 \cdot 2u + 3 \cdot v = 2 \cdot 2 + 3 = 7.$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = 2 \cdot (-2v) + 3 \cdot u = 2 \cdot (-2) + 3 \cdot 1 = -1.$$

**2E-5** a) $w_r = w_x x_r + w_y y_r = w_x\cos\theta + w_y\sin\theta$

$w_\theta = w_x x_\theta + w_y y_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$

Therefore,

$(w_r)^2 + (w_\theta/r)^2$

$= (w_x)^2(\cos^2\theta + \sin^2\theta) + (w_y)^2(\sin^2\theta + \cos^2\theta) + 2w_xw_y\cos\theta\sin\theta - 2w_xw_y\sin\theta\cos\theta$

$= (w_x)^2 + (w_y)^2.$

b) The point $r = \sqrt{2}$, $\theta = \pi/4$ in polar coordinates corresponds in rectangular coordinates to the point $x = 1,\ y = 1$. Using the chain rule equations in part (a),

$$w_r = w_x \cos\theta + w_y \sin\theta; \qquad w_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$$

but evaluating all the partial derivatives at the point, we get

$$w_r = 2\cdot\tfrac{1}{2}\sqrt{2} - 1\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}; \qquad \frac{w_\theta}{r} = 2(-\tfrac{1}{2})\sqrt{2} - \tfrac{1}{2}\sqrt{2} = -\tfrac{3}{2}\sqrt{2};$$

$$(w_r)^2 + \frac{1}{r}(w_\theta)^2 = \tfrac{1}{2} + \tfrac{9}{2} = 5; \qquad (w_x)^2 + (w_y)^2 = 2^2 + (-1)^2 = 5.$$

**2E-6** $w_u = w_x \cdot 2u + w_y \cdot 2v$; $\quad w_v = w_x \cdot (-2v) + w_y \cdot 2u$, by the chain rule. Therefore

$$\begin{aligned}(w_u)^2 + (w_v)^2 &= [4u^2(w_x) + 4v^2(w_y)^2 + 4uvw_xw_y] + [4v^2(w_x) + 4u^2(w_y)^2 - 4uvw_xw_y] \\ &= 4(u^2+v^2)[(w_x)^2 + (w_y)^2].\end{aligned}$$

**2E-7** By the chain rule, $f_u = f_x x_u + f_y y_u$, $\quad f_v = f_x x_v + f_y y_v$; $\qquad$ therefore

$$\langle f_u\ f_v\rangle = \langle f_x\ f_y\rangle \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

**2E-8** a) By the chain rule for functions of one variable,

$$\frac{\partial w}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = f'(u)\cdot -\frac{y}{x^2}; \qquad \frac{\partial w}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y} = f'(u)\cdot\frac{1}{x};$$

Therefore,

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = f'(u)\cdot -\frac{y}{x} + f'(u)\cdot\frac{y}{x} = 0.$$

## 2F. Maximum-minimum Problems

**2F-1** In these, denote by $D = x^2 + y^2 + z^2$ the square of the distance from the point $(x, y, z)$ to the origin; then the point which minimizes $D$ will also minimize the actual distance.

a) Since $z^2 = \dfrac{1}{xy}$, we get on substituting, $D = x^2 + y^2 + \dfrac{1}{xy}$. with $x$ and $y$ independent; setting the partial derivatives equal to zero, we get

$$D_x = 2x - \frac{1}{x^2y} = 0; \quad D_y = 2y - \frac{1}{y^2x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.$$

Solving, we see first that $x^2 = \dfrac{1}{2xy} = y^2$, from which $y = \pm x$.

If $y = x$, then $x^4 = \frac{1}{2}$ and $x = y = 2^{-1/4}$, and so $z = 2^{1/4}$; if $y = -x$, then $x^4 = -\frac{1}{2}$ and there are no solutions. Thus the unique point is $(1/2^{1/4}, 1/2^{1/4}, 2^{1/4})$.

b) Using the relation $x^2 = 1 + yz$ to eliminate $x$, we have $D = 1 + yz + y^2 + z^2$, with $y$ and $z$ independent; setting the partial derivatives equal to zero, we get

$$D_y = 2y + z = 0, \quad D_z = 2z + y = 0;$$

solving, these equations only have the solution $y = z = 0$; therefore $x = \pm 1$, and there are two points: $(\pm 1, 0, 0)$, both at distance 1 from the origin.

