18.02

Calculus
Notes and Exercises

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with the assistance of T.Shifrin and S. LeDuc, and including a section on non-independent variables by Bjorn Poonen.

18.02 NOTES, EXERCISES, AND SOLUTIONS

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D. Determinants

Given a square array A of numbers, we associate with it a number called the **determinant** of A, and written either $\det(A)$, or $|A|$. For 2×2 and 3×3 arrays, the number is defined by

$$(1) \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc; \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + dhc - ceg - bdi - fha\ .$$

Do not memorize these as formulas — learn instead the patterns which give the terms. The 2×2 case is easy: the product of the elements on one diagonal (the "main diagonal"), minus the product of the elements on the other (the "antidiagonal").

For the 3×3 case, three products get the + sign: those formed from the main diagonal, or having two factors parallel to the main diagonal. The other three get a negative sign: those from the antidiagonal, or having two factors parallel to it.[1] Try the following example on your own, then check your work against the solution.

Example 1.1 Evaluate $\begin{vmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & -1 & 4 \end{vmatrix}$ using (1).

Solution. Using the same order as in (1), we get $12 + (-8) + 1 - 6 - 8 - (-2) = -7$.

Important facts about $|A|$:

D-1. $|A|$ is multiplied by -1 if we interchange two rows or two columns.

D-2. $|A| = 0$ if one row or column is all zero, or if two rows or two columns are the same.

D-3. $|A|$ is multiplied by c, if every element of some row or column is multiplied by c.

D-4. The value of $|A|$ is unchanged if we add to one row (or column) a constant multiple of another row (resp. column).

All of these facts are easy to check for 2×2 determinants from the formula (1); from this, their truth also for 3×3 determinants will follow from the Laplace expansion.

Though the letters $a, b, c, \ldots$ can be used for very small determinants, they can't for larger ones; it's important early on to get used to the standard notation for the entries of determinants. This is what the common software packages and the literature use. The determinants of order two and three would be written respectively

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

[1] There is another form for this rule which requires adding two extra columns to the determinant, but this wastes too much time in practice and leads to awkward write-ups; instead, learn to evaluate each of the six products mentally, writing it down with the correct sign, and then add the six numbers, as is done in Example 1. Note that the word "determinant" is also used for the square array itself, enclosed between two vertical lines, as for example when one speaks of "the second row of the determinant".

In general, the **ij-entry**, written a_{ij}, is the number in the i-th row and j-th column.

Its **ij-minor**, written $|A_{ij}|$, is the determinant that's left after deleting from $|A|$ the row and column containing a_{ij}.

Its **ij-cofactor**, written here A_{ij}, is given as a formula by $A_{ij} = (-1)^{i+j}|A_{ij}|$. For a 3×3 determinant, it is easier to think of it this way: we put $+$ or $-$ in front of the ij-minor, according to whether $+$ or $-$ occurs in the ij-position in the checkerboard pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}. \tag{2}$$

Example 1.2 $\quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}. \quad$ Find $|A_{12}|,\ A_{12},\ |A_{22}|,\ A_{22}$.

Solution. $|A_{12}| = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1, \quad A_{12} = -1. \qquad |A_{22}| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7, \quad A_{22} = -7.$

Laplace expansion by cofactors

This is another way to evaluate a determinant; we give the rule for a 3×3. It generalizes easily to an $n \times n$ determinant.

Select any row (or column) of the determinant. Multiply each entry a_{ij} in that row (or column) by its cofactor A_{ij}, and add the three resulting numbers; you get the value of the determinant.

As practice with notation, here is the formula for the Laplace expansion of a third order (i.e., a 3×3) determinant using the cofactors of the first row:

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ = \ |A| \tag{3}$$

and the formula using the cofactors of the j-th column:

$$a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} \ = \ |A| \tag{4}$$

Example 1.3 Evaluate the determinant in Example 1.2 using the Laplace expansions by the first row and by the second column, and check by also using (1).

Solution. The Laplace expansion by the first row is

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} \ = \ 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot (-1) - 0 \cdot 1 + 3 \cdot (-3) = -10.$$

The Laplace expansion by the second column would be

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} \ = \ -0 \cdot \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = 0 + 2 \cdot (-7) - 1 \cdot (-4) = -10.$$

Checking by (1), we have $|A| \;=\; -2+0+3-12-0-(-1) \;=\; -10.$

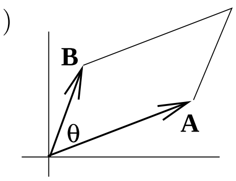
Example 1.4 Show the Laplace expansion by the first row agrees with definition (1).

Solution. We have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= \; a(ei - fh) - b(di - fg) + c(dh - eg),$$

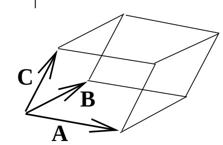
whose six terms agree with the six terms on the right of definition (1).

(A similar argument can be made for the Laplace expansion by any row or column.)



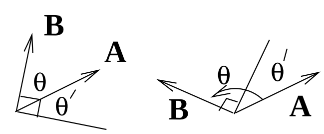
Area and volume interpretation of the determinant:

(5) $\quad \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \;=\;$ area of parallelogram with edges $\mathbf{A} = (a_1, a_2)$, $\mathbf{B} = (b_1, b_2)$.



(6) $\quad \pm \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \;=\;$ volume of parallelepiped with edges row-vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

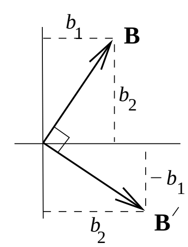
In each case, choose the sign which makes the left side non-negative.



Proof of (5). We begin with two preliminary observations.

Let θ be the positive angle from $\mathbf{A}$ to $\mathbf{B}$; we assume it is $< \pi$, so that $\mathbf{A}$ and $\mathbf{B}$ have the general positions illustrated.

Let $\theta' = \pi/2 - \theta$, as illustrated. Then $\cos\theta' = \sin\theta$.



Draw the vector $\mathbf{B}'$ obtained by rotating $\mathbf{B}$ to the right by $\pi/2$. The picture shows that $\mathbf{B}' = (b_2, -b_1)$, and $|\mathbf{B}'| = |\mathbf{B}|$.

To prove (5) now, we have a standard formula of Euclidean geometry,

$$\begin{aligned} \text{area of parallelogram} \;&=\; |\mathbf{A}||\mathbf{B}|\sin\theta \\ &=\; |\mathbf{A}||\mathbf{B}'|\cos\theta', && \text{by the above observations} \\ &=\; \mathbf{A}\cdot\mathbf{B}', && \text{by the geometric definition of dot product} \\ &=\; a_1 b_2 - a_2 b_1 && \text{by the formula for } \mathbf{B}' \end{aligned}$$

This proves the area interpretation (5) if $\mathbf{A}$ and $\mathbf{B}$ have the position shown. If their positions are reversed, then the area is the same, but the sign of the determinant is changed, so the formula has to read,

$$\text{area of parallelogram} \;=\; \pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \text{whichever sign makes the right side } \geq 0.$$

The proof of the analogous volume formula (6) will be made when we study the scalar triple product $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

For $n \times n$ determinants, the analog of definition (1) is a bit complicated, and not used to compute them; that's done by the analog of the Laplace expansion, which we give in a moment, or by using Fact **D-4** in a systematic way to make the entries below the main diagonal all zero. Generalizing (5) and (6), $n \times n$ determinants can be interpreted as the hypervolume in n-space of a n-dimensional parallelotope.

For $n \times n$ determinants, the **minor** $|A_{ij}|$ of the entry a_{ij} is defined to be the determinant obtained by deleting the i-th row and j-th column; the **cofactor** A_{ij} is the minor, prefixed by a $+$ or $-$ sign according to the natural generalization of the checkerboard pattern (2). Then the Laplace expansion by the i-th row would be

$$|A| \;=\; a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}\ .$$

This is an inductive calculation — it expresses the determinant of order n in terms of determinants of order $n-1$. Thus, since we can calculate determinants of order 3, it allows us to calculate determinants of order 4; then determinants of order 5, and so on. If we take for definiteness $i = 1$, then the above Laplace expansion formula can be used as the basis of an inductive definition of the $n \times n$ determinant.

Example 1.5 Evaluate $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 1 & 4 \\ -1 & 4 & 1 & 0 \\ 0 & 4 & 2 & -1 \end{vmatrix}$ by its Laplace expansion by the first row.

Solution. $1 \cdot \begin{vmatrix} -1 & 1 & 4 \\ 4 & 1 & 0 \\ 4 & 2 & -1 \end{vmatrix} \;-\; 0 \cdot A_{12} \;+\; 2 \cdot \begin{vmatrix} 2 & -1 & 4 \\ -1 & 4 & 0 \\ 0 & 4 & -1 \end{vmatrix} \;-\; 3 \cdot \begin{vmatrix} 2 & -1 & 1 \\ -1 & 4 & 1 \\ 0 & 4 & 2 \end{vmatrix}$

$= \; 1 \cdot 21 \; + \; 2 \cdot (-23) \; - \; 3 \cdot 2 \; = \; -31.$

Exercises: Section 1C

M. Matrices and Linear Algebra

1. Matrix algebra.

In section D we calculated the determinants of square arrays of numbers. Such arrays are important in mathematics and its applications; they are called *matrices*. In general, they need not be square, only rectangular.

A rectangular array of numbers having m rows and n columns is called an $m \times n$ **matrix**. The number in the i-th row and j-th column (where $1 \leq i \leq m,\ 1 \leq j \leq n$) is called the **ij-entry**, and denoted a_{ij}; the matrix itself is denoted by A, or sometimes by (a_{ij}).

Two matrices of the same size are *equal* if corresponding entries are equal.

Two special kinds of matrices are the **row-vectors**: the $1 \times n$ matrices $(a_1, a_2, \ldots, a_n)$; and the **column vectors**: the $m \times 1$ matrices consisting of a column of m numbers.

> From now on, row-vectors or column-vectors will be indicated by boldface small letters; when writing them by hand, put an arrow over the symbol.

Matrix operations

There are four basic operations which produce new matrices from old.

1. Scalar multiplication: Multiply each entry by c : $cA = (ca_{ij})$

2. Matrix addition: Add the corresponding entries: $A + B = (a_{ij} + b_{ij})$; the two matrices must have the same number of rows and the same number of columns.

3. Transposition: The *transpose* of the $m \times n$ matrix A is the $n \times m$ matrix obtained by making the rows of A the columns of the new matrix. Common notations for the transpose are A^T and A'; using the first we can write its definition as $A^T = (a_{ji})$.

If the matrix A is square, you can think of A^T as the matrix obtained by flipping A over around its main diagonal.

Example 1.1 Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \\ -1 & 0 \end{pmatrix}$. Find $A+B,\ A^T,\ 2A-3B$.

Solution. $A + B = \begin{pmatrix} 3 & 2 \\ -2 & 4 \\ -2 & 2 \end{pmatrix}$; $\quad A^T = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 1 & 2 \end{pmatrix}$;

$$2A + (-3B) = \begin{pmatrix} 4 & -6 \\ 0 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & -15 \\ 6 & -9 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -21 \\ 6 & -7 \\ 1 & 4 \end{pmatrix}.$$

4. Matrix multiplication This is the most important operation. Schematically, we have

$$\begin{array}{ccccc} A & \cdot & B & = & C \\ m \times n & & n \times p & & m \times p \\ & & c_{ij} & = & \displaystyle\sum_{k=1}^{n} a_{ik} b_{kj} \end{array}$$

The essential points are:

1. For the multiplication to be defined, A must have as many *columns* as B has *rows*;

2. The ij-th entry of the product matrix C is the dot product of the i-th row of A with the j-th column of B.

Example 1.2 $\quad \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2+4-2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix};$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 8 & 10 \\ -4 & -5 \end{pmatrix}; \qquad \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & -6 \\ 0 & 2 & 2 \end{pmatrix}$$

The two most important types of multiplication, for multivariable calculus and differential equations, are:

1. AB, where A and B are two *square* matrices of the same size — these can always be multiplied;

2. $A\mathbf{b}$, where A is a square $n \times n$ matrix, and $\mathbf{b}$ is a column n-vector.

Laws and properties of matrix multiplication

M-1. $A(B+C) = AB + AC, \qquad (A+B)C = AC + BC$ *distributive laws*

M-2. $(AB)C = A(BC); \qquad (cA)B = c(AB).$ *associative laws*

In both cases, the matrices must have compatible dimensions.

M-3. Let $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; then $AI = A$ and $IA = A$ for any 3×3 matrix.

I is called the **identity** matrix of order 3. There is an analogously defined square identity matrix I_n of any order n, obeying the same multiplication laws.

M-4. In general, for two square $n \times n$ matrices A and B, $AB \neq BA$: *matrix multiplication is not commutative.* (There are a few important exceptions, but they are very special — for example, the equality $AI = IA$ where I is the identity matrix.)

M-5. For two square $n \times n$ matrices A and B, we have the *determinant law*:

$$|AB| = |A||B|, \qquad \text{also written} \qquad \det(AB) = \det(A)\det(B)$$

For 2×2 matrices, this can be verified by direct calculation, but this naive method is unsuitable for larger matrices; it's better to use some theory. We will simply assume it in these notes; we will also assume the other results above (of which only the associative law **M-2** offers any difficulty in the proof).

M-6. A useful fact is this: matrix multiplication can be used to pick out a row or column of a given matrix: you multiply by a simple row or column vector to do this. Two examples should give the idea:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \qquad \text{the second column}$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \qquad \text{the first row}$$

Exercises: Section 1F

2. Solving square systems of linear equations; inverse matrices.

Linear algebra is essentially about solving systems of linear equations, an important application of mathematics to real-world problems in engineering, business, and science, especially the social sciences. Here we will just stick to the most important case, where the system is *square*, i.e., there are as many variables as there are equations. In low dimensions such systems look as follows (we give a 2×2 system and a 3×3 system):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{7}$$

In these systems, the a_{ij} and b_i are given, and we want to solve for the x_i.

As a simple mathematical example, consider the linear change of coordinates given by the equations

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ x_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ x_3 &= a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{aligned}$$

If we know the y-coordinates of a point, then these equations tell us its x-coordinates immediately. But if instead we are given the x-coordinates, to find the y-coordinates we must solve a system of equations like (7) above, with the y_i as the unknowns.

Using matrix multiplication, we can abbreviate the system on the right in (7) by

$$A\mathbf{x} = \mathbf{b}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{8}$$

where A is the square matrix of coefficients (a_{ij}). (The 2×2 system and the $n \times n$ system would be written analogously; all of them are abbreviated by the same equation $A\mathbf{x} = \mathbf{b}$, notice.)

You have had experience with solving small systems like (7) by *elimination*: multiplying the equations by constants and subtracting them from each other, the purpose being to eliminate all the variables but one. When elimination is done systematically, it is an efficient method. Here however we want to talk about another method more compatible with hand-held calculators and MatLab, and which leads more rapidly to certain key ideas and results in linear algebra.

Inverse matrices.

Referring to the system (8), suppose we can find a square matrix M, the same size as A, such that

$$MA = I \qquad \text{(the identity matrix)}. \tag{9}$$

We can then solve (8) by matrix multiplication, using the successive steps,

$$\begin{aligned} A\,\mathbf{x} &= \mathbf{b} \\ M(A\,\mathbf{x}) &= M\,\mathbf{b} \\ \mathbf{x} &= M\,\mathbf{b}; \end{aligned} \tag{10}$$

where the step $M(A\mathbf{x}) = \mathbf{x}$ is justified by

$$\begin{aligned} M(A\,\mathbf{x}) &= (MA)\mathbf{x}, && \text{by M-2;} \\ &= I\,\mathbf{x}, && \text{by (9);} \\ &= \mathbf{x}, && \text{by M-3 .} \end{aligned}$$

Moreover, the solution is unique, since (10) gives an explicit formula for it.

The same procedure solves the problem of determining the inverse to the linear change of coordinates $\mathbf{x} = A\mathbf{y}$, as the next example illustrates.

Example 2.1 Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $M = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$. Verify that M satisfies (9) above, and use it to solve the first system below for x_i and the second for the y_i in terms of the x_i:

$$\begin{aligned} x_1 + 2x_2 &= -1 \\ 2x_1 + 3x_2 &= 4 \end{aligned} \qquad\qquad \begin{aligned} x_1 &= y_1 + 2y_2 \\ x_2 &= 2y_1 + 3y_2 \end{aligned}$$

Solution. We have $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, by matrix multiplication. To solve the first system, we have by (10), $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -6 \end{pmatrix}$, so the solution is $x_1 = 11, x_2 = -6$. By reasoning similar to that used above in going from $A\mathbf{x} = \mathbf{b}$ to $\mathbf{x} = M\mathbf{b}$, the solution to $\mathbf{x} = A\mathbf{y}$ is $\mathbf{y} = M\mathbf{x}$, so that we get

$$\begin{aligned} y_1 &= -3x_1 + 2x_2 \\ y_2 &= 2x_1 - x_2 \end{aligned}$$

as the expression for the y_i in terms of the x_i.

Our problem now is: how do we get the matrix M? In practice, you mostly press a key on the calculator, or type a Matlab command. But we need to be able to work abstractly with the matrix — i.e., with symbols, not just numbers, and for this some theoretical ideas are important. The first is that M doesn't always exist.

$$M \text{ exists} \quad \Leftrightarrow \quad |A| \neq 0.$$

The implication $\Rightarrow$ follows immediately from the law **M-5**, since

$$MA = I \quad \Rightarrow \quad |M||A| = |I| = 1 \quad \Rightarrow \quad |A| \neq 0.$$

The implication in the other direction requires more; for the low-dimensional cases, we will produce a formula for M. Let's go to the formal definition first, and give M its proper name, A^{-1}:

Definition. Let A be an $n \times n$ matrix, with $|A| \neq 0$. Then the **inverse** of A is an $n \times n$ matrix, written A^{-1}, such that

$$A^{-1}A = I_n, \qquad AA^{-1} = I_n \tag{11}$$

(It is actually enough to verify either equation; the other follows automatically — see the exercises.)

Using the above notation, our previous reasoning (9) - (10) shows that

$$|A| \neq 0 \quad \Rightarrow \quad \text{the unique solution of } A\,\mathbf{x} = \mathbf{b} \text{ is } \mathbf{x} = A^{-1}\mathbf{b}; \tag{12}$$

$$|A| \neq 0 \quad \Rightarrow \quad \text{the solution of } \mathbf{x} = A\,\mathbf{y} \text{ for the } y_i \text{ is } \mathbf{y} = A^{-1}\mathbf{x}. \tag{12}$$

Calculating the inverse of a 3×3 matrix

Let A be the matrix. The formulas for its **inverse** A^{-1} and for an auxiliary matrix $\operatorname{adj} A$ called the **adjoint** of A (or in some books the **adjugate** of A) are

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T . \tag{13}$$

In the formula, A_{ij} is the cofactor of the element a_{ij} in the matrix, i.e., its minor with its sign changed by the checkerboard rule (see section 1 on determinants).