**2F-2** Letting $x$ be the length of the ends, $y$ the length of the sides, and $z$ the height, we have

$$\text{total area of cardboard} \ \ A = 3xy + 4xz + 2yz, \qquad \text{volume} \ \ V = xyz = 1.$$

Eliminating $z$ to make the remaining variables independent, and equating the partials to zero, we get

$$A = 3xy + \frac{4}{y} + \frac{2}{x}; \qquad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0.$$

From these last two equations, we get

$$3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x$$

$$\Rightarrow \quad 3x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{3^{1/3}}, \ \ y = \frac{2}{3^{1/3}}, \ \ z = \frac{1}{xy} = \frac{3^{2/3}}{2} = \frac{3}{2 \cdot 3^{1/3}};$$

therefore the proportions of the most economical box are $x : y : z = 1 : 2 : \frac{3}{2}$.

**2F-5** The cost is $C = xy + xz + 4yz + 4xz$, where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

minimize: $C = xy + 5xz + 4yz$, with the constraint: $xyz = V = 2.5$

Substituting $z = V/xy$ into $C$, we get

$$C = xy + \frac{5V}{y} + \frac{4V}{x}; \qquad \frac{\partial C}{\partial x} = y - \frac{4V}{x^2}, \quad \frac{\partial C}{\partial y} = x - \frac{5V}{y^2}.$$

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating $y$; we get $x^3 = \dfrac{16V}{5} = 8$, (using $V = 5/2$), so $x = 2$, $y = \frac{5}{2}$, $z = \frac{1}{2}$.

## 2G. Least-squares Interpolation

**2G-1** Find $y = mx + b$ that best fits $(1,1)$, $(2,3)$, $(3,2)$ .

$$\begin{aligned} D &= (m+b-1)^2 + (2m+b-3)^2 + (3m+b-2)^2 \\ \frac{\partial D}{\partial m} &= 2(m+b-1) + 4(2m+b-3) + 6(3m+b-2) \ = \ 2(14m+6b-13) \\ \frac{\partial D}{\partial b} &= 2(m+b-1) + 2(2m+b-3) + 2(3m+b-2) \ = \ 2(6m+3b-6). \end{aligned}$$

Thus the equations $\dfrac{\partial D}{\partial m} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ are $\begin{cases} 14m + 6b = 13 \\ 6m + 3b = 6 \end{cases}$, whose solution is $m = \frac{1}{2}$, $b = 1$, and the line is $y = \frac{1}{2}x + 1$ .

**2G-4** $D = \sum_i (a + bx_i + cy_i - z_i)^2$. The equations are

$$\begin{aligned} \partial D/\partial a &= \textstyle\sum 2(a + bx_i + cy_i - z_i) \ = \ 0 \\ \partial D/\partial b &= \textstyle\sum 2x_i(a + bx_i + cy_i - z_i) \ = \ 0 \\ \partial D/\partial c &= \textstyle\sum 2y_i(a + bx_i + cy_i - z_i) \ = \ 0 \end{aligned}$$

Cancel the 2's; the equations become (on the right, $\mathbf{x} = [x_1, \dots, x_n]$, $\mathbf{1} = [1, \dots, 1]$, etc.)