Formula (13) shows that the steps in calculating the inverse matrix are:

1. Calculate the matrix of minors.
2. Change the signs of the entries according to the checkerboard rule.
3. Transpose the resulting matrix; this gives $\operatorname{adj} A$.
4. Divide every entry by $|A|$.

(If inconvenient, for example if it would produce a matrix having fractions for every entry, you can just leave the $1/|A|$ factor outside, as in the formula. Note that step 4 can only be taken if $|A| \neq 0$, so if you haven't checked this before, you'll be reminded of it now.)

The notation A_{ij} for a cofactor makes it look like a matrix, rather than a signed determinant; this isn't good, but we can live with it.

Example 2.2 Find the inverse to $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

Solution. We calculate that $|A| = 2$. Then the steps are (T means transpose):

$$\underset{\text{matrix } A}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \rightarrow \underset{\text{cofactor matrix}}{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}} \underset{\mathrm{T}}{\rightarrow} \underset{\text{adj } A}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}} \rightarrow \underset{\text{inverse of } A}{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}$$

To get practice in matrix multiplication, check that $A \cdot A^{-1} = I$, or to avoid the fractions, check that $A \cdot \text{adj } (A) = 2I$.

The same procedure works for calculating the inverse of a 2×2 matrix A. We do it for a general matrix, since it will save you time in differential equations if you can learn the resulting formula.

$$\underset{\text{matrix } A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \rightarrow \underset{\text{cofactors}}{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}} \underset{\mathrm{T}}{\rightarrow} \underset{\text{adj } A}{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \rightarrow \underset{\text{inverse of } A}{\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Example 2.3 Find the inverses to: a) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Solution. a) Use the formula: $|A| = 2$, so $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$.

b) Follow the previous scheme:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & -3 & 7 \\ 2 & 0 & -1 \\ 4 & 3 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} \rightarrow \frac{1}{3}\begin{pmatrix} -5 & 2 & 4 \\ -3 & 0 & 3 \\ 7 & -1 & -5 \end{pmatrix} = A^{-1}.$$

Both solutions should be checked by multiplying the answer by the respective A.

Proof of formula (13) for the inverse matrix.

We want to show $A \cdot A^{-1} = I$, or equivalently, $A \cdot \text{adj } A = |A|I$; when this last is written out using (13) (remembering to transpose the matrix on the right there), it becomes

$$(14) \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}.$$

To prove (14), it will be enough to look at two typical entries in the matrix on the right — say the first two in the top row. According to the rule for multiplying the two matrices on the left, what we have to show is that

$$(15) \qquad a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|;$$

$$(16) \qquad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

These two equations are both evaluating determinants by Laplace expansions: the first equation (15) evaluates the determinant on the left below by the cofactors of the first row; the second equation (16) evaluates the determinant on the right below by the cofactors of the second row (notice that the cofactors of the second row don't care what's actually in the second row, since to calculate them you only need to know the other two rows).

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The two equations (15) and (16) now follow, since the determinant on the left is just $|A|$, while the determinant on the right is 0, since two of its rows are the same. □

The procedure we have given for calculating an inverse works for $n \times n$ matrices, but gets to be too cumbersome if $n > 3$, and other methods are used. The calculation of A^{-1} for reasonable-sized A is a standard package in computer algebra programs and MatLab. Unfortunately, social scientists often want the inverses of very large matrices, and for this special techniques have had to be devised, which produce approximate but acceptable results.

Exercises: Section 1G

3. Cramer's rule *(some 18.02 classes omit this)*

The general square system and its solution may be written

$$A\mathbf{x} = \mathbf{b}, \qquad |A| \neq 0 \qquad \Rightarrow \qquad \mathbf{x} = A^{-1}\mathbf{b}\ . \tag{17}$$

When this solution is written out and simplified, it becomes a rule for solving the system $A\mathbf{x} = \mathbf{b}$ known as *Cramer's rule.* We illustrate with the 2×2 case first; the system is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad\qquad |A| \neq 0\ .$$

The solution is, according to (17),

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ a_{11}b_2 - a_{21}b_1 \end{pmatrix}.$$

If we write out the answer using determinants, it becomes **Cramer's rule**:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|}\ ; \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|}\ . \tag{18}$$

The formulas in the 3×3 case are similar, and may be expressed this way:

Cramer's rule. *If $|A| \neq 0$, the solution of $A\mathbf{x} = \mathbf{b}$ is given by*

$$x_i = \frac{|A_i|}{|A|}, \qquad\qquad i = 1, 2, 3, \tag{19}$$

where $|A_i|$ is the determinant obtained by replacing the i-th column of $|A|$ by the column vector $\mathbf{b}$.

Cramer's rule is particularly useful if one only wants one of the x_i, as in the next example.

Example 3.1. Solve for x, using Cramer's rule (19):

$$\begin{aligned} 2x - 3y + z &= 1 \\ -x + y - z &= 2 \\ 4x + 3y - 2z &= -1 \ . \end{aligned}$$

Solution. We rewrite the system on the left below, then use Cramer's rule (19):

$$\begin{pmatrix} 2 & -3 & 1 \\ -1 & 1 & -1 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} ; \qquad x = \frac{\begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ -1 & 1 & -1 \\ 4 & 3 & -2 \end{vmatrix}} = \frac{-7}{13} \ .$$

Proof of (19). Since the solution to the system is $\mathbf{x} = A^{-1}\mathbf{b}$, when we write it out explicitly, it becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} .$$

We show that this gives formula (19) for x_1; the arguments for the other x_i go similarly. From the definition of matrix multiplication, we get from the above

$$\begin{aligned} x_1 &= \frac{1}{|A|}(A_{11}b_1 + A_{21}b_2 + A_{31}b_3), \\ x_1 &= \frac{1}{|A|}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \end{aligned}$$

according to the Laplace expansion of the determinant by its first column. But this last equation is exactly Cramer's rule for finding x_1. □

Cramer's rule is also valid for $n \times n$ systems; it is not normally used for systems larger than 3×3 however. You would use A^{-1}, or systematic elimination of variables. Nonetheless, the formula (19) is important as a theoretical tool in proofs and derivations.

Exercises: Section 1H

4. Theorems about homogeneous and inhomogeneous systems.

On the basis of our work so far, we can formulate a few general results about square systems of linear equations. They are the theorems most frequently referred to in the applications.

Definition. The linear system $A\mathbf{x} = \mathbf{b}$ is called **homogeneous** if $\mathbf{b} = \mathbf{0}$; otherwise, it is called **inhomogeneous**.

Theorem 1. *Let A be an $n \times n$ matrix.*

(20) $|A| \neq 0 \quad \Rightarrow \quad A\,\mathbf{x} = \mathbf{b}$ *has the unique solution,* $\mathbf{x} = A^{-1}\mathbf{b}$.

(21) $|A| \neq 0 \quad \Rightarrow \quad A\,\mathbf{x} = \mathbf{0}$ *has only the trivial solution,* $\mathbf{x} = \mathbf{0}$.

Notice that (21) is the special case of (20) where $\mathbf{b} = \mathbf{0}$. Often it is stated and used in the contrapositive form:

(21′) $A\mathbf{x} = \mathbf{0}$ has a non-zero solution $\quad \Rightarrow \quad |A| = 0$.

(The contrapositive of a statement $P \Rightarrow Q$ is not-$Q \Rightarrow$ not-P; the two statements say the same thing.)

Theorem 2. *Let A be an $n \times n$ matrix.*

(22) $|A| = 0 \quad \Rightarrow \quad A\,\mathbf{x} = \mathbf{0}$ *has non-trivial (i.e., non-zero) solutions.*

(23) $|A| = 0 \quad \Rightarrow \quad A\,\mathbf{x} = \mathbf{b}$ *usually has no solutions, but has solutions for some* $\mathbf{b}$.

In (23), we call the system **consistent** if it has solutions, **inconsistent** otherwise.

This probably seems like a maze of similar-sounding and confusing theorems. Let's get another perspective on these ideas by seeing how they apply separately to homogeneous and inhomogeneous systems.

Homogeneous systems: $A\mathbf{x} = \mathbf{0}$ *has non-trivial solutions* $\Leftrightarrow |A| = 0$.

Inhomogeneous systems: $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$, *if* $|A| \neq 0$. *If* $|A| = 0$, *then* $A\mathbf{x} = \mathbf{b}$ *usually has no solutions, but does have solutions for some* $\mathbf{b}$.

The statements (20) and (21) are proved, since we have a formula for the solution, and it is easy to see by multiplying $A\,\mathbf{x} = \mathbf{b}$ by A^{-1} that if $\mathbf{x}$ is a solution, it must be of the form $\mathbf{x} = A^{-1}\mathbf{b}$.

We prove (22) just for the 3×3 case, by interpreting it geometrically. We will give a partial argument for (23), based on both algebra and geometry.

Proof of (22).

We represent the three rows of A by the row vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and we let $\mathbf{x} = (x, y, z)$; think of all these as origin vectors, i.e., place their tails at the origin. Then, considering the homogeneous system first,

(24) $A\,\mathbf{x} = \mathbf{0}$ is the same as the system $\quad \mathbf{a} \cdot \mathbf{x} = 0, \quad \mathbf{b} \cdot \mathbf{x} = 0, \quad \mathbf{c} \cdot \mathbf{x} = 0.$

In other words, we are looking for a row vector $\mathbf{x}$ which is orthogonal to three given vectors, namely the three rows of A. By Section 1, we have

$$|A| = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \ = \ \text{volume of parallelepiped spanned by } \mathbf{a}, \mathbf{b}, \mathbf{c}.$$

It follows that if $|A| = 0$, the parallelepiped has zero volume, and therefore the origin vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in a plane. Any non-zero vector $\mathbf{x}$ which is orthogonal to this plane will then be orthogonal to $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and therefore will be a solution to the system (24). This proves (22): if $|A| = 0$, then $A\,\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Partial proof of (23). We write the system as $A\,\mathbf{x} = \mathbf{d}$, where $\mathbf{d}$ is the column vector $\mathbf{d} = (d_1, d_2, d_3)^T$.

Writing this out as we did in (24), it becomes the system

$$\mathbf{a}\cdot\mathbf{x} = d_1, \quad \mathbf{b}\cdot\mathbf{x} = d_2, \quad \mathbf{c}\cdot\mathbf{x} = d_3 \ . \tag{25}$$

If $|A| = 0$, the three origin vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in a plane, which means we can write one of them, say $\mathbf{c}$, as a linear combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{c} = r\mathbf{a} + s\mathbf{b}, \qquad r.s \text{ real numbers.} \tag{26}$$

Then if $\mathbf{x}$ is any vector, it follows that

$$\mathbf{c}\cdot\mathbf{x} = r(\mathbf{a}\cdot\mathbf{x}) + s(\mathbf{b}\cdot\mathbf{x}) \ . \tag{27}$$

Now if $\mathbf{x}$ is also a solution to (25), we see from (25) and (27) that

$$d_3 = rd_1 + sd_2; \tag{28}$$

this shows that unless the components of $\mathbf{d}$ satisfy the relation (28), there cannot be a solution to (25); thus in general there are no solutions.

If however, $\mathbf{d}$ does satisfy the relation (28), then the last equation in (25) is a consequence of the first two and can be discarded, and we get a system of two equations in three unknowns, which will in general have a non-zero solution, unless they represent two planes which are parallel.

Singular matrices; computational difficulties.

Because so much depends on whether $|A|$ is zero or not, this property is given a name. We say the square matrix A is **singular** if $|A| = 0$, and **nonsingular** or **invertible** if $|A| \neq 0$.

> Indeed, we know that A^{-1} exists if and only if $|A| \neq 0$, which explains the term "invertible"; the use of "singular" will be familiar to Sherlock Holmes fans: it is the 19th century version of "peculiar" or the late 20th century word "special".

Even if A is nonsingular, the solution of $A\mathbf{x} = \mathbf{b}$ is likely to run into trouble if $|A| \approx 0$, or as one says, A is *almost-singular*. Namely, in the formulas given in Cramer's rule (19), the $|A|$ occurs in the denominator, so that unless there is some sort of compensation for this in the numerator, the solutions are likely to be very sensitive to small changes in the coefficients of A, i.e., to the coefficients of the equations. Systems (of any kind) whose solutions behave this way are said to be **ill-conditioned**; it is difficult to solve such systems numerically and special methods must be used.

To see the difficulty geometrically, think of a 2×2 system $A\mathbf{x} = \mathbf{b}$ as representing a pair of lines; the solution is the point in which they intersect. If $|A| \approx 0$, but its entries are not small, then its two rows must be vectors which are almost parallel (since they span a parallelogram of small area). The two lines are therefore almost parallel; their intersection point exists, but its position is highly sensitive to the exact positions of the two lines, i.e., to the values of the coefficients of the system of equations.

Exercises: Section 1H

K. Kepler's Second Law

By studying the Danish astronomer Tycho Brahe's data about the motion of the planets, Kepler formulated three empirical laws; two of them can be stated as follows:

Second Law A planet moves in a plane, and the radius vector (from the sun to the planet) sweeps out equal areas in equal times.

First Law The planet's orbit in that plane is an ellipse, with the sun at one focus.

From these laws, Newton deduced that the force keeping the planets in their orbits had magnitude $1/d^2$, where d was the distance to the sun; moreover, it was directed toward the sun, or as was said, *central*, since the sun was placed at the origin.

Using a little vector analysis (without coordinates), this section is devoted to showing that *the Second Law is equivalent to the force being central.*

> It is harder to show that an elliptical orbit implies the magnitude of the force is of the form K/d^2, and vice-versa; this uses vector analysis in polar coordinates and requires the solution of non-linear differential equations.

1. Differentiation of products of vectors

Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be two differentiable vector functions in 2- or 3-space. Then

$$(1) \qquad \frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) \;=\; \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} \;+\; \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}; \qquad \frac{d}{dt}(\mathbf{r}\times\mathbf{s}) \;=\; \frac{d\mathbf{r}}{dt}\times\mathbf{s} \;+\; \mathbf{r}\times\frac{d\mathbf{s}}{dt}.$$

These rules are just like the product rule for differentiation. Be careful in the second rule to get the multiplication order correct on the right, since $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$ in general. The two rules can be proved by writing everything out in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and differentiating. They can also be proved directly from the definition of derivative, without resorting to components, as follows:

Let t increase by Δt. Then $\mathbf{r}$ increases by $\Delta\mathbf{r}$, and $\mathbf{s}$ by $\Delta\mathbf{s}$, and the corresponding change in $\mathbf{r}\cdot\mathbf{s}$ is given by

$$\Delta(\mathbf{r}\cdot\mathbf{s}) \;=\; (\mathbf{r}+\Delta\mathbf{r})\cdot(\mathbf{s}+\Delta\mathbf{s}) \;-\; \mathbf{r}\cdot\mathbf{s}\,,$$

so if we expand the right side out and divide all terms by Δt, we get

$$\frac{\Delta(\mathbf{r}\cdot\mathbf{s})}{\Delta t} \;=\; \frac{\Delta\mathbf{r}}{\Delta t}\cdot\mathbf{s} \;+\; \mathbf{r}\cdot\frac{\Delta\mathbf{s}}{\Delta t} \;+\; \frac{\Delta\mathbf{r}}{\Delta t}\cdot\Delta\mathbf{s}\,.$$

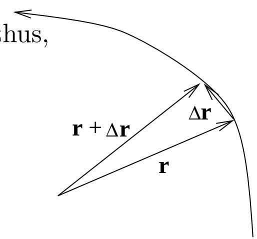
Now let $\Delta t \to 0$; then $\Delta\mathbf{s} \to 0$ since $\mathbf{s}(t)$ is continuous, and we get the first equation in (1). The second equation in (1) is proved the same way, replacing $\cdot$ by $\times$ everywhere.

2. Kepler's second law and the central force. To show that the force being central (i.e., directed toward the sun) is equivalent to Kepler's second law, we need to translate that law into calculus. "Sweeps out equal areas in equal times" means:

the radius vector sweeps out area at a constant rate .

The first thing therefore is to obtain a mathematical expression for this rate. Referring to the picture, we see that as the time increases from t to $t+\Delta t$, the corresponding change in the area A is given approximately by

$$\Delta A \approx \text{area of the triangle} = \frac{1}{2}|\mathbf{r}\times\Delta\mathbf{r}|\ ,$$



since the triangle has half the area of the parallelogram formed by $\mathbf{r}$ and $\Delta\mathbf{r}$; thus,

$$2\,\frac{\Delta A}{\Delta t} \approx \left|\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}\right|,$$

and as $\Delta t\to 0$, this becomes

(2) $$2\,\frac{dA}{dt} = \left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right| = |\mathbf{r}\times\mathbf{v}|. \qquad \text{where } \mathbf{v}=\frac{d\mathbf{r}}{dt}\ .$$

Using (2), we can interpret Kepler's second law mathematically. Since the area is swept out at a constant rate, dA/dt is constant, so according to (2),

(3) $$|\mathbf{r}\times\mathbf{v}| \quad \textit{is a constant.}$$

Moreover, since Kepler's law says $\mathbf{r}$ lies in a plane, the velocity vector $\mathbf{v}$ also lies in the same plane, and therefore

(4) $\mathbf{r}\times\mathbf{v}$ *has constant direction (perpendicular to the plane of motion).*

Since the direction and magnitude of $\mathbf{r}\times\mathbf{v}$ are both constant,

(5) $$\mathbf{r}\times\mathbf{v} = \mathbf{K}, \text{ a constant vector,}$$

and from this we see that

(6) $$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{0}\ .$$

But according to the rule (1) for differentiating a vector product,

(7) $$\begin{aligned}\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) &= \mathbf{v}\times\mathbf{v}+\mathbf{r}\times\mathbf{a}, &&\text{where } \mathbf{a}=\frac{d\mathbf{v}}{dt},\\ &= \mathbf{r}\times\mathbf{a}, &&\text{since } \mathbf{s}\times\mathbf{s}=\mathbf{0} \text{ for any vector } \mathbf{s}.\end{aligned}$$

Now (6) and (7) together imply

(8) $$\mathbf{r}\times\mathbf{a} = \mathbf{0},$$

which shows that *the acceleration vector* $\mathbf{a}$ *is parallel to* $\mathbf{r}$, but in the opposite direction, since the planets do go around the sun, not shoot off to infinity.