$$\begin{aligned} na + (\textstyle\sum x_i)b + (\sum y_i)c &= \textstyle\sum z_i & \qquad & n\ a + (\mathbf{x}\cdot\mathbf{1})\ b + (\mathbf{y}\cdot\mathbf{1})\ c = \mathbf{z}\cdot\mathbf{1} \\ (\textstyle\sum x_i)a + (\sum x_i^2)b + (\sum x_i y_i)c &= \textstyle\sum x_i z_i & \text{or} \qquad & (\mathbf{x}\cdot\mathbf{1})\ a + (\mathbf{x}\cdot\mathbf{x})\ b + (\mathbf{x}\cdot\mathbf{y})\ c = \mathbf{x}\cdot\mathbf{z} \\ (\textstyle\sum y_i)a + (\sum x_i y_i)b + (\sum y_i^2)c &= \textstyle\sum y_i z_i & \qquad & (\mathbf{y}\cdot\mathbf{1})\ a + (\mathbf{x}\cdot\mathbf{y}\ )b + (\mathbf{y}\cdot\mathbf{y})\ c = \mathbf{y}\cdot\mathbf{z} \end{aligned}$$

## 2H. Max-min: 2nd Derivative Criterion; Boundary Curves

**2H-1**

a) $f_x = 0: \ 2x - y = 3; \quad f_y = 0: \ -x - 4y = 3 \qquad$ critical point: $(1, -1)$
$A = f_{xx} = 2; \ B = f_{xy} = -1; \ C = f_{yy} = -4; \qquad AC - B^2 = -9 < 0$; saddle point

b) $f_x = 0: \ 6x + y = 1; \quad f_y = 0: \ x + 2y = 2 \qquad$ critical point: $(0, 1)$
$A = f_{xx} = 6; \ B = f_{xy} = 1; \ C = f_{yy} = 2; \qquad AC - B^2 = 11 > 0$; local minimum

c) $f_x = 0: \ 8x^3 - y = 0; \quad f_y = 0: \ 2y - x = 0; \qquad$ eliminating $y$, we get

$16x^3 - x = 0$, or $x(16x^2 - 1) = 0 \ \Rightarrow \ x = 0, \ x = \frac{1}{4}, \ x = -\frac{1}{4}$, giving the critical points

$$(0,0), \quad (\tfrac{1}{4}, \tfrac{1}{8}), \quad (-\tfrac{1}{4}, -\tfrac{1}{8}).$$

Since $f_{xx} = 24x^2, \quad f_{xy} = -1, \quad f_{yy} = 2$, we get for the three points respectively:

$(0,0): \ \Delta = -1$ (saddle); $\quad (\frac{1}{4}, \frac{1}{8}): \ \Delta = 2$ (minimum); $\qquad (-\frac{1}{4}, -\frac{1}{8}): \ \Delta = 2$ (minimum)

d) $f_x = 0: \quad 3x^2 - 3y = 0; \qquad f_y = 0: \quad -3x + 3y^2 = 0$. Eliminating $y$ gives

$-x + x^4 = 0$, or $x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0, \ y = 0$ or $x = 1, \ y = 1$.

Since $f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 6y$, we get for the two critical points respectively:

$(0,0): \quad AC - B^2 = -9$ (saddle); $\qquad (1,1): \quad AC - B^2 = 27$ (minimum)

e) $f_x = 0: \quad 3x^2(y^3 + 1) = 0; \qquad f_y = 0: \quad 3y^2(x^3 + 1) = 0$; solving simultaneously, we get from the first equation that either $x = 0$ or $y = -1$; finding in each case the other coordinate then leads to the two critical points $(0, 0)$ and $(-1, -1)$.

Since $f_{xx} = 6x(y^3 + 1), \quad f_{xy} = 3x^2 \cdot 3y^2, \quad f_{yy} = 6y(x^3 + 1)$ , we have

$(-1, -1): \quad AC - B^2 = -9$ (saddle); $\qquad (0,0): \quad AC - B^2 = 0$, test fails.

(By studying the behavior of $f(x, y)$ on the lines $y = mx$, for different values of $m$, it is possible to see that also $(0, 0)$ is a saddle point.)

**2H-3** The region $R$ has no critical points; namely, the equations $f_x = 0$ and $f_y = 0$ are

$$2x + 2 = 0, \quad 2y + 4 = 0 \quad \Rightarrow \quad x = -1, \ y = -2,$$

but this point is not in $R$. We therefore investigate the diagonal boundary of $R$, using the parametrization $x = t, \ y = -t$. Restricted to this line, $f(x, y)$ becomes a function of $t$ alone, which we denote by $g(t)$, and we look for its maxima and minima.

$$g(t) = f(t, -t) = 2t^2 - 2t - 1; \qquad g'(t) = 4t - 2, \text{ which is 0 at } t = 1/2.$$

This point is evidently a minimum for $g(t)$; there is no maximum: $g(t)$ tends to $\infty$. Therefore for $f(x, y)$ on $R$, the minimum occurs at the point $(1/2, -1/2)$, and there is no maximum; $f(x, y)$ tends to infinity in different directions in $R$.

**2H-4** We have $f_x = y - 1, \quad f_y = x - 1,$ so the only critical point is at $(1, 1)$.

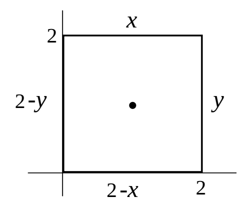
a) On the two sides of the boundary, the function $f(x, y)$ becomes respectively

$$y = 0: \quad f(x, y) \;=\; -x + 2; \qquad x = 0: \quad f(x, y) \;=\; -y + 2.$$

Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is $-\infty$). Since $f(1, 1) = 1$ and $f(x, x) = x^2 - 2x + 2 \to \infty$ as $x \to \infty$, the maximum of $f$ on the first quadrant is $\infty$.

b) Continuing the reasoning of (a) to find the maximum and minimum points of $f(x, y)$ on the boundary, on the other two sides of the boundary square, the function $f(x, y)$ becomes

$$y = 2: \quad f(x, y) = x \qquad\qquad x = 2: \quad f(x, y) = y$$




Since $f(x, y)$ is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square $R$ can only occur at the four corner points; evaluating $f(x, y)$ at these four points, we find

$$f(0, 0) = 2; \qquad f(2, 2) = 2; \qquad\qquad f(2, 0) = 0; \qquad f(0, 2) = 0.$$

As in (a), since $f(1, 1) = 1$,
maximum points of $f$ on $R$: $(0, 0)$ and $(2, 2)$; minimum points: $(2, 0)$ and $(0, 2)$.

c) The data indicates that $(1, 1)$ is probably a saddle point. Confirming this, we have $f_{xx} = 0, \; f_{xy} = 1, \; f_{yy} = 0$ for all $x$ and $y$; therefore $AC - B^2 = -1 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

**2H-5** Since $f(x, y)$ is linear, it will not have critical points: namely, for all $x$ and $y$ we have $f_x = 1, \; f_y = \sqrt{3}$. So any maxima or minima must occur on the boundary circle.

We parametrize the circle by $x = \cos\theta, \; y = \sin\theta$; restricted to this boundary circle, $f(x, y)$ becomes a function of $\theta$ alone which we call $g(\theta)$:

$$g(\theta) = f(\cos\theta, \sin\theta) = \cos\theta + \sqrt{3}\sin\theta + 2.$$

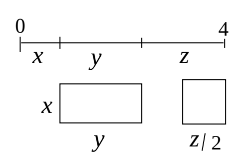
Proceeding in the usual way to find the maxima and minima of $g(\theta)$, we get

$$g'(\theta) = -\sin\theta + \sqrt{3}\cos\theta = 0, \qquad \text{or} \qquad \tan\theta = \sqrt{3}.$$

It follows that the two critical points of $g(\theta)$ are $\theta = \dfrac{\pi}{3}$ and $\dfrac{4\pi}{3}$; evaluating $g$ at these two points, we get $g(\pi/3) = 4$ (the maximum), and $g(4\pi/3) = 0$ (the minimum).

Thus the maximum of $f(x, y)$ in the circular disc $R$ is at $(1/2, \sqrt{3}/2)$, while the minimum is at $(-1/2, -\sqrt{3}/2)$.