Thus $\mathbf{a}$ is directed toward the center (i.e., the sun), and since $\mathbf{F}=m\mathbf{a}$, the force $\mathbf{F}$ is also directed toward the sun. (Note that "center" does not mean the center of the elliptical orbits, but the mathematical origin, i.e., the tail of the radius vector $\mathbf{r}$, which we are taking to be the sun's position.)

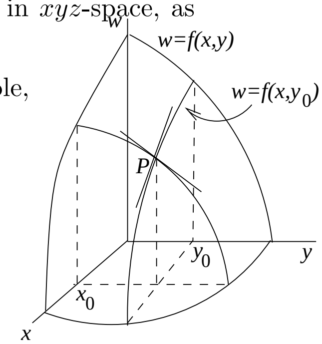
The reasoning is reversible, so for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate.

Exercises: Section 1K

TA. The Tangent Approximation

1. Partial derivatives

Let $w = f(x, y)$ be a function of two variables. Its graph is a surface in xyz-space, as pictured.



Fix a value $y = y_0$ and just let x vary. You get a function of *one* variable,

(1) $$w = f(x, y_0), \qquad \text{the } \textbf{partial function} \text{ for } y = y_0.$$

Its graph is a curve in the vertical plane $y = y_0$, whose slope at the point P where $x = x_0$ is given by the derivative

(2) $$\frac{d}{dx} f(x, y_0)\Big|_{x_0}, \quad \text{or} \quad \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}.$$

We call (2) the **partial derivative** of f with respect to x at the point (x_0, y_0); the right side of (2) is the standard notation for it. The partial derivative is just the ordinary derivative of the partial function — it is calculated by holding one variable fixed and differentiating with respect to the other variable. Other notations for this partial derivative are

$$f_x(x_0, y_0), \qquad \frac{\partial w}{\partial x}\Big|_{(x_0, y_0)}, \qquad \left(\frac{\partial f}{\partial x}\right)_0, \qquad \left(\frac{\partial w}{\partial x}\right)_0;$$

the first is convenient for including the specific point; the second is common in science and engineering, where you are just dealing with relations between variables and don't mention the function explicitly; the third and fourth indicate the point by just using a single subscript.

Analogously, fixing $x = x_0$ and letting y vary, we get the partial function $w = f(x_0, y)$, whose graph lies in the vertical plane $x = x_0$, and whose slope at P is the *partial derivative of f with respect to y*; the notations are

$$\frac{\partial f}{\partial y}\Big|_{(x_0, y_0)}, \qquad f_y(x_0, y_0), \qquad \frac{\partial w}{\partial y}\Big|_{(x_0, y_0)}, \qquad \left(\frac{\partial f}{\partial y}\right)_0, \qquad \left(\frac{\partial w}{\partial y}\right)_0.$$

The partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ depend on (x_0, y_0) and are therefore functions of x and y.

Written as $\partial w/\partial x$, the partial derivative gives the rate of change of w with respect to x alone, at the point (x_0, y_0): it tells how fast w is increasing as x increases, when y is held constant.

For a function of three or more variables, $w = f(x, y, z, \dots)$, we cannot draw graphs any more, but the idea behind partial differentiation remains the same: to define the partial derivative with respect to x, for instance, hold all the other variables constant and take the ordinary derivative with respect to x; the notations are the same as above:

$$\frac{d}{dx} f(x, y_0, z_0, \dots) \;=\; f_x(x_0, y_0, z_0, \dots), \qquad \left(\frac{\partial f}{\partial x}\right)_0, \qquad \left(\frac{\partial w}{\partial x}\right)_0.$$

Your book has examples illustrating the calculation of partial derivatives for functions of two and three variables.

2. The tangent plane.

For a function of one variable, $w = f(x)$, the tangent line to its graph at a point (x_0, w_0) is the line passing through (x_0, w_0) and having slope $\left(\frac{dw}{dx}\right)_0$.

For a function of two variables, $w = f(x, y)$, the natural analogue is the *tangent plane* to the graph, at a point (x_0, y_0, w_0). What's the equation of this tangent plane? Referring to the picture of the graph on the preceding page, we see that the tangent plane

(i) must pass through (x_0, y_0, w_0), where $w_0 = f(x_0, y_0)$;

(ii) must contain the tangent lines to the graphs of the two partial functions — this will hold if the plane has the same slopes in the $\mathbf{i}$ and $\mathbf{j}$ directions as the surface does.

Using these two conditions, it is easy to find the equation of the tangent plane. The general equation of a plane through (x_0, y_0, w_0) is

$$A(x - x_0) + B(y - y_0) + C(w - w_0) \;=\; 0\;.$$

Assume the plane is not vertical; then $C \neq 0$, so we can divide through by C and solve for $w - w_0$, getting

$$w - w_0 \;=\; a(x - x_0) + b(y - y_0), \qquad a = A/C,\; b = B/C. \tag{3}$$

The plane passes through (x_0, y_0, w_0); what values of the coefficients a and b will make it also tangent to the graph there? We have

$$\begin{aligned} a \;&=\; \text{slope of plane (3) in the } \mathbf{i}\text{-direction} && \text{(by putting } y = y_0 \text{ in (3));} \\ &=\; \text{slope of graph in the } \mathbf{i}\text{-direction,} && \text{(by (ii) above)} \\ &=\; \left(\frac{\partial w}{\partial x}\right)_0; && \text{(by the definition of partial derivative);} \quad \text{similarly,} \\ b \;&=\; \left(\frac{\partial w}{\partial y}\right)_0. \end{aligned}$$

Therefore the equation of the **tangent plane** to $w = f(x, y)$ at (x_0, y_0) is

$$w - w_0 \;=\; \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial w}{\partial y}\right)_0 (y - y_0) \tag{4}$$

Your book has examples of calculating tangent planes, using (4).

3. The approximation formula.

The most important use for the tangent plane is to give an approximation that is the basic formula in the study of functions of several variables — almost everything follows in one way or another from it.

The intuitive idea is that if we stay near (x_0, y_0, w_0), the graph of the tangent plane (4) will be a good approximation to the graph of the function $w = f(x, y)$. Therefore if the point (x, y) is close to (x_0, y_0),

$$(5) \qquad \underset{\text{height of graph}}{f(x,y)} \quad \approx \quad \underset{\text{height of tangent plane}}{w_0 + \left(\frac{\partial w}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial w}{\partial y}\right)_0 (y - y_0)}$$

The function on the right side of (5) whose graph is the tangent plane is often called the **linearization** of $f(x, y)$ at (x_0, y_0): it is the linear function which gives the best approximation to $f(x, y)$ for values of (x, y) close to (x_0, y_0).

An equivalent form of the approximation (5) is obtained by using Δ notation; if we put

$$\Delta x = x - x_0, \qquad \Delta y = y - y_0, \qquad \Delta w = w - w_0,$$

then (5) becomes

$$(6) \qquad \Delta w \approx \left(\frac{\partial w}{\partial x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right)_0 \Delta y, \qquad \text{if} \quad \Delta x \approx 0,\ \Delta y \approx 0\ .$$

This formula gives the approximate change in w when we make a small change in x and y. We will use it often.

The analogous approximation formula for a function $w = f(x, y, z)$ of three variables would be

$$(7) \qquad \Delta w \approx \left(\frac{\partial w}{\partial x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right)_0 \Delta y + \left(\frac{\partial w}{\partial z}\right)_0 \Delta z, \qquad \text{if} \quad \Delta x, \Delta y \Delta z \approx 0\ .$$

Unfortunately, for functions of three or more variables, we can't use a geometric argument for the approximation formula (7); for this reason, it's best to recast the argument for (6) in a form which doesn't use tangent planes and geometry, and therefore can be generalized to several variables. This is done at the end of this Chapter TA; for now let's just assume the truth of (7) and its higher-dimensional analogues.

Here are two typical examples of the use of the approximation formula. Other examples are in the Exercises. In the rest of your study of partial differentiation, you will see how the approximation formula is used to derive the important theorems and formulas.

Example 1. Give a reasonable square, centered at $(1, 1)$, over which the value of $w = x^3 y^4$ will not vary by more than $\pm$ .1 .

Solution. We use (6). We calculate for the two partial derivatives

$$w_x = 3x^2 y^4 \qquad\qquad w_y = 4x^3 y^3$$

and therefore, evaluating the partials at $(1, 1)$ and using (6), we get

$$\Delta w \approx 3\Delta x + 4\Delta y.$$

Thus if $|\Delta x| \le .01$ and $\Delta y| \le .01$, we should have

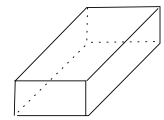
$$|\Delta w| \le 3|\Delta x| + 4|\Delta y| \le .07,$$

which is within the bounds. So the answer is the square with center at (1,1) given by

$$|x - 1| \le .01, \qquad |y - 1| \le .01\ .$$

Example 2. The sides a, b, c of a rectangular box have lengths measured to be respectively 1, 2, and 3. To which of these measurements is the volume V most sensitive?

Solution. $V = abc$, and therefore by the approximation formula (7),



$$\begin{aligned} \Delta V &\approx bc\,\Delta a + ac\,\Delta b + ab\,\Delta c \\ &\approx 6\,\Delta a + 3\,\Delta b + 2\,\Delta c, \qquad \text{at } (1,2,3); \end{aligned}$$

thus it is most sensitive to small changes in side a, since Δa occurs with the largest coefficient. (That is, if one at a time the measurement of each side were changed by say .01, it is the change in a which would produce the biggest change in V, namely .06 .)

> The result may seem paradoxical — the value of V is most sensitive to the length of the *shortest* side — but it's actually intuitive, as you can see by thinking about how the box looks.

Sensitivity Principle *The numerical value of $w = f(x, y, \dots)$, calculated at some point $(x_0, y_0, \dots)$, will be most sensitive to small changes in that variable for which the corresponding partial derivative $w_x, w_y, \dots$ has the largest absolute value at the point.*

4. Critique of the approximation formula.

First of all, the approximation formula (6) is not a precise mathematical statement, since the symbol $\approx$ does not specify exactly how close the quantitites on either side of the formula are to each other. To fix this up, one would have to specify the error in the approximation. (This can be done, but it is not often used.)

A more fundamental objection is that our discussion of approximations was based on the assumption that the tangent plane is a good approximation to the surface at (x_0, y_0, w_0). Is this really so?

Look at it this way. The tangent plane was determined as the plane which has the same slope as the surface in the $\mathbf{i}$ and $\mathbf{j}$ directions. This means the approximation (6) will be good if you move away from (x_0, y_0) in the $\mathbf{i}$ direction (by taking $\Delta y = 0$), or in the $\mathbf{j}$ direction (putting $\Delta x = 0$). But does the tangent plane have the same slope as the surface in all the other directions as well?

Intuitively, we should expect that this will be so if the graph of $f(x, y)$ is a "smooth" surface at (x_0, y_0) — it doesn't have any sharp points, folds, or look peculiar. Here is the mathematical hypothesis which guarantees this.

Smoothness hypothesis. We say $f(x, y)$ is **smooth** at (x_0, y_0) if

(8) $\qquad$ *f_x and f_y are continuous in some rectangle centered at (x_0, y_0).*

If (8) holds, the approximation formula (6) will be valid.

Though pathological examples can be constructed, in general the normal way a function fails to be smooth (and in turn (6) fails to hold) is that one or both partial derivatives fail to exist at (x_0, y_0). This means of course that you won't even be able to write the formula (6), unless you're sleepy. Here is a simple example.

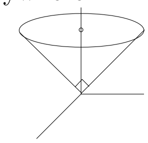
Example 3. Where is $w = \sqrt{x^2 + y^2}$ smooth? Discuss.

Solution. Calculating formally, we get

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\ , \qquad\qquad \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

These are continuous at all points except $(0,0)$, where they are undefined. So the function is smooth except at the origin; the approximation formula (6) should be valid everywhere except at the origin.

Indeed, investigating the graph of this function, since $w = \sqrt{x^2+y^2}$ says that



$$\text{height of graph over } (x,y) \;=\; \text{distance of } (x,y) \text{ from } w\text{-axis},$$

the graph is a right circular cone, with vertex at $(0,0)$, axis along the w-axis, and vertex angle a right angle. Geometrically the graph has a sharp point at the origin, so there should be no tangent plane there, and no valid approximation formula (6) — there is no linear function which approximates a cone at its vertex.

A non-geometrical argument for the approximation formula

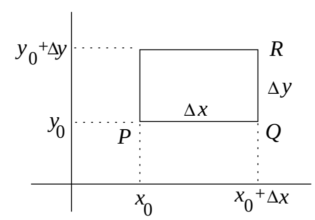
We promised earlier a non-geometrical approach to the approximation formula (6) that would generalize to higher-dimensions, in particular to the 3-variable formula (7). This approach will also show why the hypothesis (8) of *smoothness* is needed. The argument is still imprecise, since it uses the symbol $\approx$, but it can be refined to a proof (which you will find in your book, though it's not easy reading).

It uses the one-variable approximation formula for a differentiable function $w = f(u)$:

$$\Delta w \approx f'(u_0)\Delta u, \qquad \text{if } \Delta u \approx 0\ . \tag{9}$$

We wish to justify — without using reasoning based on 3-space — the approximation formula

$$\Delta w \;\approx\; \left(\frac{\partial w}{\partial x}\right)_0 \Delta x + \left(\frac{\partial w}{\partial y}\right)_0 \Delta y, \qquad\qquad \text{if} \quad \Delta x \approx 0,\ \Delta y \approx 0\ . \tag{6}$$



We are trying to calculate the change in w as we go from P to R in the picture, where $P = (x_0, y_0)$, $R = (x_0+\Delta x, y_0+\Delta y)$. This change can be thought of as taking place in two steps:

$$\Delta w \;=\; \Delta w_1 \;+\; \Delta w_2, \tag{10}$$

the first being the change in w as you move from P to Q, the second the change as you move from Q to R. Using the one-variable approximation formula (9) :

$$\Delta w_1 \;\approx\; \frac{d}{dx} f(x, y_0)\bigg|_{x_0} \cdot \Delta x \;=\; f_x(x_0, y_0)\,\Delta x; \tag{11}$$

similarly,

$$\begin{aligned} \Delta w_2 \;&\approx\; \frac{d}{dy} f(x_0+\Delta x, y)\bigg|_{y_0} \cdot \Delta y \;=\; f_y(x_0+\Delta x, y_0)\,\Delta y \\ &\approx f_y(x_0, y_0)\,\Delta y, \end{aligned} \tag{12}$$

if we assume that f_y is continuous (i.e., f is smooth), since the difference between the two terms on the right in the last two lines will then be like $\epsilon\,\Delta y$, which is negligible compared with either term itself. Substituting the two approximate values (11) and (12) into (10) gives us the approximation formula (6). $\square$

To make this a proof, the error terms in the approximations have to be analyzed, or more simply, one has to replace the $\approx$ symbol by equalities based on the Mean-Value Theorem of one-variable calculus.

This argument readily generalizes to the higher-dimensional approximation formulas, such as (7); again the essential hypothesis would be smoothness: the three partial derivatives w_x, w_y, w_z should be continuous in a neighborhood of the point (x_0, y_0, z_0).

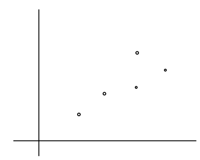
Exercises: Section 2B

LS. Least Squares Interpolation

1. The least-squares line.

Suppose you have a large number n of experimentally determined points, through which you want to pass a curve. There is a formula (the Lagrange interpolation formula) producing a polynomial curve of degree $n-1$ which goes through the points exactly. But normally one wants to find a simple curve, like a line, parabola, or exponential, which goes approximately through the points, rather than a high-degree polynomial which goes exactly through them. The reason is that the location of the points is to some extent determined by experimental error, so one wants a smooth-looking curve which averages out these errors, not a wiggly polynomial which takes them seriously.

In this section, we consider the most common case — finding a line which goes approximately through a set of data points.



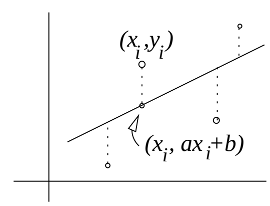
Suppose the data points are

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

and we want to find the line

$$y = ax + b \tag{1}$$

which "best" passes through them. Assuming our errors in measurement are distributed randomly according to the usual bell-shaped curve (the so-called "Gaussian distribution"), it can be shown that the right choice of a and b is the one for which the sum D of the squares of the deviations



$$D \;=\; \sum_{i=1}^{n} \big(y_i - (ax_i + b)\big)^2 \tag{2}$$

is a *minimum*. In the formula (2), the quantities in parentheses (shown by dotted lines in the picture) are the **deviations** between the observed values y_i and the ones $ax_i + b$ that would be predicted using the line (1).

The deviations are squared for theoretical reasons connected with the assumed Gaussian error distribution; note however that the effect is to ensure that we sum only positive quantities; this is important, since we do not want deviations of opposite sign to cancel each other out. It also weights more heavily the larger deviations, keeping experimenters honest, since they tend to ignore large deviations ("I had a headache that day").

This prescription for finding the line (1) is called the **method of least squares**, and the resulting line (1) is called the **least-squares** line or the **regression** line.

To calculate the values of a and b which make D a minimum, we see where the two partial derivatives are zero:

$$
\begin{aligned}
\frac{\partial D}{\partial a} \;&=\; \sum_{i=1}^{n} 2(y_i - ax_i - b)(-x_i) \;=\; 0 \\
\frac{\partial D}{\partial b} \;&=\; \sum_{i=1}^{n} 2(y_i - ax_i - b)(-1) \;=\; 0\,.
\end{aligned}
\tag{3}
$$

These give us a pair of *linear* equations for determining a and b, as we see by collecting terms and cancelling the 2's:

$$\begin{aligned} \Big(\sum x_i^2\Big) a \;+\; \Big(\sum x_i\Big) b \;&=\; \sum x_i y_i \\ \Big(\sum x_i\Big) a \;+\; n\, b \;&=\; \sum y_i \;. \end{aligned} \tag{4}$$

(Notice that it saves a lot of work to differentiate (2) using the chain rule, rather than first expanding out the squares.)

The equations (4) are usually divided by n to make them more expressive:

$$\begin{aligned} \bar{s}\, a \;+\; \bar{x}\, b \;&=\; \frac{1}{n}\sum x_i y_i \\ \bar{x}\, a \;+\; b \;&=\; \bar{y}\;, \end{aligned} \tag{5}$$

where $\bar{x}$ and $\bar{y}$ are the average of the x_i and y_i, and $\bar{s} = \sum x_i^2/n$ is the average of the squares.

From this point on use linear algebra to determine a and b. It is a good exercise to see that the equations are always solvable unless all the x_i are the same (in which case the best line is vertical and can't be written in the form (1)).

In practice, least-squares lines are found by pressing a calculator button, or giving a MatLab command. Examples of calculating a least-squares line are in the exercises in your book and these notes. Do them from scratch, starting from (2), since the purpose here is to get practice with max-min problems in several variables; don't plug into the equations (5). Remember to differentiate (2) using the chain rule; don't expand out the squares, which leads to messy algebra and highly probable error.

2. Fitting curves by least squares.

If the experimental points seem to follow a curve rather than a line, it might make more sense to try to fit a second-degree polynomial

$$y = a_0 + a_1 x + a_2 x^2 \tag{6}$$

to them. If there are only three points, we can do this exactly (by the Lagrange interpolation formula). For more points, however, we once again seek the values of a_0, a_1, a_2 for which the sum of the squares of the deviations

$$D \;=\; \sum_1^n \big(y_i - (a_0 + a_1 x_i + a_2 x_i^2)\big)^2 \tag{7}$$

is a minimum. Now there are three unknowns, a_0, a_1, a_2. Calculating (remember to use the chain rule!) the three partial derivatives $\partial D/\partial a_i,\ i = 0, 1, 2$, and setting them equal to zero leads to a square system of three linear equations; the a_i are the three unknowns, and the coefficients depend on the data points (x_i, y_i). They can be solved by finding the inverse matrix, elimination, or using a calculator or MatLab.

If the points seem to lie more and more along a line as $x \to \infty$, but lie on one side of the line for low values of x, it might be reasonable to try a function which has similar behavior, like

$$y \;=\; a_0 + a_1 x + a_2\,\frac{1}{x} \tag{8}$$

and again minimize the sum of the squares of the deviations, as in (7). In general, this method of least squares applies to a trial expression of the form

$$y = a_0 f_0(x) + a_1 f_1(x) + \ldots + a_r f_r(x), \tag{9}$$

where the $f_i(x)$ are given functions (usually simple ones like $1, x, x^2, 1/x, e^{kx}$, etc. Such an expression (9) is called a **linear combination** of the functions $f_i(x)$. The method produces a square inhomogeneous system of linear equations in the unknowns $a_0, \ldots, a_r$ which can be solved by finding the inverse matrix to the system, or by elimination.

The method also applies to finding a linear function

$$z = a_1 + a_2 x + a_3 y \tag{10}$$

to fit a set of data points

$$(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n) \ . \tag{11}$$

where there are two independent variables x and y and a dependent variable z (this is the quantity being experimentally measured, for different values of (x, y)). This time after differentiation we get a 3×3 system of linear equations for determining a_1, a_2, a_3 .

The essential point in all this is that the unknown coefficients a_i should occur *linearly* in the trial function. Try fitting a function like ce^{kx} to data points by using least squares, and you'll see the difficulty right away. (Since this is an important problem — fitting an exponential to data points — one of the Exercises explains how to adapt the method to this type of problem.)

Exercises: Section 2G

SD. Second Derivative Test

1. The Second Derivative Test

We begin by recalling the situation for twice differentiable functions $f(x)$ of one variable. To find their local (or "relative") maxima and minima, we

1. find the critical points, i.e., the solutions of $f'(x) = 0$;
2. apply the second derivative test to each critical point x_0:

$$f''(x_0) > 0 \;\Rightarrow\; x_0 \text{ is a local minimum point;}$$
$$f''(x_0) < 0 \;\Rightarrow\; x_0 \text{ is a local maximum point.}$$

The idea behind it is: at x_0 the slope $f'(x_0) = 0$; if $f''(x_0) > 0$, then $f'(x)$ is strictly increasing for x near x_0, so that the slope is negative to the left of x_0 and positive to the right, which shows that x_0 is a minimum point. The reasoning for the maximum point is similar.

If $f''(x_0) = 0$, the test fails and one has to investigate further, by taking more derivatives, or getting more information about the graph. Besides being a maximum or minimum, such a point could also be a horizontal point of inflection.

The analogous test for maxima and minima of functions of two variables $f(x, y)$ is a little more complicated, since there are several equations to satisfy, several derivatives to be taken into account, and another important geometric possibility for a critical point, namely a **saddle point**. This is a local minimax point; around such a point the graph of $f(x, y)$ looks like the central part of a saddle, or the region around the highest point of a mountain pass. In the neighborhood of a saddle point, the graph of the function lies both above and below its horizontal tangent plane at the point. (Your textbook has illustrations.)

The second-derivative test for maxima, minima, and saddle points has two steps.

1. Find the critical points by solving the simultaneous equations $\begin{cases} f_x(x, y) = 0, \\ f_y(x, y) = 0. \end{cases}$

Since a critical point (x_0, y_0) is a solution to both equations, both partial derivatives are zero there, so that the tangent plane to the graph of $f(x, y)$ is horizontal.

2. To test such a point to see if it is a local maximum or minimum point, we calculate the three second derivatives at the point (we use subscript 0 to denote evaluation at (x_0, y_0), so for example $(f)_0 = f(x_0, y_0)$), and denote the values by A, B, and C:

$$A = (f_{xx})_0, \qquad B = (f_{xy})_0 = (f_{yx})_0, \qquad C = (f_{yy})_0, \tag{1}$$

(we are assuming the derivatives exist and are continuous).

Second-derivative test. *Let (x_0, y_0) be a critical point of $f(x, y)$, and A, B, and C be as in (1). Then*

$$AC - B^2 > 0, \quad A > 0 \text{ or } C > 0 \;\Rightarrow\; (x_0, y_0) \text{ is a minimum point;}$$
$$AC - B^2 > 0, \quad A < 0 \text{ or } C < 0 \;\Rightarrow\; (x_0, y_0) \text{ is a maximum point;}$$
$$AC - B^2 < 0 \;\Rightarrow\; (x_0, y_0) \text{ is a saddle point.}$$

If $AC - B^2 = 0$, the test fails and more investigation is needed.

Note that if $AC - B^2 > 0$, then $AC > 0$, so that A and C must have the same sign.

Example 1. Find the critical points of $w = 12x^2 + y^3 - 12xy$ and determine their type.

Solution. We calculate the partial derivatives easily:

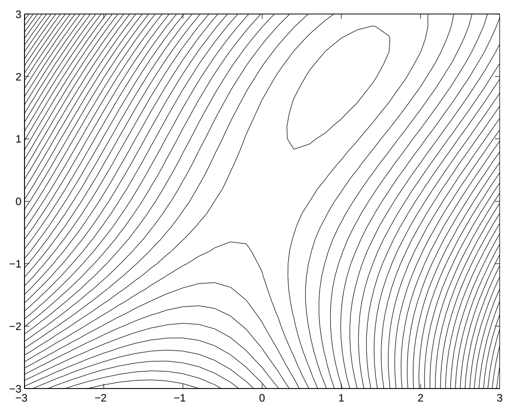
$$
\begin{aligned} w_x &= 24x - 12y \\ w_y &= 3y^2 - 12x \end{aligned} \qquad \begin{aligned} A &= w_{xx} = 24 \\ B &= w_{xy} = -12 \\ C &= w_{yy} = 6y \end{aligned} \tag{2}
$$

To find the critical points we solve simultaneously the equations $w_x = 0$ and $w_y = 0$; we get

$$
\begin{aligned} w_x &= 0 \\ w_y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= 2x \\ y^2 &= 4x \end{aligned} \quad \Rightarrow \quad 4x^2 = 4x \quad \Rightarrow \quad x = 0, 1 \quad \Rightarrow \quad \begin{aligned} (x,y) &= (0,0) \\ (x,y) &= (1,2) \end{aligned}.
$$

Thus there are two critical points: $(0,0)$ and $(1,2)$. To determine their type, we use the second derivative test: we have $AC - B^2 = 144y - 144$, so that

at $(0,0)$, we have $AC - B^2 = -144$, so it is a saddle point;
at $(1,2)$, we have $AC - B^2 = 144$ and $A > 0$, so it is a a minimum point.



A plot of the level curves is given at the right, which confirms the above. Note that the behavior of the level curves near the origin can be determined by using the approximation $w \approx 12x^2 - 12xy$; this shows the level curves near $(0,0)$ look like those of the function $x(x-y)$: the family of hyperbolas $x(x-y) = c$, with asymptotes given by the degenerate hyperbola $x(x-y) = 0$, i.e., the pair of lines $x = 0$ (the y-axis) and $x - y = 0$ (the diagonal line $y = x$).

2. Justification for the Second-derivative Test.

The test involves the quantity $AC - B^2$. In general, whenever we see the expressions $B^2 - 4AC$ or $B^2 - AC$ or their negatives, it means the quadratic formula is involved, in one of its two forms (the second is often used to get rid of the excess two's):

$$
Ax^2 + Bx + C = 0 \quad \Rightarrow \quad x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{3}
$$

$$
Ax^2 + 2Bx + C = 0 \quad \Rightarrow \quad x = \frac{-B \pm \sqrt{B^2 - AC}}{A} \tag{4}
$$

This is what is happening here. We want to know whether, near a critical point P_0, the graph of our function $w = f(x,y)$ always stays on one side of its horizontal tangent plane (P_0 is then a maximum or minimum point), or whether it lies partly above and partly below the tangent plane (P_0 is then a saddle point). As we will see, this is determined by how the graph of a quadratic function $f(x)$ lies with respect to the x-axis. Here is the basic lemma.

Lemma. For the quadratic function $Ax^2 + 2Bx + C$,

$$
AC - B^2 > 0, \quad A > 0 \text{ or } C > 0 \quad \Rightarrow \quad Ax^2 + 2Bx + C > 0 \quad \text{for all } x; \tag{5}
$$

$$
AC - B^2 > 0, \quad A < 0 \text{ or } C < 0 \quad \Rightarrow \quad Ax^2 + 2Bx + C < 0 \quad \text{for all } x; \tag{6}
$$

$$
AC - B^2 < 0 \quad \Rightarrow \quad \begin{cases} Ax^2 + 2Bx + C > 0, & \text{for some } x; \\ Ax^2 + 2Bx + C < 0, & \text{for some } x. \end{cases} \tag{7}
$$

Proof of the Lemma. To prove (5), we note that the quadratic formula in the form (4) shows that the zeros of $Ax^2 + 2Bx + C$ are imaginary, i.e., it has no real zeros. Therefore its graph must lie entirely on one side of the x-axis; which side can be determined from either A or C, since

$$A > 0 \;\Rightarrow\; \lim_{x\to\infty} Ax^2 + 2Bx + C = \infty; \qquad C > 0 \;\Rightarrow\; Ax^2 + 2Bx + C > 0 \text{ when } x = 0.$$

If $A < 0$ or $C < 0$, the reasoning is analogous and proves (6).

If on the other hand $AC - B^2 < 0$, formula (4) shows the quadratic function has two real roots, so that its parabolic graph crosses the x-axis twice, and hence lies partly above and partly below it. This proves (7). □

Proof of the Second-derivative Test in a special case.

The simplest function is a linear function, $w = w_0 + ax + by$, but it does not in general have maximum or minimum points and its second derivatives are all zero. The simplest functions to have interesting critical points are the quadratic functions, which we write in the form (the 2's will be explained momentarily):

$$w = w_0 + ax + by + \frac{1}{2}(Ax^2 + 2Bxy + Cy^2). \tag{8}$$

Such a function has in general a unique critical point, which we will assume is $(0,0)$; this gives the function a special form, which we can determine by evaluating its partial derivatives at $(0,0)$:

$$\begin{array}{ll} (w_x)_0 = a & \quad w_{xx} = A \\ (w_y)_0 = b & \quad w_{xy} = B \\ & \quad w_{yy} = C \end{array} \tag{9}$$

(The neat look of the above explains the $\frac{1}{2}$ and $2B$ in (8).) Since $(0,0)$ is a critical point, (9) shows that $a = 0$ and $b = 0$, so our quadratic function has the form

$$w - w_0 = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2). \tag{10}$$

We moved w_0 to the left side since the tangent plane at $(0,0)$ is the horizontal plane $w = w_0$, and we are interested in whether the graph of the quadratic function lies above or below this tangent plane, i.e., whether $w - w_0 > 0$ or $w - w_0 < 0$ at points other than the origin.

If $(x,y) \neq (0,0)$, then either $x \neq 0$ or $y \neq 0$; say $y \neq 0$. Then we write (10) as

$$w - w_0 \;=\; \frac{y^2}{2}\left[A\left(\frac{x}{y}\right)^2 + 2B\left(\frac{x}{y}\right) + C\right] \tag{11}$$

We know that $y^2 > 0$ if $y \neq 0$; applying our previous lemma to the factor on the right of (11), (or if $y = 0$, switching the roles of x and y in (11) and applying the lemma), we get

$$\begin{array}{lcl}
AC - B^2 > 0, \quad A > 0 \text{ or } C > 0 & \Rightarrow & w - w_0 > 0 \quad \text{for all } (x,y) \neq (0,0); \\
 & \Rightarrow & (0,0) \text{ is a minimum point;} \\
AC - B^2 > 0, \quad A < 0 \text{ or } C < 0 & \Rightarrow & w - w_0 < 0 \quad \text{for all } (x,y) \neq (0,0); \\
 & \Rightarrow & (0,0) \text{ is a maximum point;} \\
AC - B^2 < 0 & \Rightarrow & \begin{cases} w - w_0 > 0, & \text{for some } (x,y); \\ w - w_0 < 0, & \text{for some } (x,y); \end{cases} \\
 & \Rightarrow & (0,0) \text{ is a saddle point.}
\end{array}$$

Argument for the Second-derivative Test for a general function.

This part won't be rigorous, only suggestive, but it will give the right idea.

We consider a general function $w = f(x, y)$, and assume it has a critical point at (x_0, y_0), and continuous second derivatives in the neighborhood of the critical point. Then by a generalization of Taylor's formula to functions of several variables, the function has a best quadratic approximation at the critical point. To simplify the notation, we will move the critical point to the origin by making the change of variables

$$u = x - x_0, \qquad v = y - y_0.$$

Then the best quadratic approximation is (if the x, y on the left and u, v on the right is upsetting, just imagine u and v replaced everywhere by $x - x_0$ and $y - y_0$):

$$w \; = \; f(x,y) \quad \approx \quad w_0 + \frac{1}{2}\big(Au^2 + 2Buv + Cv^2\big); \tag{13}$$

here the coefficients A, B, C are given as in (1) by the second partial derivatives with respect to u and v at $(0,0)$, or what is the same (according to the chain rule—see the footnote below), by the second partial derivatives with respect to x and y at (x_0, y_0).

(Intuitively, one can see the coefficients have these values by differentiating both sides of (13) and pretending the approximation is an equality. There are no linear terms in u and v on the right since $(0,0)$ is a critical point.)

Since the quadratic function on the right of (13) is the best approximation to $w = f(x, y)$ for (x, y) close to (x_0, y_0), it is reasonable to suppose that their graphs are essentially the same near (x_0, y_0), so that if the quadratic function has a maximum, minimum or saddle point there, so will $f(x, y)$. Thus our results for the special case of a quadratic function having the origin as critical point carry over to the general function $f(x, y)$ at a critical point (x_0, y_0), if we interpret $A,\ B,\ C$ as the second partial derivatives at (x_0, y_0).
This is what the second derivative test says. □

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Footnote: Using $u = x - x_0$ and $v = y - y_0$, we can apply the chain rule for partial derivatives, which tells us that for all x, y and the corresponding u, v, we have

$$w_x = w_u \frac{\partial u}{\partial x} + w_v \frac{\partial v}{\partial x} = w_u, \;\text{ since } u_x = 1 \text{ and } v_x = 0,$$

and similarly, $w_y = w_v$. Therefore at the corresponding points,

$$(w_x)_{(x_0,y_0)} = (w_u)_{(0,0)}, \qquad\qquad (w_y)_{(x_0,y_0)} = (w_v)_{(0,0)},$$

and differentiating once more and using the same reasoning,

$$(w_{xx})_{(x_0,y_0)} = (w_{uu})_{(0,0)}, \qquad (w_{xy})_{(x_0,y_0)} = (w_{uv})_{(0,0)}, \qquad (w_{yy})_{(x_0,y_0)} = (w_{vv})_{(0,0)}.$$

NON-INDEPENDENT VARIABLES

1. Introduction

Up to now, we have considered partial derivatives of n-variable functions defined on all of $\mathbb{R}^n$ or on an n-dimensional subset defined by inequalities (for example, the orthant defined by $x_1, \ldots, x_n > 0$). In such a situation, the variables are free to change independently, and in particular it makes sense to vary one variable while holding all the others constant.

But if a function is defined only on a subset defined by constraint equations, such as the unit sphere $x^2 + y^2 + z^2 = 1$, then it might be impossible to vary one variable while holding the others constant. In such a situation, extra care is needed in defining partial derivatives and working with them.

2. Constraint equations

Part of the specification of a function is its domain, the set of inputs on which it is being considered.

Example 2.1. A meteorologist studying the current world temperature is probably not interested in the temperature in deep outer space, and hence would be more likely to model temperature as a function defined on the sphere $x^2+y^2+z^2 = 1$ instead of a function defined on all of $\mathbb{R}^3$. In this case, the domain is the sphere $x^2 + y^2 + z^2 = 1$, and $x^2 + y^2 + z^2 = 1$ is called a constraint equation.

Example 2.2. The pressure P, volume V, and temperature T of a fixed amount of gas satisfy the ideal gas law $PV = nRT$, where n and R are constants (here n is the amount of gas measured in moles, and R is a universal constant). A thermodynamic quantity expressible in terms of P, V, and T (such as internal energy U or entropy S) is represented mathematically as a function whose domain is the set of points in the 3-dimensional (P, V, T)-space satisfying the constraint equation $PV = nRT$ and the constraint inequalities $P, V, T > 0$; this domain is a 2-dimensional surface inside the first orthant of $\mathbb{R}^3$. The state of the gas at a particular time is given by the numerical values of (P, V, T) then; mathematically, a state is just a point in the domain. In thermodynamics, the domain is also called the state space because it is the set of all physically possible states.

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3. Constraint equations and dimension

Although the sphere $x^2 + y^2 + z^2 = 1$ is contained in 3-dimensional space, it is really a 2-dimensional object, with surface area, not volume (the interior is not included).

Rule of thumb:

$$\boxed{\text{Each constraint equation usually reduces the dimension of the domain by 1.}}$$

Example 3.1. The space $\mathbb{R}^3$ has dimension 3, but the set of points in $\mathbb{R}^3$ satisfying the constraint equation

$$x + 2y + 3z = 5$$

is only 2-dimensional (as you know, it's a plane).

Example 3.2. Similarly, the solution set to the system

$$x^2 + y^2 + z^2 = 100$$
$$x + 2y + 3z = 5$$

with two constraint equations is only 1-dimensional. (It is the intersection of a sphere and a plane, and it turns out to be a circle.)