**2H-6** a) Since $z = 4 - x - y$, the problem is to find on $R$ the maximum and minimum of the total area




$$f(x, y) = xy + \tfrac{1}{4}(4 - x - y)^2$$

where $R$ is the triangle given by$R: \; 0 \le x, \quad 0 \le y, \quad x + y \le 4.$

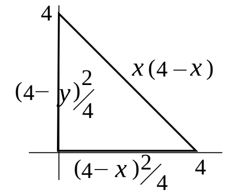
To find the critical points of $f(x, y)$, the equations $f_x = 0$ and $f_y = 0$ are respectively

$$y - \tfrac{1}{2}(4 - x - y) = 0; \quad x - \tfrac{1}{2}(4 - x - y) = 0,$$

which imply first that $x = y$, and from this, $x - \frac{1}{2}(4 - 2x)$; the unique solution is $x = 1, \; y = 1$.

The region $R$ is a triangle, on whose sides $f(x,y)$ takes respectively the values

bottom: $y=0$; $f=\frac{1}{4}(4-x)^2$; left side: $x=0$; $f=\frac{1}{4}(4-y)^2$;
diagonal $y=4-x$; $f=x(4-x)$.

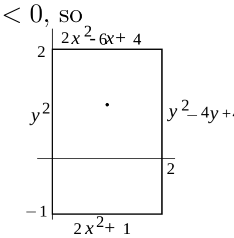



On the bottom and side, $f$ is decreasing; on the diagonal, $f$ has a maximum at $x=2,\ y=2$. Therefore we need to examine the three corner points and $(2,2)$ as candidates for maximum and minimum points, as well as the critical point $(1,1)$. We find

$$f(0,0)=4; \quad f(4,0)=0; \quad f(0,4)=0; \quad f(2,2)=4 \quad f(1,1)=2.$$

It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x=y=0,\ z=4$, or $x=y=2,\ z=0$, either of which gives a point "rectangle" and a square of side 2; for the minimum total area 0, take for example $x=0, y=4, z=0$, which gives a "rectangle" of length 4 with zero area, and a point square.

b) We have $f_{xx}=\frac{1}{2},\ f_{xy}=\frac{3}{2},\ f_{yy}=\frac{1}{2}$ for all $x$ and $y$; therefore $AC-B^2=-2<0$, so $(1,1)$ is a saddle point, by the 2nd-derivative criterion.




**2H-7** a) $f_x=4x-2y-2, \quad f_y=-2x+2y$; setting these $=0$ and solving simultaneously, we get $x=1,\ y=1$, which is therefore the only critical point.

On the four sides of the boundary rectangle $R$, the function $f(x,y)$ becomes:

on $y=-1$: $f(x,y)=2x^2+1$; on $y=2$: $f(x,y)=2x^2-6x+4$
on $x=0$: $f(x,y)=y^2$; on $x=2$: $f(x,y)=y^2-4y+4$

By one-variable calculus, $f(x,y)$ is increasing on the bottom and decresing on the right side; on the left side it has a minimum at $(0,0)$, and on the top a minimum at $(\frac{3}{2},2)$. Thus the maximum and minimum points on the boundary rectangle $R$ can only occur at the four corner points, or at $(0,0)$ or $(\frac{3}{2},2)$. At these we find:

$$f(0,-1)=1; \quad f(0,2)=4; \quad f(2,-1)=9; \quad f(2,2)=0; \quad f(\tfrac{3}{2},2)=-\tfrac{1}{2}, \quad f(0,0)=0.$$

At the critical point $f(1,1)=-1$; comparing with the above, it is a minimum; therefore,
maximum point of $f(x,y)$ on $R$: $(2,-1)$ minimum point of $f(x,y)$ on $R$: $(1,1)$

b) We have $f_{xx}=4,\ f_{xy}=-2,\ f_{yy}=2$ for all $x$ and $y$; therefore $AC-B^2=4>0$ and $A=4>0$, so $(1,1)$ is a minimum point, by the 2nd-derivative criterion.