The rule of thumb implies that if the domain is defined by e equations in n variables, it will usually be $(n - e)$-dimensional. If $e > n$ (more equations than variables), so that $n - e$ is negative, usually this means that the constraint equations are inconsistent, so they define an *empty* domain.

Warning 3.3. It is not *always* true that each constraint equation reduces the dimension by 1. Sometimes the rule of thumb fails because of redundancy in the equations, as in the following examples.

Example 3.4. The system

$$x + 2y + 3z = 5$$
$$2x + 4y + 6z = 10$$

has two constraint equations, so one might expect them to reduce the dimension from 3 to 1, but in fact the solution set has dimension 2, because the second equation is just double the first one, so it contains no new information; it is as if there were only one constraint equation. (Geometrically, the solution is the intersection of two planes, but the two planes are the same!)

Sometimes the redundancy is a little more subtle:

Example 3.5. Usually a system of 3 linear equations in 3 variables has 0-dimensional solution set (in fact, a single point), but the system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 5 \\ 2x + 3y + 4z &= 6 \end{aligned}$$

has a 1-dimensional solution set since the third equation is a consequence of the first two (it is their sum) and hence contains no new information. On the other hand, the system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 5 \\ 2x + 3y + 4z &= 7 \end{aligned}$$

is inconsistent (no solutions), because the first two equations imply that $2x + 3y + 4z = 6$, making it impossible to satisfy all three equations simultaneously.

Remark 3.6. For a square system of linear equations, redundancy or inconsistency occurs when the coefficient matrix A satisfies $\det A = 0$. Most square matrices have *nonzero* determinant, so most square systems have a 0-dimensional solution set (in fact, we know that if $\det A \neq 0$, then the solution set to $A\mathbf{x} = \mathbf{b}$ consists of a single point, namely $A^{-1}\mathbf{b}$).

4. Constraint inequalities and dimension

What about constraint *inequalities*? These usually do not affect the dimension. For the example, the region in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 < 9$ is still 3-dimensional (it is the interior of a ball of radius 3, and has volume).

5. Independent variables

Example 5.1. On the upper half of the circle $x^2 + y^2 = 9$ in $\mathbb{R}^2$ one can express y in terms of x, namely

$$y = \sqrt{9 - x^2}.$$

Here we think of x as an independent variable, and y depends on x. On the lower half one would use

$$y = -\sqrt{9 - x^2}$$

instead.

Example 5.2. Similarly, on the sphere $x^2 + y^2 + z^2 = 1$, one can locally express z as a function of independent variables x and y. (We said "locally" because there is not a single function that works on the whole sphere: one function works on the upper half, and a different function on the lower half.)

In general, for a D-dimensional domain defined by e constraint equations in n variables, usually $D = n - e$ and locally one can choose D variables to be the independent ones so that each of the remaining e variables can be expressed as a function of the D independent variables. In particular,

$$\boxed{\text{number of independent variables} = \text{dimension of the domain.}}$$

The precise mathematical statement along these lines is called the implicit function theorem; it is discussed in more advanced math courses.

6. Functions on a constrained domain

Consider the function

$$f(x, y, z) := x^2y^3z^5 \quad \textit{restricted to the sphere } x^2 + y^2 + z^2 = 1.$$

Since there is one equation in three variables, and $3 - 1 = 2$, the domain should be 2-dimensional, so we should be able to express f locally in terms of two independent variables.

On the part of the sphere where $z > 0$, we may choose x and y as independent variables, so that $z = \sqrt{1 - x^2 - y^2}$ and

$$f = x^2y^3(1 - x^2 - y^2)^{5/2}. \tag{1}$$

The point of eliminating z is that now f is expressed in terms of variables x and y that are not related by any constraint equation.

Alternatively, on a suitable part of the sphere we could express f in terms of independent variables y and z, by substituting $x = \sqrt{1 - y^2 - z^2}$:

$$f = (1 - y^2 - z^2)y^3z^5. \tag{2}$$

Formulas (1) and (2) represent (pieces of) the *same function*, just expressed in terms of different variables.

7. Derivatives

Question 7.1. Consider the function $f(x, y) := xy$ where (x, y) is constrained to lie on the line $2x + y = 7$. What is $\dfrac{df}{dx}$ at the point $(3, 1)$?

(We wrote $\dfrac{df}{dx}$ instead of $\dfrac{\partial f}{\partial x}$ because there is only one independent variable: y depends on x.)

Incorrect solution: The derivative of xy with respect to x is y, whose value at $(3, 1)$ is 1.

What makes this wrong? It is true that if $f(x, y) := xy$ on $\mathbb{R}^2$, then $\dfrac{\partial f}{\partial x} = y$. But the definition of partial derivative assumes that it makes sense to hold y constant while varying x, which is impossible if (x, y) is required to satisfy the constraint $2x + y = 7$.

Correct solution 1 (elimination of dependent variable): There are two variables and one constraint equation, and $2 - 1 = 1$, so we should be looking to express f in terms of one independent variable. In fact, we can choose x as the independent variable, which is what we should do if we are interested in $\frac{df}{dx}$. Now use the constraint equation to eliminate the dependent variable y and express everything in terms of x:

$$\begin{aligned} y &= 7 - 2x \\ f &= x(7-2x) = 7x - 2x^2 \\ \frac{df}{dx} &= 7 - 4x, \end{aligned}$$

and the value of $\frac{df}{dx}$ at $(x, y) = (3, 1)$ is $7 - 4(3) = -5$. □

In Correct Solution 1, we were lucky that it was easy to solve for y in terms of x. In more complicated situations, this might not be possible, but one can still determine how quickly y changes as x changes, by taking the differential of the constraint equation. To see how this works, let's solve the same problem again.

Correct solution 2 (differentials): If $f = xy$ is viewed as a function on $\mathbb{R}^2$, the definition of df gives

$$df = y\,dx + x\,dy. \tag{3}$$

This expresses how f changes as x and y change.

If f is restricted to a function on the domain defined by the constraint equation, then (3) still holds, but now any change in x causes a change in y, so dx and dy are related. To find the relation, *take the differential of the constraint equation* $2x + y = 7$; this gives

$$2\,dx + dy = 0$$

so

$$dy = -2\,dx$$

(which makes sense since (x, y) is constrained to lie on the line $2x + y = 7$ of slope -2). To compute $\frac{df}{dx}$, we want to consider f as a function of the independent variable x alone, so we should express df in terms of dx alone. To eliminate the dy term, substitute $dy = -2\,dx$ into (3) to get

$$\begin{aligned} df &= y\,dx + x(-2\,dx) \\ &= (y - 2x)\,dx. \end{aligned}$$

This means that

$$\frac{df}{dx} = y - 2x.$$

At $(3,1)$, this is $1-2(3)=-5$. $\square$

8. Partial derivatives

Now let's consider what happens in a question like Question 7.1 when there is more than one independent variable.

Question 8.1. Consider the function $f(x,y,z) := x+y+x^2z$ where (x,y,z) is constrained to lie on the surface $xyz=6$. What is $\dfrac{\partial f}{\partial x}$ at the point $(1,2,3)$?

Answer: In the presence of the constraint equation $xyz=6$, the notation $\dfrac{\partial f}{\partial x}$ is meaningless, so the question does not make sense!

Here is why: Usually $\dfrac{\partial f}{\partial x}$ means the rate of change of f as x varies while all the other variables are held constant. But we can't hold both y and z constant while varying x, if we want the constraint equation $xyz=6$ to remain true.

Conclusions:

1. We are allowed to talk about partial derivatives of f only if f is expressed as a function of *independent* variables (independence guarantees that we can vary one variable while holding the others constant).
2. If f is initially expressed in terms of variables satisfying constraint equations, *we must choose some of the variables to be the independent ones*, and view f and all other variables as functions of the independent variables (as in Section 6), before talking about partial derivatives of f. The notation for the partial derivatives must indicate which variables are being used as the independent ones.

The notational convention is that all the independent variables are listed at the bottom of the partial derivative notation, with the variables being held constant listed as subscripts outside parentheses:

Definition 8.2. The notation

$$\left(\frac{\partial f}{\partial x}\right)_y$$

means that we are viewing f as a function of independent variables x and y, and measuring the rate of change of f as x varies while holding y constant.

Similarly, $\left(\dfrac{\partial f}{\partial x}\right)_z$ means that we are viewing f as a function of independent variables x and z, and measuring the rate of change of f as x varies while holding z constant.

Example 8.3 (analogous to Correct Solution 1 in Section 7)**.** In Question 8.1, we can use the constraint equation to eliminate z and express f in terms of independent variables x and y:

$$f = x + y + x^2 \left(\frac{6}{xy}\right) = x + y + \frac{6x}{y}.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + \frac{6}{y},$$

so $\left(\frac{\partial f}{\partial x}\right)_y$ at $(1, 2, 3)$ is

$$1 + \frac{6}{2} = 4.$$

Example 8.4 (analogous to Correct Solution 1 in Section 7 again)**.** In Question 8.1, we can use the constraint equation to eliminate y and express f in terms of independent variables x and z:

$$f = x + \frac{6}{xz} + x^2 z.$$

Then

$$\left(\frac{\partial f}{\partial x}\right)_z = 1 - \frac{6}{x^2 z} + 2xz,$$

so $\left(\frac{\partial f}{\partial x}\right)_z$ at $(1, 2, 3)$ is

$$1 - \frac{6}{1^2(3)} + 2(1)(3) = 5.$$

Remark 8.5. The values of $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial x}\right)_z$ are rates of change as one moves along two different paths in the domain: along the first path y is constant while z varies in response to x varying, but along the second path z is constant while y varies in response to x varying. So it is not surprising that the two values are different.

Remark 8.6. In Question 7.1, there was only one independent variable, namely the variable x with respect to which the derivative was being taken, so it was not necessary to specify the independent variables in the notation $\frac{df}{dx}$.

Example 8.7. Suppose that g is a function of variables s, t, u, v related by one constraint equation. Then usually g would be locally expressible as a function of three independent variables. Thus one might have partial derivatives such as $\left(\frac{\partial g}{\partial u}\right)_{t,v}$, in which g is viewed as a function of independent variables t, u, v.

9. Partial derivatives and differentials

If we cannot solve the constraint equations to eliminate the dependent variables, we can try the method of differentials.

Question 9.1. Suppose that $f(x, y, z) := x + y + x^2 z$, where x, y, z are constrained to lie on the surface $xyz = 6$. What is $\left(\frac{\partial f}{\partial x}\right)_y$ at the point $(1, 2, 3)$?

This is the same question as in Example 8.3, but this time we are going to answer it using differentials, in a manner similar to Correct Solution 2 in Section 7.

Solution: To compute $\left(\frac{\partial f}{\partial x}\right)_y$, in which the independent variables are x and y, we need to express df in the form

$$df = ?\, dx + ?\, dy,$$

where each ? represents a function; then $\left(\frac{\partial f}{\partial x}\right)_y$ is the first ? (and $\left(\frac{\partial f}{\partial y}\right)_x$ is the second ?).

But f is initially given as a function of dependent variables x, y, z. If $f = x + y + x^2 z$ is viewed as a function on $\mathbb{R}^3$, the definition of df gives

$$df = (1 + 2xz)\, dx + dy + x^2\, dz. \tag{4}$$

If f is restricted to a function on the domain defined by the constraint equation, then (4) still holds, but taking the differential of the constraint equation $xyz = 6$ gives a relation between dx, dy, dz:

$$yz\, dx + xz\, dy + xy\, dz = 0. \tag{5}$$

Because we want df in terms of dx and dy only, we solve (5) for dz,

$$\begin{aligned} xy\, dz &= -yz\, dx - xz\, dy \\ dz &= -\frac{z}{x}\, dx - \frac{z}{y}\, dy, \end{aligned}$$

and substitute into (4):

$$\begin{aligned} df &= (1 + 2xz)\, dx + dy + x^2\, dz \\ &= (1 + 2xz)\, dx + dy + x^2 \left(-\frac{z}{x}\, dx - \frac{z}{y}\, dy\right) \\ &= (1 + 2xz)\, dx + dy - xz\, dx - \frac{x^2 z}{y}\, dy \\ &= (1 + xz)\, dx + \left(1 - \frac{x^2 z}{y}\right) dy. \end{aligned}$$

This means that
$$\left(\frac{\partial f}{\partial x}\right)_y = 1 + xz,$$
and the value of $\left(\dfrac{\partial f}{\partial x}\right)_y$ at $(1, 2, 3)$ is $1 + 1(3) = 4$.

10. Proving rules concerning partial derivatives

There are many rules relating different partial derivatives, but they all follow from the method of differentials, so there is no need to memorize the rules. The purpose of this section is not to list rules to be memorized, but to give practice in using the method of differentials.

Problem 10.1. The cyclic rule states that if variables x, y, z are related by a constraint equation such that (as expected) any two of the variables may be taken as the independent variables, then
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$
Prove this rule.

Proof. Let $f(x, y, z) = 0$ be the constraint equation, where f is a function that makes sense on $\mathbb{R}^3$. Taking the differential of the constraint equation gives
$$f_x\, dx + f_y\, dy + f_z\, dz = 0,$$
where f_x, f_y, f_z are the partial derivatives of f viewed as a function on $\mathbb{R}^3$ (or at least a 3-dimensional part of $\mathbb{R}^3$). Solving for dx gives
$$dx = -\frac{f_y}{f_x}\, dy - \frac{f_z}{f_x}\, dz, \tag{6}$$
which means that when x is viewed as a function of independent variables y, z, then
$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x},$$
the coefficient of dy in (6). A similar argument shows that
$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{f_z}{f_y}$$
$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z},$$
and multiplying all three gives
$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{f_y}{f_x}\right)\left(-\frac{f_z}{f_y}\right)\left(-\frac{f_x}{f_z}\right) = -1.$$ □

Another example is the two-Jacobian rule. To state it, we need a definition:

Definition 10.2. If $u = u(x, y)$ and $v = v(x, y)$, then the Jacobian of (u, v) with respect to (x, y) is the function

$$\frac{\partial(u,v)}{\partial(x,y)} := \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Here u_x means $\left(\frac{\partial u}{\partial x}\right)_y$, and so on.

Two-Jacobian rule: If u, v, w, x, y are related by constraint equations such that any two of the variables may be taken as the independent variables, then

$$\left(\frac{\partial u}{\partial v}\right)_w = \frac{\partial(u,w)/\partial(x,y)}{\partial(v,w)/\partial(x,y)}.$$

(The right side could also be written out as

$$\frac{u_x w_y - w_x u_y}{v_x w_y - w_x v_y},$$

a ratio of determinants.)

How is the two-Jacobian rule used? It says that if one knows the partial derivatives of all the variables with respect to independent variables x, y, then one can calculate the partial derivatives using any other variables as the independent ones.

The two-Jacobian rule can be proved using differentials.

11. Summary

Here is a summary of some of the key points.

- The dimension of a domain in $\mathbb{R}^n$ is defined by e constraint equations (in n variables) is usually $D := n - e$.
- In that case, usually it is possible locally to choose D of the variables to be the independent variables so that the other variables becomes functions of the independent variables. Then a function f on the domain can be locally re-expressed as a function in only the independent variables.
- When discussing partial derivatives of a function defined on a domain defined by constraint equations, one must specify which variables are being used as the independent variables (and specify which independent variable we are changing while holding the other independent variables constant). For example, $\left(\frac{\partial f}{\partial s}\right)_{r,t}$ means that we are viewing f as a function of independent variables r, s, t and measuring the rate of change of f as s varies while r and t are held constant.
- There are two methods for computing a partial derivative like $\left(\frac{\partial f}{\partial s}\right)_{r,t}$:

– Method 1: Eliminate the dependent variables to express f as an explicit function of the independent variables r, s, t.

– Method 2: Start with

$$df = f_r\, dr + f_s\, ds + f_t\, dt + f_u\, du + \cdots ,$$

in which f is viewed as a function on (part of) $\mathbb{R}^n$ before taking the constraint equations into account. Use the differential of the constraint equation(s) to eliminate the differentials of the dependent variables $(du, \dots)$, so as to re-express df in terms of the differentials of the independent variables only:

$$df = ?\, dr + ?\, ds + ?\, dt.$$

Then $\left(\dfrac{\partial f}{\partial s}\right)_{r,t}$ is the function ? in front of ds.

P. Partial Differential Equations

An important application of the higher partial derivatives is that they are used in partial differential equations to express some laws of physics which are basic to most science and engineering subjects. In this section, we will give examples of a few such equations. The reason is partly cultural, so you meet these equations early and learn to recognize them, and partly technical: to give you a little more practice with the chain rule and computing higher derivatives.

A **partial differential equation**, PDE for short, is an equation involving some unknown function of several variables and one or more of its partial derivatives. For example,

$$x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} = 0$$

is such an equation. Evidently here the unknown function is a function of two variables

$$w = f(x,y) ;$$

we infer this from the equation, since only x and y occur in it as independent variables. In general a **solution** of a partial differential equation is a differentiable function that satisfies it. In the above example, the functions

$$w = x^n y^n \qquad \text{any } n$$

all are solutions to the equation. In general, PDE's have many solutions, far too many to find all of them. The problem is always to find the one solution satisfying some extra conditions, usually called either *boundary conditions* or *initial conditions* depending on their nature.

Our first important PDE is the **Laplace equation** in three dimensions:

$$(1) \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0 .$$

Any steady-state temperature distribution in three-space

$$(2) \qquad w = T(x,y,z), \qquad T = \text{temperature at the point } (x,y,z)$$

satisfies Laplace's equation. (Here *steady-state* means that it is unchanging over time, here reflected in the fact that T is not a function of time. For example, imagine a solid object made of some uniform heat-conducting material (say a solid metal ball), and imagine a steady temperature distribution on its surface is maintained somehow (say with some arrangement of wires and thermostats). Then after a while the temperature at each point inside the ball will come to equilibrium — reach a steady state — and the resulting temperature function (2) inside the ball will then satisfy Laplace's equation.

As another example, the *gravitational potential*

$$w = \phi(x,y,z)$$

resulting from some arrangement of masses in space satisfies Laplace's equation in any region R of space not containing masses. The same is true of the *electrostatic potential* resulting from some collection of electric charges in space: (1) is satisfied in any region which is free of charge. This potential function measures the work done (against the field) carrying a unit test mass (or charge) from a fixed reference point to the point (x, y, z) in the gravitational (or electrostatic) field. Knowing ϕ, the field itself can be recovered as its negative gradient:

$$\mathbf{F} = -\nabla\phi .$$

All of this is just to stress the fundamental character of Laplace's equation — we live our lives surrounded by its solutions.

The *two-dimensional* Laplace equation is similar — you just drop the term involving z. The steady-state temperature distribution in a flat metal plate would satisfy the two-dimensional Laplace equation, if the faces of the plate were kept insulated and a steady-state temperature distribution maintained around the edges of the plate.

If in the temperature model we include also heat sources and sinks in the region, unchanging over time, the temperature function satisfies the closely related **Poisson equation**

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = f(x, y, z), \tag{3}$$

where f is some given function related to the sources and sinks.

Another important PDE is the **wave equation**; given below are the one-dimensional and two-dimensional versions; the three dimensional version would add a similar term in z to the left:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}; \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}. \tag{4}$$

Here $x, y, \ldots$ are the space variables, t is the time, and c is the velocity with which the wave travels — this depends on the medium and the type of wave (light, sound, etc.). A solution, respectively

$$w = w(x, t), \qquad w = w(x, y, t),$$

gives for each moment t_0 of time the shape $w(x, t_0)$, $w(x, y, t_0)$ of the wave.

The third PDE goes by two names, depending on the context: **heat equation** or **diffusion equation**. The one- and two-dimensional versions are respectively

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{a^2}\frac{\partial w}{\partial t}; \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{a^2}\frac{\partial w}{\partial t}. \tag{5}$$

It looks a lot like the wave equation (4), but the right-hand side this time involves only the first derivative, which gives it mathematically and physically an entirely different character.

When it is called the (one-dimensional) heat equation, a solution $w(x, t)$ represents a time-varying temperature distribution in say a uniform conducting metal rod, with insulated sides. In the same way, $w(x, y, t)$ would be the time-varying temperature distribution in a flat metal plate with insulated faces. For each moment t_0 in time, $w(x, y, t_0)$ gives the temperature distribution at that moment.

For example, if we assume the distribution is steady-state, i.e., not changing with time, then

$$\frac{\partial w}{\partial t} = 0 \qquad \text{(steady-state condition)}$$

and the two-dimensional heat equation would turn into the two-dimensional Laplace equation (1).

When (5) is referred to as the *diffusion equation*, say in one dimension, then $w(x,t)$ represents the concentration of a dissolved substance diffusing along a uniform tube filled with liquid, or of a gas diffusing down a uniform pipe.

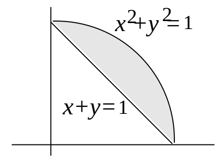
Notice that all of these PDE's are second-order, that is, involve derivatives no higher than the second. There is an important fourth-order PDE in elasticity theory (the bilaplacian equation), but by and large the general rule seems to be either that Nature is content with laws that only require second partial derivatives, or that these are the only laws that humans are intelligent enough to formulate.

Exercises: Section 2K

I. Limits in Iterated Integrals

For most students, the trickiest part of evaluating multiple integrals by iteration is to put in the limits of integration. Fortunately, a fairly uniform procedure is available which works in any coordinate system. *You must always begin by sketching the region; in what follows we'll assume you've done this.*

1. Double integrals in rectangular coordinates.



Let's illustrate this procedure on the first case that's usually taken up: double integrals in rectangular coordinates. Suppose we want to evaluate over the region R pictured the integral

$$\iint_R f(x,y)\,dy\,dx\ , \qquad\qquad R = \ \text{ region between } x^2+y^2=1 \quad \text{and} \quad x+y=1\ ;$$

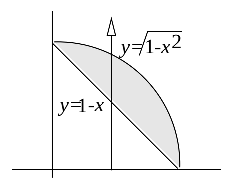
we are integrating first with respect to y. Then to put in the limits,

1. Hold x fixed, and let y increase (since we are integrating with respect to y). As the point (x,y) moves, it traces out a vertical line.

2. Integrate from the y-value where this vertical line enters the region R, to the y-value where it leaves R.

3. Then let x increase, integrating from the lowest x-value for which the vertical line intersects R, to the highest such x-value.

Carrying out this program for the region R pictured, the vertical line enters R where $y = 1-x$, and leaves where $y = \sqrt{1-x^2}$.



The vertical lines which intersect R are those between $x = 0$ and $x = 1$. Thus we get for the limits:

$$\iint_R f(x,y)\,dy\,dx \ = \ \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

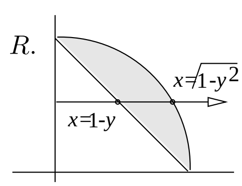
To calculate the double integral, integrating in the reverse order $\iint_R f(x,y)\,dx\,dy$,

1. Hold y fixed, let x increase (since we are integrating first with respect to x). This traces out a horizontal line.

2. Integrate from the x-value where the horizontal line enters R to the x-value where it leaves.

3. Choose the y-limits to include all of the horizontal lines which intersect R.

Following this prescription with our integral we get:



$$\iint_R f(x,y)\,dx\,dy \ = \ \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Exercises: 3A-2

2. Double integrals in polar coordinates

The same procedure for putting in limits works for these also. Suppose we want to evaluate over the same region R as before

$$\iint_R dr\, d\theta\ .$$

(The integrand, including the r that usually goes with $r\,dr\,d\theta$, is irrelevant, and therefore omitted.)

As usual, we integrate first with respect to r. Therefore, we

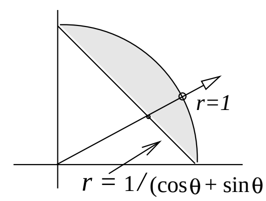
1. Hold θ fixed, and let r increase (since we are integrating with respect to r). As the point moves, it traces out a ray going out from the origin.

2. Integrate from the r-value where the ray enters R to the r-value where it leaves. This gives the limits on r.

3. Integrate from the lowest value of θ for which the corresponding ray intersects R to the highest value of θ.

To follow this procedure, we need the equation of the line in polar coordinates. We have

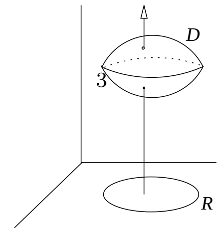
$$x + y = 1 \qquad \to \qquad r\cos\theta + \mathbf{r}\sin\theta = 1, \qquad \text{or} \quad r \;=\; \frac{1}{\cos\theta + \sin\theta}.$$



This is the r value where the ray enters the region; it leaves where $r = 1$. The rays which intersect R lie between $\theta = 0$ and $\theta = \pi/2$. Thus the double iterated integral in polar coordinates has the limits

$$\int_0^{\pi/2}\int_{1/(\cos\theta+\sin\theta)}^{1} dr\, d\theta\ .$$

Exercises: 3B-1



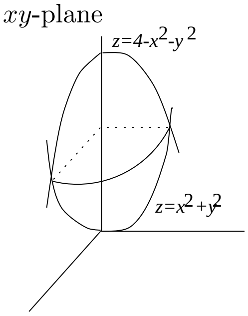
3. Triple integrals in rectangular and cylindrical coordinates.

You do these the same way, basically. To supply limits for $\iiint_D dz\,dy\,dx$ over the region D, we integrate first with respect to z. Therefore we

1. Hold x and y fixed, and let z increase. This gives us a vertical line.

2. Integrate from the z-value where the vertical line enters the region D to the z-value where it leaves D.

3. Supply the remaining limits (in either xy-coordinates or polar coordinates) so that you include all vertical lines which intersect D. This means that you will be integrating the remaining double integral over the region R in the xy-plane which D projects onto.



For example, if D is the region lying between the two paraboloids

$$z = x^2 + y^2 \qquad\qquad z = 4 - x^2 - y^2,$$

we get by following steps 1 and 2,

$$\iiint_D dz\,dy\,dx \;=\; \iint_R \int_{x^2+y^2}^{4-x^2-y^2} dz\,dA$$

where R is the projection of D onto the xy-plane. To finish the job, we have to determine what this projection is. From the picture, what we should determine is the xy-curve over which the two surfaces intersect. We find this curve by eliminating z from the two equations, getting

$$\begin{aligned} x^2+y^2 &= 4 - x^2 - y^2, \qquad \text{which implies} \\ x^2+y^2 &= 2. \end{aligned}$$

Thus the xy-curve bounding R is the circle in the xy-plane with center at the origin and radius $\sqrt{2}$.

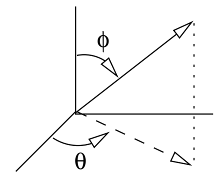
This makes it natural to finish the integral in polar coordinates. We get

$$\iiint_D dz\,dy\,dx \;=\; \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,r\,dr\,d\theta\ ;$$

the limits on z will be replaced by r^2 and $4 - r^2$ when the integration is carried out.

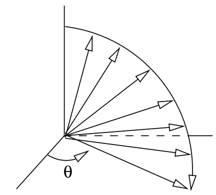
Exercises: 5A-2

4. Spherical coordinates.



Once again, we use the same procedure. To calculate the limits for an iterated integral $\iiint_D d\rho\, d\phi\, d\theta$ over a region D in 3-space, we are integrating first with respect to ρ. Therefore we

1. Hold ϕ and θ fixed, and let ρ increase. This gives us a ray going out from the origin.

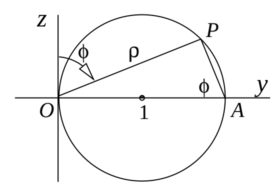


2. Integrate from the ρ-value where the ray enters D to the ρ-value where the ray leaves D. This gives the limits on ρ.

3. Hold θ fixed and let ϕ increase. This gives a family of rays, that form a sort of fan. Integrate over those ϕ-values for which the rays intersect the region D.

4. Finally, supply limits on θ so as to include all of the fans which intersect the region D.

For example, suppose we start with the circle in the yz-plane of radius 1 and center at $(1,0)$, rotate it about the z-axis, and take D to be that part of the resulting solid lying in the first octant.



First of all, we have to determine the equation of the surface formed by the rotated circle. In the yz-plane, the two coordinates ρ and ϕ are indicated. To see the relation between them when P is on the circle, we see that also angle $OAP = \phi$, since both the angle ϕ and OAP are complements of the same angle, AOP. From the right triangle, this shows the relation is $\rho = 2\sin\phi$.

As the circle is rotated around the z-axis, the relationship stays the same, so $\rho = 2\sin\phi$ is the equation of the whole surface.

To determine the limits of integration, when ϕ and θ are fixed, the correpsonding ray enters the region where $\rho = 0$ and leaves where $\rho = 2\sin\phi$.

As ϕ increases, with θ fixed, it is the rays between $\phi = 0$ and $\phi = \pi/2$ that intersect D, since we are only considering the portion of the surface lying in the first octant (and thus above the xy-plane).

Again, since we only want the part in the first octant, we only use θ values from 0 to $\pi/2$. So the iterated integral is

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi} d\rho\, d\phi\, d\theta\ .$$

Exercises: 5B-1

CV. Changing Variables in Multiple Integrals

1. Changing variables.

Double integrals in x, y coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates:

$$(1) \qquad \iint_R f(x,y)\, dA = \iint_R g(r,\theta)\, r\, dr d\theta \ .$$

This involves introducing the new variables r and θ, together with the equations relating them to x, y in both the forward and backward directions:

$$(2) \qquad r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}(y/x); \qquad x = r\cos\theta, \qquad y = r\sin\theta \ .$$

Changing the integral to polar coordinates then requires three steps:

A. Changing the integrand $f(x,y)$ to $g(r,\theta)$, by using (2);

B. Supplying the area element in the r, θ system: $dA = \ r\, dr\, d\theta$;

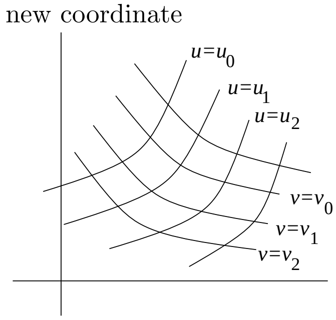
C. Using the region R to determine the limits of integration in the r, θ system.

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from x, y to one better adapted to the region or integrand. Let's call the new coordinates u and v; then there will be equations introducing the new coordinates, going in both directions:

$$(3) \qquad u = u(x,y), \qquad v = v(x,y); \qquad x = x(u,v), \qquad y = y(u,v)$$

(often one will only get or use the equations in one of these directions). To change the integral to u, v-coordinates, we then have to carry out the three steps $\mathbf{A}, \mathbf{B}, \mathbf{C}$ above. A first step is to picture the new coordinate system; for this we use the same idea as for polar coordinates, namely, we consider the grid formed by the level curves of the new coordinate functions:

$$(4) \qquad u(x,y) = u_0, \qquad v(x,y) = v_0 \ .$$



Once we have this, algebraic and geometric intuition will usually handle steps **A** and **C**, but for **B** we will need a formula: it uses a determinant called the **Jacobian**, whose notation and definition are

$$(5) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \ .$$

Using it, the formula for the area element in the u, v-system is

$$(6) \qquad dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \ ,$$

so the change of variable formula is

$$\iint_R f(x,y)\,dx\,dy \;=\; \iint_R g(u,v)\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv\;, \tag{7}$$

where $g(u,v)$ is obtained from $f(x,y)$ by substitution, using the equations (3).

We will derive the formula (5) for the new area element in the next section; for now let's check that it works for polar coordinates.

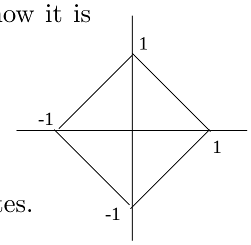
Example 1. Verify (1) using the general formulas (5) and (6).

Solution. Using (2), we calculate:

$$\frac{\partial(x,y)}{\partial(r,\theta)} \;=\; \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} \;=\; \begin{vmatrix} \cos\theta & -r\,\sin\theta \\ \sin\theta & r\,\cos\theta \end{vmatrix} \;=\; r(\cos^2\theta+\sin^2\theta) \;=\; r\;,$$

so that $dA \;=\; r\,dr\,d\theta$, according to (5) and (6); note that we can omit the absolute value, since by convention, in integration problems we always assume $r \geq 0$, as is implied already by the equations (2).

We now work an example illustrating why the general formula is needed and how it is used; it illustrates step **C** also — putting in the new limits of integration.



Example 2. Evaluate $\displaystyle\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$ over the region R pictured.

Solution. This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$$x+y=\pm 1, \qquad x-y=\pm 1 \tag{8}$$

and the integrand also contains the combinations $x-y$ and $x+y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for x and y):

$$u=x+y, \qquad v=x-y; \qquad\qquad x=\frac{u+v}{2}, \qquad y=\frac{u-v}{2}\;. \tag{9}$$

We will also need the new area element; using (5) and (9) above. we get

$$\frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} \;=\; -\frac{1}{2}\;; \tag{10}$$

note that it is the second pair of equations in (9) that were used, not the ones introducing u and v. Thus the new area element is (this time we do need the absolute value sign in (6))

$$dA \;=\; \frac{1}{2}\,du\,dv\;. \tag{11}$$

We now combine steps **A** and **B** to get the new double integral; substituting into the integrand by using the first pair of equations in (9), we get

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy \;=\; \iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\,du\,dv\;. \tag{12}$$

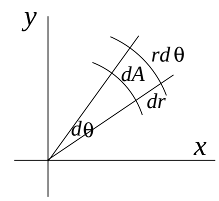
In uv-coordinates, the boundaries (8) of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral (12) becomes

$$\iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv \;=\; \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv$$

We have

$$\text{inner integral} \;=\; -\frac{v^2}{2(u+2)}\bigg]_{u=-1}^{u=1} \;=\; \frac{v^2}{3}\;; \qquad \text{outer integral} \;=\; \frac{v^3}{9}\bigg]_{-1}^{1} \;=\; \frac{2}{9}\;.$$

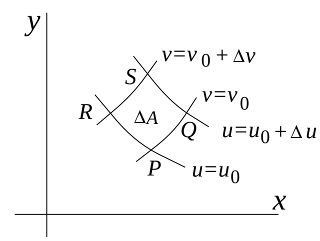
2. The area element.



In polar coordinates, we found the formula $dA = r\,dr\,d\theta$ for the area element by drawing the grid curves $r = r_0$ and $\theta = \theta_0$ for the r,θ-system, and determining (see the picture) the infinitesimal area of one of the little elements of the grid.

For general u, v-coordinates, we do the same thing. The grid curves (4) divide up the plane into small regions ΔA bounded by these contour curves. If the contour curves are close together, they will be approximately parallel, so that the grid element will be approximately a small parallelogram, and

$$(13) \qquad \Delta A \;\approx\; \text{area of parallelogram PQRS} \;=\; |PQ \times PR|$$



In the uv-system, the points P, Q, R have the coordinates

$$P : (u_0, v_0), \qquad Q : (u_0 + \Delta u, v_0), \qquad R : (u_0, v_0 + \Delta v)\;;$$

to use the cross-product however in (13), we need PQ and PR in $\mathbf{i}\,\mathbf{j}$- coordinates. Consider PQ first; we have

$$(14) \qquad PQ \;=\; \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}\;,$$

where Δx and Δy are the changes in x and y as you hold $v = v_0$ and change u_0 to $u_0 + \Delta u$. According to the definition of partial derivative,

$$\Delta x \;\approx\; \left(\frac{\partial x}{\partial u}\right)_0 \Delta u, \qquad \Delta y \;\approx\; \left(\frac{\partial y}{\partial u}\right)_0 \Delta u;$$

so that by (14),

$$(15) \qquad PQ \;\approx\; \left(\frac{\partial x}{\partial u}\right)_0 \Delta u\;\mathbf{i} \;+\; \left(\frac{\partial y}{\partial u}\right)_0 \Delta u\;\mathbf{j}\;.$$

In the same way, since in moving from P to R we hold u fixed and increase v_0 by Δv,

$$(16) \qquad PR \;\approx\; \left(\frac{\partial x}{\partial v}\right)_0 \Delta v\;\mathbf{i} \;+\; \left(\frac{\partial y}{\partial v}\right)_0 \Delta v\;\mathbf{j}\;.$$

We now use (13); since the vectors are in the xy-plane, $PQ \times PR$ has only a $\mathbf{k}$-component, and we calculate from (15) and (16) that

$$\mathbf{k}\text{-component of } \; PQ \times PR \;\approx\; \begin{vmatrix} x_u \Delta u & y_u \Delta u \\ x_v \Delta v & y_v \Delta v \end{vmatrix}_0$$

$$(17) \qquad\qquad = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}_0 \Delta u \Delta v \;,$$

where we have first taken the transpose of the determinant (which doesn't change its value), and then factored the Δu and Δv out of the two columns. Finally, taking the absolute value, we get from (13) and (17), and the definition (5) of Jacobian,

$$\Delta A \;\approx\; \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_0 \Delta u \Delta v \;;$$

passing to the limit as $\Delta u, \Delta v \to 0$ and dropping the subscript 0 (so that P becomes any point in the plane), we get the desired formula for the area element,

$$dA \;=\; \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \;.$$

3. Examples and comments; putting in limits.

If we write the change of variable formula as

$$(18) \qquad \iint_R f(x,y)\, dx\, dy \;=\; \iint_R g(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \;,$$

where

$$(19) \qquad \frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \;, \qquad g(u,v) = f(x(u,v), y(u,v)),$$

it looks as if the essential equations we need are the inverse equations:

$$(20) \qquad x = x(u,v), \qquad y = y(u,v)$$

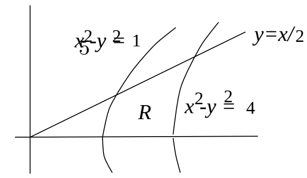
rather than the direct equations we are usually given:

$$(21) \qquad u = u(x,y), \qquad v = v(x,y) \;.$$

If it is awkward to get (20) by solving (21) simultaneously for x and y in terms of u and v, sometimes one can avoid having to do this by using the following relation (whose proof is an application of the chain rule, and left for the Exercises):

$$(22) \qquad \frac{\partial(x,y)}{\partial(u,v)} \, \frac{\partial(u,v)}{\partial(x,y)} \;=\; 1$$

The right-hand Jacobian is easy to calculate if you know $u(x,y)$ and $v(x,y)$; then the left-hand one — the one needed in (19) — will be its reciprocal. Unfortunately, it will be in terms of x and y instead of u and v, so (20) still ought to be needed, but sometimes one gets lucky. The next example illustrates.