## 2I. Lagrange Multipliers

**2I-1** Letting $P:(x,y,z)$ be the point, in both problems we want to maximize $V=xyz$, subject to a constraint $f(x,y,z)=c$. The Lagrange equations for this, in vector form, are

$$\nabla(xyz)=\lambda\cdot\nabla f(x,y,z), \qquad f(x,y,z)=c.$$

a) Here $f=c$ is $x+2y+3z=18$; equating components, the Lagrange equations become

$$yz=\lambda, \quad xz=2\lambda, \quad xy=3\lambda; \qquad x+2y+3z=18.$$

To solve these symmetrically, multiply the left sides respectively by $x$, $y$, and $z$ to make them equal; this gives

$$\lambda x=2\lambda y=3\lambda z, \quad \text{or} \quad x=2y=3z=6, \text{ since the sum is 18.}$$

We get therefore as the answer $x = 6, \quad y = 3, \quad z = 2$. This is a maximum point, since if $P$ lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here $f = c$ is $x^2 + 2y^2 + 4z^2 = 12$; equating components, the Lagrange equations become

$$yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \qquad x^2 + 2y^2 + 4z^2 = 12.$$

To solve these symmetrically, multiply the left sides respectively by $x$, $y$, and $z$ to make them equal; this gives

$$\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad \text{or} \quad x^2 = 2y^2 = 4z^2 = 4, \quad \text{since the sum is 12.}$$

We get therefore as the answer $x = 2, \quad y = \sqrt{2}, \quad z = 1$. This is a maximum point, since if $P$ lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

**2I-2** Since we want to minimize $x^2 + y^2 + z^2$, subject to the constraint $x^3y^2z = 6\sqrt{3}$, the Lagrange multiplier equations are

$$2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^3y^2z = 6\sqrt{3}.$$

To solve them symmetrically, multiply the first three equations respectively by $x$, $y$, and $z$, then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

$$\frac{x^2}{3} = \frac{y^2}{2} = z^2; \quad \text{therefore } x = z\sqrt{3},\ y = z\sqrt{2}.$$

Substituting into $x^3y^2z = 6\sqrt{3}$, we get $3\sqrt{3}z^3 \cdot 2z^2 \cdot z = 6\sqrt{3}$, which gives as the answer, $x = \sqrt{3},\ y = \sqrt{2},\ z = 1$.

This is clearly a minimum, since if $P$ is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since $x^3y^2z = 6\sqrt{3}$; thus $P$ will be far from the origin.

**2I-3** Referring to the solution of 2F-2, we let $x$ be the length of the ends, $y$ the length of the sides, and $z$ the height, and get

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \qquad \text{volume } V = xyz = 1.$$

The Lagrange multiplier equations $\nabla A = \lambda \cdot \nabla(xyz)$; $xyz = 1$, then become

$$3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.$$

To solve these equations for $x, y, z, \lambda$, treat them symmetrically. Divide the first equation through by $yz$, and treat the next two equations analogously, to get

$$3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,$$

which by subtracting the equations in pairs leads to $3/z = 4/y = 2/x$; setting these all equal to $k$, we get $x = 2/k, y = 4/k, z = 3/k$, which shows the proportions using least cardboard are $x : y : z = 2 : 4 : 3$.

To find the actual values of $x, y$, and $z$, we set $1/k = m$; then substituting into $xyz = 1$ gives $(2m)(4m)(3m) = 1$, from which $m^3 = 1/24$, $m = 1/2 \cdot 3^{1/3}$, giving finally

$$x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.$$

**2I-4** The equations for the cost $C$ and the volume $V$ are $xy+4yz+6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

a) $$yz = \lambda(y+6z), \quad xz = \lambda(x+4z), \quad xy = \lambda(4y+6x); \qquad xy+4yz+6xz = 72$$

b) $$y+6z = \mu\cdot yz, \quad x+4z = \mu\cdot xz, \quad 4y+6x = \mu\cdot xy; \qquad xyz = 24$$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by $x$, $y$, and $z$ respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the $\lambda$,

$$xy+6xz = xy+4yz = 4yz+6xz, \quad \text{which implies} \quad xy = 4yz = 6xz.$$

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \quad yz = 6, \quad xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4$, $y = 6$, $z = 1$.

b) Dividing $xy = 4yz = 6xz$ by $xyz$ leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4$, $y = 6$, $z = 1$.