Example 3. Evaluate $\displaystyle\iint_R \frac{y}{x}\,dx\,dy$, where R is the region pictured, having as boundaries the curves $x^2 - y^2 = 1, \quad x^2 - y^2 = 4, \quad y = 0, \quad y = x/2$.

Solution. Since the boundaries of the region are contour curves of $x^2 - y^2$ and y/x , and the integrand is y/x, this suggests making the change of variable

$$u = x^2 - y^2, \qquad v = \frac{y}{x} . \tag{23}$$

We will try to get through without solving these backwards for x, y in terms of u, v. Since changing the integrand to the u, v variables will give no trouble, the question is whether we can get the Jacobian in terms of u and v easily. It all works out, using (22):

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{vmatrix} = 2 - 2y^2/x^2 = 2 - 2v^2; \qquad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(1-v^2)} ,$$

according to (22). We use now (18), put in the limits, and evaluate; note that the answer is positive, as it should be, since the integrand is positive.

$$\begin{aligned} \iint_R \frac{y}{x}\,dx\,dy &= \iint_R \frac{v}{2(1-v^2)}\,du\,dv \\ &= \int_0^{1/2}\int_1^4 \frac{v}{2(1-v^2)}\,du\,dv \\ &= -\frac{3}{4}\ln(1-v^2)\Big]_0^{1/2} = -\frac{3}{4}\ln\frac{3}{4} . \end{aligned}$$

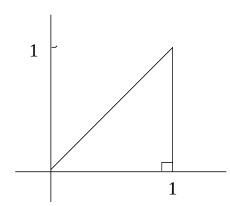
Putting in the limits

In the examples worked out so far, we had no trouble finding the limits of integration, since the region R was bounded by contour curves of u and v, which meant that the limits were constants.

If the region is not bounded by contour curves, maybe you should use a different change of variables, but if this isn't possible, you'll have to figure out the uv-equations of the boundary curves. The two examples below illustrate.

Example 4. Let $u = x + y, \ v = x - y$; change $\displaystyle\int_0^1\int_0^x dy\,dx$ to an iterated integral $du\,dv$.

Solution. Using (19) and (22), we calculate $\dfrac{\partial(x,y)}{\partial(u.v)} = -1/2$, so the Jacobian factor in the area element will be $1/2$.



To put in the new limits, we sketch the region of integration, as shown at the right. The diagonal boundary is the contour curve $v = 0$; the horizontal and vertical boundaries are not contour curves — what are their uv-equations? There are two ways to answer this; the first is more widely applicable, but requires a separate calculation for each boundary curve.

Method 1 *Eliminate x and y from the three simultaneous equations $u = u(x,y)$, $v = v(x,y)$, and the xy-equation of the boundary curve.* For the x-axis and $x = 1$, this gives

$$\begin{cases} u = x + y \\ v = x - y \\ y = 0 \end{cases} \Rightarrow \quad u = v; \qquad \begin{cases} u = x + y \\ v = x - y \\ x = 1 \end{cases} \Rightarrow \quad \begin{cases} u = 1 + y \\ v = 1 - y \end{cases} \Rightarrow \quad u + v = 2.$$

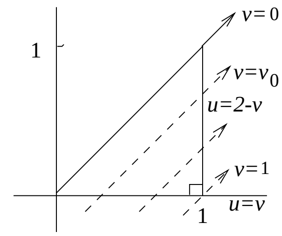
Method 2 *Solve for x and y in terms of u, v; then substitute $x = x(u,v)$, $y = y(u,v)$ into the xy-equation of the curve.*

Using this method, we get $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$; substituting into the xy-equations:

$$y = 0 \;\Rightarrow\; \frac{1}{2}(u-v) = 0 \;\Rightarrow\; u = v; \qquad\qquad x = 1 \;\Rightarrow\; \frac{1}{2}(u+v) = 1 \;\Rightarrow\; u + v = 2.$$

To supply the limits for the integration order $\iint du\,dv$, we

1. first hold v fixed, let u increase; this gives us the dashed lines shown;
2. integrate with respect to u from the u-value where a dashed line enters R (namely, $u = v$), to the u-value where it leaves (namely, $u = 2 - v$).
3. integrate with respect to v from the lowest v-values for which the dashed lines intersect the region R (namely, $v = 0$), to the highest such v-value (namely, $v = 1$).



Therefore the integral is $\displaystyle\int_0^1 \int_v^{2-v} \frac{1}{2}\,du\,dv$.

(As a check, evaluate it, and confirm that its value is the area of R. Then try setting up the iterated integral in the order $dv\,du$; you'll have to break it into two parts.)

Example 5. Using the change of coordinates $u = x^2 - y^2$, $v = y/x$ of Example 3, supply limits and integrand for $\displaystyle\iint_R \frac{dxdy}{x^2}$, where R is the infinite region in the first quadrant under $y = 1/x$ and to the right of $x^2 - y^2 = 1$.

Solution. We have to change the integrand, supply the Jacobian factor, and put in the right limits.

To change the integrand, we want to express x^2 in terms of u and v; this suggests eliminating y from the u, v equations; we get

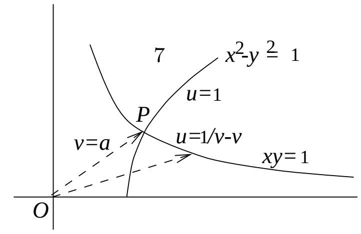
$$u = x^2 - y^2, \quad y = vx \quad\Rightarrow\quad u = x^2 - v^2x^2 \quad\Rightarrow\quad x^2 = \frac{u}{1-v^2}\ .$$

From Example 3, we know that the Jacobian factor is $\dfrac{1}{2(1-v^2)}$; since in the region R we have by inspection $0 \le v < 1$, the Jacobian factor is always positive and we don't need the absolute value sign. So by (18) our integral becomes

$$\iint_R \frac{dx\,dxy}{x^2} \;=\; \iint_R \frac{1-v^2}{2u(1-v^2)}\,du\,dv \;=\; \iint_R \frac{du\,dv}{2u}$$

Finally, we have to put in the limits. The x-axis and the left-hand boundary curve $x^2 - y^2 = 1$ are respectively the contour curves $v = 0$ and $u = 1$; our problem is the upper boundary curve $xy = 1$. To change this to $u - v$ coordinates, we follow Method 1:

$$\begin{cases} u = x^2 - y^2 \\ y = vx \\ xy = 1 \end{cases} \quad\Rightarrow \begin{cases} u = x^2 - 1/x^2 \\ v = 1/x^2 \end{cases} \quad\Rightarrow\; u = \frac{1}{v} - v\ .$$



The form of this upper limit suggests that we should integrate first with respect to u. Therefore we hold v fixed, and let u increase; this gives the dashed ray shown in the picture; we integrate from where it enters R at $u = 1$ to where it leaves, at $u = \dfrac{1}{v} - v$.

The rays we use are those intersecting R: they start from the lowest ray, corresponding to $v = 0$, and go to the ray $v = a$, where a is the slope of OP. Thus our integral is

$$\int_0^a \int_1^{1/v-v} \frac{du\,dv}{2u} \ .$$

To complete the work, we should determine a explicitly. This can be done by solving $xy = 1$ and $x^2 - y^2 = 1$ simultaneously to find the coordinates of P. A more elegant approach is to add $y = ax$ (representing the line OP) to the list of equations, and solve all three simultaneously for the slope a. We substitute $y = ax$ into the other two equations, and get

$$\begin{cases} ax^2 = 1 \\ x^2(1-a^2) = 1 \end{cases} \qquad \Rightarrow \qquad a = 1 - a^2 \qquad \Rightarrow \qquad a = \frac{-1+\sqrt{5}}{2}\ ,$$

by the quadratic formula.

4. Changing coordinates in triple integrals

Here the coordinate change will involve three functions

$$u = u(x, y, z), \quad v = v(x, y, z) \quad w = w(x, y, z)$$

but the general principles remain the same. The new coordinates u, v, and w give a three-dimensional grid, made up of the three families of contour surfaces of u, v, and w. Limits are put in by the kind of reasoning we used for double integrals. What we still need is the formula for the new volume element dV.

To get the volume of the little six-sided region ΔV of space bounded by three pairs of these contour surfaces, we note that nearby contour surfaces are approximately parallel, so that ΔV is approximately a parallelepiped, whose volume is (up to sign) the 3×3 determinant whose rows are the vectors forming the three edges of ΔV meeting at a corner. These vectors are calculated as in section 2; after passing to the limit we get

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\ , \tag{24}$$

where the key factor is the **Jacobian**

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}\ . \tag{25}$$

As an example, you can verify that this gives the correct volume element for the change from rectangular to spherical coordinates:

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi; \tag{26}$$

while this is a good exercise, it will make you realize why most people prefer to derive the volume element in spherical coordinates by geometric reasoning.

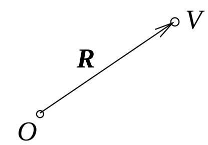
Exercises: Section 3D

G. Gravitational Attraction

We use triple integration to calculate the gravitational attraction that a solid body V of mass M exerts on a unit point mass placed at the origin.

If the solid V is also a point mass, then according to Newton's law of gravitation, the force it exerts is given by

$$\mathbf{F} \;=\; \frac{GM}{|\mathbf{R}|^2}\,\mathbf{r}, \tag{1}$$



where $\mathbf{R}$ is the position vector from the origin $\mathbf{0}$ to the point V, and the unit vector $\mathbf{r} = \mathbf{R}/|\mathbf{R}|$ is its direction.

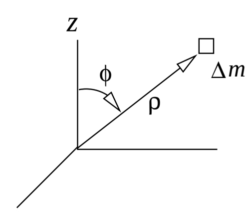
If however the solid body V is not a point mass, we have to use integration. We concentrate on finding just the $\mathbf{k}$ component of the gravitational attraction — all our examples will have the solid body V placed symmetrically so that its pull is all in the $\mathbf{k}$ direction anyway.

To calculate this force, we divide up the solid V into small pieces having volume ΔV and mass Δm. If the density function is $\delta(x, y, z)$, we have for the piece containing the point (x, y, z)

$$\Delta m \;\approx\; \delta(x,y,z)\Delta V, \tag{2}$$

Thinking of this small piece as being essentially a point mass at (x, y, z), the force $\Delta\mathbf{F}$ it exerts on the unit mass at the origin is given by (1), and its $\mathbf{k}$ component ΔF_z is therefore

$$\Delta F_z = \; G\,\frac{\Delta m}{|\mathbf{R}|^2}\,\mathbf{r}\cdot\mathbf{k}\;\;,$$



which in spherical coordinates becomes, using (2), and the picture,

$$\Delta F_z = \; G\,\frac{\cos\phi}{\rho^2}\,\delta\,\Delta V \;\;=\;\; G\,\frac{\delta\,\Delta V}{\rho^2}\;\cos\phi\;.$$

If we sum all the contributions to the force from each of the mass elements Δm and pass to the limit, we get for the $\mathbf{k}$-component of the gravitational force

$$F_z = G\iiint_V \frac{\cos\phi}{\rho^2}\,\delta\,dV\;. \tag{3}$$

If the integral is in spherical coordinates, then $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and the integral becomes

$$F_z = G\iiint_V \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta\;. \tag{4}$$

Example 1. Find the gravitational attraction of the upper half of a solid sphere of radius a centered at the origin, if its density is given by $\delta = \sqrt{x^2+y^2}$.

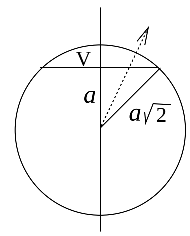
Solution. Since the solid and its density are symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction, and we can use (3) or (4). Since

$$\sqrt{x^2+y^2} = r = \rho \sin\phi \ ,$$

the integral is

$$F_z = G \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho \, \sin^2\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

which evaluates easily to $\pi G a^2/3$.



Example 2. Let V be the solid spherical cap obtained by slicing a solid sphere of radius $a\sqrt{2}$ by a plane at a distance a from the center of the sphere. Find the gravitational attraction of V on a unit point mass at the center of the sphere. (Take the density to be 1.)

Solution. To take advantage of the symmetry, place the origin at the center of the sphere, and align the axis of the cap along the z-axis (so the flat side of the cap is parallel to the xy-plane).

We use spherical coordinates; the main problem is determining the limits of integration. If we fix ϕ and θ and let ρ vary, we get a ray which enters V at its flat side

$$z = a, \quad \text{or} \quad \rho \cos\phi = a,$$

and leaves V on its spherical side, $\rho = a\sqrt{2}$. The rays which intersect V in this way are those for which $0 \le \phi \le \pi/4$, as one sees from the picture. Thus by (4),

$$F_z \ = \ G \int_0^{2\pi} \int_0^{\pi/4} \int_{a/\cos\phi}^{a\sqrt{2}} \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta,$$

which after integrating with respect to ρ (and θ) becomes

$$\begin{aligned} &= \ 2\pi G \int_0^{\pi/4} a\Big(\sqrt{2} - \frac{1}{\cos\phi}\Big) \sin\phi \cos\phi \, d\phi \\ &= \ 2\pi G a \Big(\frac{3\sqrt{2}}{4} - 1\Big) \ . \end{aligned}$$

Remark. Newton proved that a solid sphere of uniform density and mass M exerts the same force on an external point mass as would a point mass M placed at the center of the sphere. (See Problem 6a).

This does *not* however generalize to other uniform solids of mass M — it is not true that the gravitational force they exert is the same as that of a point mass M at their center of mass. For if this were so, a unit test mass placed on the axis between two equal point masses M and M' ought to be pulled toward the midposition, whereas actually it will be pulled toward the closer of the two masses.

Exercises: Section 5C

V. VECTOR INTEGRAL CALCULUS

V1. Plane Vector Fields

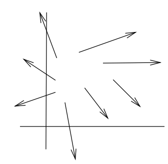
1. Vector fields in the plane; gradient fields.

We consider a function of the type

$$\mathbf{F}(x,y) \;=\; M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}\ . \tag{1}$$

where M and N are both functions of two variables. To each pair of values (x_0, y_0) for which both M and N are defined, such a function assigns a vector $\mathbf{F}(x_0, y_0)$ in the plane. $\mathbf{F}$ is therefore called a **vector function of two variables**. The set of points (x, y) for which $\mathbf{F}$ is defined is called the *domain* of $\mathbf{F}$.

To visualize the function $\mathbf{F}(x, y)$, at each point (x_0, y_0) in the domain we place the corresponding vector $\mathbf{F}(x_0, y_0)$ so that its tail is at (x_0, y_0). Thus each point of the domain is the tail end of a vector, and what we get is called a **vector field**. This vector field gives a picture of the vector function $\mathbf{F}(x, y)$.



Conversely, given a vector field in a region of the xy-plane, it determines a vector function of the type (1), by expressing each vector of the field in terms of its $\mathbf{i}$ and $\mathbf{j}$ components. Thus there is no real distinction between "vector function" and "vector field". Mindful of the applications to physics, in these notes we will mostly use "vector field". We will use the same symbol $\mathbf{F}$ to denote both the field and the function, saying "the vector field $\mathbf{F}$", rather than "the vector field corresponding to the vector function $\mathbf{F}$".

We say the vector field $\mathbf{F}$ is *continuous* in some region of the plane if both $M(x, y)$ and $N(x, y)$ are continuous functions in that region. The intuitive picture of a continuous vector field is that the vectors associated to points sufficiently near (x_0, y_0) should have direction and magnitude very close to that of $\mathbf{F}(x_0, y_0)$ — in other words, as you move around the field, the vectors should change direction and magnitude smoothly, without sudden jumps in size or direction.

In the same way, we say $\mathbf{F}$ is *differentiable* in some region if M and N are differentiable, that is, if all the partial derivatives

$$\frac{\partial M}{\partial x}\,, \qquad \frac{\partial M}{\partial y}\,, \qquad \frac{\partial N}{\partial x}\,, \qquad \frac{\partial N}{\partial y}$$

exist in the region. We say $\mathbf{F}$ is *continuously differentiable* in the region if all these partial derivatives are themselves continuous there. In general, all the commonly used vector fields are continuously differentiable, except perhaps at isolated points, or along certain curves. But as you will see, these points or curves affect the properties of the field in very important ways.

Where do vector fields arise in science and engineering?

One important way is as **gradient vector fields**. If

$$w = f(x,y) \tag{2}$$

is a differentiable function of two variables, then its *gradient*

$$\nabla w = \frac{\partial w}{\partial x}\,\mathbf{i} + \frac{\partial w}{\partial y}\,\mathbf{j} \tag{3}$$

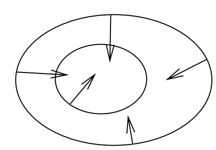
is a vector field, since both partial derivatives are functions of x and y. We recall the geometric interpretation of the gradient:

$$\begin{aligned} \text{dir } \nabla w &= \text{the direction } \mathbf{u} \text{ in which } \left.\frac{dw}{ds}\right|_{\mathbf{u}} \text{ is greatest;} \\ |\nabla w| &= \text{this greatest value of } \left.\frac{dw}{ds}\right|_{\mathbf{u}}, \end{aligned} \tag{4}$$

where $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \nabla w \cdot \mathbf{u}$ is the directional derivative of w in the direction $\mathbf{u}$.