## 2J. Non-independent Variables

**2J-1** a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ means that $x$ is the dependent variable; get rid of it by writing $w = (z-y)^2 + y^2 + z^2 = z + z^2$. This shows that $\left(\dfrac{\partial w}{\partial y}\right)_z = 0$.

b) To calculate $\left(\dfrac{\partial w}{\partial z}\right)_y$, once again $x$ is the dependent variable; as in part (a), we have $w = z + z^2$ and so $\left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z$.

**2J-2** a) Differentiating $z = x^2 + y^2$ w.r.t. $y$: $\quad 0 = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y$; so $\left(\dfrac{\partial x}{\partial y}\right)_z = -\dfrac{y}{x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial y}\right)_z = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y = 2x\left(\dfrac{-y}{x}\right) + 2y = 0$.

Differentiating $z = x^2 + y^2$ with respect to $z$: $\quad 1 = 2x\left(\dfrac{\partial x}{\partial z}\right)_y$; so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{1}{2x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial z}\right)_y = 2x\left(\dfrac{\partial x}{\partial z}\right)_y + 2z = 1 + 2z$.

b) Using differentials, $\quad dw = 2xdx + 2ydy + 2zdz, \quad dz = 2xdx + 2ydy;$
since the independent variables are $y$ and $z$, we eliminate $dx$ by substracting the second equation from the first, which gives $\quad dw = 0\,dy + (1+2z)\,dz;$
therefore we get $\quad \left(\dfrac{\partial w}{\partial y}\right)_z = 0, \quad \left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z.$

**2J-3** a) To calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, we see that $y$ is the dependent variable; solving for it, we get $y = \frac{zt}{x}$; using the chain rule, $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2 = x^3\frac{z}{x} - z^2 = x^2z - z^2$.

b) Similarly, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ means that $t$ is the dependent variable; since $t = \frac{xy}{z}$, we have by the chain rule, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2zt - z^2\left(\frac{\partial t}{\partial z}\right)_{x,y} = -2zt - z^2\cdot\frac{-xy}{z^2} = -zt$.

**2J-4** The differentials are calculated in equation (4).

a) Since $x, z, t$ are independent, we eliminate $dy$ by solving the second equation for $x\,dy$, substituting this into the first equation, and grouping terms:
$dw = 2x^2y\,dx + (x^2z - z^2)dt + (x^2t - 2zt)dz$, which shows that $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2z - z^2$.

b) Since $x, y, z$ are independent, we eliminate $dt$ by solving the second equation for $z\,dt$, substituting this into the first equation, and grouping terms:
$dw = (3x^2y - zy)dx + (x^3 - zx)dy - zt\,dz$, which shows that $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt$.

**2J-5** a) If $pv = nRT$, then $\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T\cdot\left(\frac{\partial T}{\partial p}\right)_v = S_p + S_T\cdot\frac{v}{nR}$.

b) Similarly, we have $\left(\frac{\partial S}{\partial T}\right)_v = S_T + S_p\cdot\left(\frac{\partial p}{\partial T}\right)_v = S_T + S_p\cdot\frac{nR}{v}$.

**2J-6** a) $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - u\cdot 2v\left(\frac{\partial v}{\partial u}\right)_x = 3u^2 - v^2 - 2uv$.

$$\left(\frac{\partial w}{\partial x}\right)_u = -u\cdot 2v\left(\frac{\partial v}{\partial x}\right)_u = -2uv.$$

b) $\quad dw = (3u^2 - v^2)du - 2uvdv; \qquad du = x\,dy + y\,dx; \qquad dv = du + dx;$
for both derivatives, $u$ and $x$ are the independent variables, so we eliminate $dv$, getting

$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv\,dx,$$

whose coefficients are $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$.

**2J-7** Since we need both derivatives for the gradient, we use differentials.

$$df = 2dx + dy - 3dz \quad \text{at } P; \qquad dz = 2x\,dx + dy = 2\,dx + dy \quad \text{at } P;$$

the independent variables are to be $x$ and $z$, so we eliminate $dy$, getting

$$df = 0\,dx - 2\,dz \quad \text{at the point } (x,z) = (1,1). \qquad \text{So } \nabla g = \langle 0, -2\rangle \text{ at } (1,1).$$

**2J-8** To calculate $\left(\frac{\partial w}{\partial r}\right)_\theta$, note that $w = r|\sin\theta|$. Therefore, $\left(\frac{\partial w}{\partial r}\right)_\theta = |\sin\theta|$.

## 2K. Partial Differential Equations

**2K-1** $w = \frac{1}{2}\ln(x^2+y^2)$. If $(x,y) \neq (0,0)$, then

$$w_{xx} = \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}\ ,$$
$$w_{yy} = \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}\ ,$$

Therefore $w$ satisfies the two-dimensional Laplace equation, $w_{xx}+w_{yy}=0$; we exclude the point $(0,0)$ since $\ln 0$ is not defined.

**2K-2** If $w = (x^2+y^2+z^2)^n$, then $\frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\big(2x\cdot n(x^2+y^2+z^2)^{n-1}\big)$

$$= 2n(x^2+y^2+z^2)^{n-1} + 4x^2n(n-1)(x^2+y^2+z^2)^{n-2}$$

We get $w_{yy}$ and $w_{zz}$ by symmetry; adding and combining, we get

$$w_{xx}+w_{yy}+w_{zz} = 6n(x^2+y^2+z^2)^{n-1} + 4(x^2+y^2+z^2)n(n-1)(x^2+y^2+z^2)^{n-2}$$

$= 2n(2n+1)(x^2+y^2+z^2)^{n-1}$, which is identically zero if $n=0$, or if $n=-1/2$.

**2K-3** a) $w = ax^2+bxy+cy^2$; $\quad w_{xx}=2a, \quad w_{yy}=2c.$

$$w_{xx}+w_{yy}=0 \quad \Rightarrow \quad 2a+2c=0, \text{ or } c=-a.$$

Therefore all quadratic polynomials satisfying the Laplace equation are of the form

$$ax^2+bxy-ay^2 = a(x^2-y^2)+bxy;$$

i.e., linear combinations of the two polynomials $f(x,y)=x^2-y^2$ and $g(x,y)=xy$ .

**2K-4** The one-dimensional wave equation is $w_{xx} = \frac{1}{c^2}w_{tt}$. So

$$\begin{aligned} w = f(x+ct)+g(x-ct) \quad &\Rightarrow \quad w_{xx} = f''(x+ct)+g''(x-ct) \\ &\Rightarrow \quad w_t = cf'(x+ct)+-cg'(x-ct). \\ &\Rightarrow \quad w_{tt} = c^2f''(x+ct)+c^2g''(x-ct) = c^2w_{xx}, \end{aligned}$$

which shows $w$ satisfies the wave equation.

**2K-5** The one-dimensional heat equation is $w_{xx} = \frac{1}{\alpha^2}w_t$. So if $w(x,t) = \sin kx e^rt$, then

$$w_{xx} = e^{rt}\cdot k^2(-\sin kx) = -k^2\,w.$$
$$w_t = re^{rt}\sin kx = r\,w.$$

Therefore, we must have $-k^2w = \dfrac{1}{\alpha^2}\,r\,w$, or $r = -\alpha^2k^2$.

However, from the additional condition that $w=0$ at $x=1$, we must have

$$\sin k\ e^{rt} = 0\ ;$$

Therefore $\sin k = 0$, and so $k = n\pi$, where $n$ is an integer.

To see what happens to $w$ as $t\to\infty$, we note that since $|\sin kx|\le 1$,

$$|w| = e^{rt}|\sin kx| \le e^{rt}.$$

Now, if $k\neq 0$, then $r=-\alpha^2k^2$ is negative and $e^{rt}\to 0$ as $t\to\infty$; therefore $|w|\to 0$.

Thus $w$ will be a solution satisfying the given side conditions if $k=n\pi$, where $n$ is a non-zero integer, and $r=-\alpha^2k^2$.

**18.02 Notes and Exercises by A. Mattuck with the assistance of T.Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.**