Another important fact about the gradient is that if one draws the contour curves of $f(x,y)$, which by definition are the curves

$$f(x,y) = c, \qquad c \text{ constant},$$

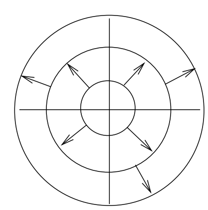


then at every point (x_0, y_0), the gradient vector ∇w at this point is perpendicular to the contour line passing through this point, i.e.,

(5) *the gradient field of f is perpendicular to the contour curves of f .*

Example 1. Let $w = \sqrt{x^2+y^2} = r$. Using the definition (3) of gradient, we find

$$\nabla w = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{r}.$$



The domain of ∇w is the xy-plane with $(0,0)$ deleted, and it is continuously differentiable in this region. Since $|x\,i + y\,j| = r$, we see that $|\nabla w| = 1$. Thus all the vectors of the vector field ∇w are unit vectors, and they point radially outward from the origin. This makes sense by (4), since the definition of w shows that dw/ds should be greatest in the radially outward direction, and have the value 1 in that direction.

Finally, the contour curves for w are circles centered at $(0,0)$, which are perpendicular to the vectors ∇w everywhere, as (5) predicts.

2. Force and velocity fields.

Continuing our search for ways in which vector fields arise, here are two physical situations which are described mathematically by vector fields. We shall refer to them often in the sequel, using our physical intuition to suggest the sort of mathematical properties that vector fields ought to have.

Force fields.

From physics, we have the two-dimensional electrostatic force fields arising from a distribution of static (i.e., not moving) charges in the plane. At each point (x_0, y_0) of the plane, we put a vector representing the force which would act on a unit positive charge placed at that point.

In the same way, we get vector fields arising from a distribution of masses in the xy-plane, representing the gravitational force acting at each point on a unit mass. There are also the electromagnetic fields arising from moving electric charges and/or a distribution of magnets, representing the magnetic force at each point.

Any of these we shall simply refer to as a **force field**.

Example 2. Express in $\mathbf{i} - \mathbf{j}$ form the electrostatic force field $\mathbf{F}$ in the xy-plane arising from a unit positive charge placed at the origin, given that the force vector at (x, y) is directed radially away from the origin and that it has magnitude c/r^2, c constant.

Solution. Since the vector $x\,\mathbf{i} + y\,\mathbf{j}$ with tail at (x, y) is directed radially outward and has magnitude r, it has the right direction, and we need only change its magnitude to c/r^2. We do this by multiplying it by c/r^3, which gives

$$\mathbf{F} \;=\; \frac{cx}{r^3}\,\mathbf{i} + \frac{cy}{r^3}\,\mathbf{j} \;=\; c\,\frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2+y^2)^{3/2}}\;.$$

Flow fields and velocity fields

A second way vector fields arise is as the steady-state *flow fields* and *velocity fields.*

Imagine a fluid flowing in a horizontal shallow tank of uniform depth, and assume that the flow pattern at any point is purely horizontal and not changing with time. We will call this a *two-dimensional steady-state flow* or for short, simply a *flow.* The fluid can either be compressible (like a gas), or incompressible (like water). We also allow for the possibility that at various points, fluid is beiong added to or subtracted from the flow; for instance, someone could be standing over the tank pouring in water at a certain point, or over a certain area. We also allow the density to vary from point to point, as it would for an unevenly heated gas.

With such a flow we can associate two vector fields.

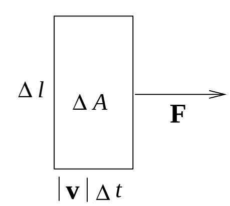
There is the **velocity field** $\mathbf{v}(x, y)$ where the vector $\mathbf{v}(x, y)$ at the point (x, y) represents the velocity vector of the flow at that point — that is, its direction gives the direction of flow, and its magnitude gives the speed of the flow.

Then there is the **flow field**, defined by

$$\mathbf{F} \;=\; \delta(x, y)\,\mathbf{v}(x, y) \tag{6}$$

where $\delta(x, y)$ gives the density of the fluid at the point (x, y), in terms of mass per unit area. Assuming it is not 0 at a point (x, y), we can interpret $\mathbf{F}(x, y)$ as follows:

$$\begin{aligned} \text{dir}\,\mathbf{F} \;&=\; \text{direction of fluid flow at } (x,y); \\ |\mathbf{F}| \;&=\; \begin{cases} \text{rate (per unit length per second) of mass transport} \\ \text{across a line perpendicular to the flow direction at } (x,y). \end{cases} \end{aligned} \tag{7}$$



Namely, we see that first by (6) and then by the picture,

$$|\mathbf{F}|\,\Delta l\,\Delta t \;=\; \delta\,|\mathbf{v}|\,\Delta t\,\Delta l \;=\; \text{mass in } \Delta A,$$

from which (7) follows by dividing by $\Delta l\,\Delta t$ and letting Δl and $\Delta t \to 0$.

If the density is a constant δ_0, as it would be for an incompressible fluid at a uniform temperature, then the flow field and velocity field are essentially the same, by (6) — the vectors of one are just a constant scalar multiple of the vectors of the other.

Example 3. Describe and interpret $\mathbf{F} \;=\; \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$ as a flow field and a force field.

Solution. As in Example 2, the field $\mathbf{F}$ is defined everywhere except $(0,0)$ and its direction is radially outward; now, however, its magnitude is r/r^2, i.e., $|\mathbf{F}| = 1/r$.

$\mathbf{F}$ is the *flow field* for a source of magnitude 2π at the origin. To see this, look at a circle of radius a centered at the origin. At each point P on this circle, the flow is radially outward and by (7),

$$\begin{aligned} \text{mass transport rate at P} \;&=\; \frac{1}{a}, \qquad \text{so that} \\ \text{mass transport rate across circle} \;&=\; \frac{1}{a}\cdot 2\pi a \;=\; 2\pi\;. \end{aligned}$$

This shows that in one second, 2π mass flows out through every circle centered at the origin. This is the flow field for a source of magnitude 2π at the origin — for example, one could imagine a narrow pipe placed over the tank, introducting 2π mass units per second at the point $(0,0)$.

We know that $|\mathbf{F}| \;=\; \delta\,|\mathbf{v}| \;=\; 1/r$. Two important cases are:

• if the fluid is incompressible, like water, then its density is constant, so the flow speed must decrease like $1/r$ — the flow outward gets slower the further you are from the origin;

• if it is compressible like a gas, and its flow speed stays constant, then the density must decrease like $1/r$.

We now interpret the same field as a *force field*.

Suppose we think of the z-axis in space as a long straight wire, bearing a uniform positive electrostatic charge. This gives us a vector field in space, representing the electrostatic force field.

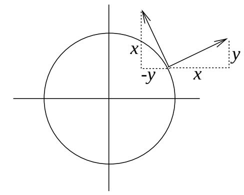
Since one part of the wire looks just like any other part, radial symmetry shows first that the vectors in the force field have 0 as their $\mathbf{k}$-component, i.e., they are pointed radially outward from the wire, and second that their magnitude depends only on their distance r from the wire. It can be shown in fact that the resulting force field is $\mathbf{F}$, up to a constant factor.

Such a field is called "two-dimensional", even though it is a vector field in space, because z and $\mathbf{k}$ don't enter into its description — once you know how it looks in the xy-plane, you know how it looks all through space.

The important thing to notice is that the magnitude of the force field in the xy-plane decreases like $1/r$, *not* like $1/r^2$, as it would if the charge were all at a point.

In the same way, the gravitational field of a uniform mass distribution along the z-axis would be $-\mathbf{F}$, up to a constant factor, and would be called a "two-dimensional gravitational field". Naturally, we don't have infinite long straight wires, but if you have a long straight wire, and stay away from its ends, or have only a short straight wire, but stay close to it, the force field will look like **F** near the wire.

Example 4. Find the velocity field of a fluid with density 1 in a shallow tank, rotating with constant angular velocity ω counterclockwise around the origin.



Solution. First we find the field direction at each point (x, y).

We know the vector $x\,\mathbf{i} + y\,\mathbf{j}$ is directed radially outward. Therefore a vector perpendicular to it in the counterclockwise direction (see picture) will be $-y\,\mathbf{i} + x\,\mathbf{j}$ (since its scalar product with $x\,\mathbf{i} + y\,\mathbf{j}$ is 0 and the signs are correct).

The preceding vector has magnitude r. If the angular velocity is ω, then the linear velocity is given by

$$|\mathbf{v}| = \omega r,$$

so to get the velocity field, we should multiply the above field by ω :

$$\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j} \ .$$

Exercises: Section 4A

V2. Gradient Fields and Exact Differentials

1. Criterion for gradient fields.

Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ be a two-dimensional vector field, where M and N are continuous functions. There are three equivalent ways of saying that $\mathbf{F}$ is conservative, i.e., a gradient field:

$$(1) \qquad \mathbf{F} = \nabla f \;\Leftrightarrow\; \int_P^Q \mathbf{F}\cdot d\mathbf{r} \;\text{ is path-independent }\; \Leftrightarrow\; \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \;\text{ for any closed } C$$

Unfortunately, these equivalent formulations don't give us any effective way of deciding if a given field $\mathbf{F}$ is a conservative field or not. However, if we assume that $\mathbf{F}$ is not just continuous but is even continuously differentiable (meaning: M_x, M_y, N_x, N_y all exist and are continuous), then there is a simple and elegant criterion for deciding whether or not $\mathbf{F}$ is a gradient field in some region.

Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a region D. Then, in D,

$$(2) \qquad \mathbf{F} = \nabla f \;\text{ for some } f(x,y) \quad\Rightarrow\quad M_y = N_x\ .$$

Proof. Since $\mathbf{F} = \nabla f$, this means

$$\begin{array}{lcll} M = f_x & \text{and} & N = f_y\ . & \quad\text{Therefore,} \\ M_y = f_{xy} & \text{and} & N_x = f_{yx}\ . & \end{array}$$

But since these two mixed partial derivatives are continuous (since M_y and N_x are, by hypothesis), a standard 18.02 theorem says they are equal. Thus $M_y = N_x$. $\square$

This theorem may be expressed in a slightly different form, if we define the scalar function called the **two-dimensional curl** of $\mathbf{F}$ by

$$(3) \qquad \operatorname{curl} \mathbf{F} \;=\; N_x - M_y\ .$$

Then (2) becomes

$$(2') \qquad \mathbf{F} = \nabla f \quad\Rightarrow\quad \operatorname{curl} \mathbf{F} = 0\ .$$

This criterion allows us to test $\mathbf{F}$ to see if it is a gradient field. Naturally, we would also like to know that the converse is true: if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is a gradient field. As we shall see, however, this requires some additional hypotheses on the region D. For now, we will assume D is the whole plane. Then we have

Converse to Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable for all x, y.

$$(4) \qquad M_y = N_x \text{ for all } x,y \quad\Rightarrow\quad \mathbf{F} = \nabla f \;\text{ for some differentiable } f \text{ and all } x,y.$$

The proof of (4) will be postponed until we have more technique. For now we will illustrate the use of the criterion and its converse.

Example 1. For which value(s), if any of the constants a, b will $axy\,\mathbf{i} + (x^2+by)\,\mathbf{j}$ be a gradient field?

Solution. The partial derivatives are continuous for all x, y and $M_y = ax,\ N_x = 2x$. Thus by (2) and (4), $\mathbf{F} = \nabla f \ \Leftrightarrow\ a = 2;\ b$ is arbitrary.

Example 2. Are the fields $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$, $\mathbf{G} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$ conservative?

Solution. We have (the second line follows from the first by interchanging x and y):

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{-x^2+y^2}{(x^2+y^2)^2}; \qquad \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2};$$
$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}; \qquad \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{2yx}{(x^2+y^2)^2};$$

from this, we see immediately that

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right); \qquad \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right);$$

the two equations in the last line show respectively that $\mathbf{F}$ and $\mathbf{G}$ satisfy the criterion (2). However, neither field is defined at $(0,0)$, so that the converse (4) is not applicable. So the question cannot be decided just on the basis of (2) and (4). In fact, it turns out that $\mathbf{F}$ is a gradient field, since one can check that

$$\mathbf{F} = \nabla \ln \sqrt{x^2+y^2} = \nabla \ln r\ .$$

On the other hand, $\mathbf{G}$ is not conservative, since if C is the unit circle $x = \cos t,\ y = \sin t$, we have

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{\sin^2 t\,dt + \cos^2 t\,dt}{\sin^2 t + \cos^2 t} = 2\pi \neq 0.$$

We will return later on in these notes to this example.

2. Finding the potential function.

Example 2 above raises the question of how we found the function $\frac{1}{2}\ln(x^2+y^2)$. More generally, if we know that $\mathbf{F} = \nabla f$ — for example if curl $\mathbf{F} = 0$ in the whole xy-plane — how do we find the function $f(x,y)$? There are two methods; some students prefer one, some the other.

Method 1. Suppose $\mathbf{F} = \nabla f$. By the Fundamental Theorem for Line Integrals,

$$\text{(5)} \qquad \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} = f(x,y) - f(x_0,y_0)\ .$$

Read from left to right, (5) gives us an easy way of finding the line integral in terms of $f(x,y)$. But read right to left, it gives us a way of finding $f(x,y)$ by using the line integral:

$$\text{(5}'\text{)} \qquad f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} + c\ .$$

(Here c is an arbitrary constant of integration; as (5′) shows, $c = f(x_0, y_0)$.)

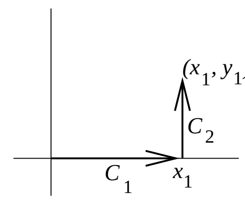
Remark. It is common to refer to $f(x, y)$ as the (mathematical) **potential function**. The potential function used in physics is $-f(x, y)$. The negative sign is used by physicists so that the potential difference will represent work done *against* the field $\mathbf{F}$, rather than work done *by* the field, as the convention is in mathematics.

Example 3. Let $\mathbf{F} = (x + y^2)\,\mathbf{i} + (2xy + 3y^2)\,\mathbf{j}$. Verify that $\mathbf{F}$ satisfies the Criterion (2), and use method 1 above to find the potential function $f(x, y)$.

Solution. We verify (2) immediately: $\dfrac{\partial(y^2)}{\partial y} = 2y = \dfrac{\partial(2xy)}{\partial x}$.

We use (5′). The point (x_0, y_0) can be any convenient starting point; $(0, 0)$ is the usual choice, if the integrand is defined there. (We will subscript the variables to avoid confusion with the variables of integration, but you don't have to.) By (5′),

$$f(x_1, y_1) = \int_{(0,0)}^{(x_1,y_1)} (x + y^2)\,dx + (2xy + 3y^2)\,dy \; . \tag{6}$$



Since the integral is path-independent, we can choose any path we like. The usual choice is the one on the right, as it simplifies the computations. (Most of what follows you can do mentally, with a little practice.)

On C_1, we have $y = 0,\; dy = 0$, so the integral on C_1 becomes $\displaystyle\int_0^{x_1} x\,dx = \frac{1}{2}\,x_1^2$.

On C_2, we have $x = x_1,\; dx = 0$, so the integral is $\displaystyle\int_0^{y_1} (2x_1 y + 3y^2)\,dy = x_1 y_1^2 + y_1^3$.

Adding the integrals on C_1 and C_2 to get the integral along the entire path, and dropping the subscripts, we get by (6) and (5′)

$$f(x, y) = \frac{1}{2}x^2 + xy^2 + y^3 + c \; .$$

(The constant of integration is added by (5′), since the choice of starting point was arbitrary. You should always confirm the answer by checking that $\nabla f = \mathbf{F}$.) □

Method 2. Once again suppose $\mathbf{F} = \nabla f$, that is $M\,\mathbf{i} + N\,\mathbf{j} = f_x\,\mathbf{i} + f_y\,\mathbf{j}$. It follows that

$$f_x = M \qquad \text{and} \qquad f_y = N \; . \tag{7}$$

These are two equations involving partial derivatives, which we can solve simultaneously by integration. We illustrate using the previous example: $\mathbf{F} = (x + y^2,\; 2xy + 3y^2)$.

Solution by Method 2. Using the first equation in (7),

$$\frac{\partial f}{\partial x} = x + y^2. \qquad \text{Hold } y \text{ fixed, integrate with respect to } x\text{:}$$

$$f = \frac{1}{2}\,x^2 + y^2 x + g(y). \qquad \text{where } g(y) \text{ is an arbitrary function of } y. \tag{8}$$

To find $g(y)$, we calculate $\dfrac{\partial f}{\partial y}$ two ways:

$$\begin{aligned} \frac{\partial f}{\partial y} &= 2yx + g'(y) && \text{by (8), while} \\ \frac{\partial f}{\partial y} &= 2xy + 3y^2 && \text{from (7), second equation.} \end{aligned}$$

Comparing these two expressions, we see that $g'(y) = 3y^2$, so $g(y) = y^3 + c$. Putting it all together, using (8), we get $f(x,y) = \frac{1}{2}x^2 + y^2x + y^3 + c$, as before. $\square$

In the first method, the answer is written down immediately as a line integral; the rest of the work is in evaluating the integral, which goes quickly, since on a horizontal or vertical path either $dx = 0$ or $dy = 0$.

In the second method, the answer is obtained by an algorithm involving several steps which should be carried out in the right order.

The first method has the advantage of reminding you each time how $f(x,y)$ is defined and what it means, facts of theoretical and practical importance. The second method has the advantage of requiring no knowledge of line integrals, which makes it popular with students; on the other hand, when done in three dimensions, the bookkeeping gets more complicated, whereas in the first method it does not; overall, the first method is faster, provided you are confident enough to do some of the work mentally.

3. Exact differentials.

The formal expressions $M(x,y)\,dx + N(x,y)\,dy$ which have appeared as the integrands in our line integrals are called **differentials**. In some applications, most notably thermodynamics, one usually works directly with the differential $M\,dx + N\,dy$ and its line integral $\int M\,dx + N\,dy$, without considering or using the associated vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. Therefore it is important to have the results about gradient fields in this section translated into the language of differentials. We do this now.

If $f(x,y)$ is a differentiable function, its **total differential** df (or simply *differential*) is by definition the expression

$$df = f_x dx + f_y dy\ . \tag{9}$$

For example, if $f(x,y) = x^2y^3$, then $d(x^2y^3) = 2xy^3dx + 3x^2y^2dy$.

We call the differential $M\,dx + N\,dy$ **exact**, in a region D where M and N are defined, if it is the total differential of some function $f(x,y)$ in this region, i.e., if in D,

$$M = f_x \qquad \text{and} \qquad N = f_y, \qquad \text{for some}\ \ f(x,y). \tag{10}$$

From this we see that the relation between differentials and vector fields is

$$\begin{aligned} M\,dx + N\,dy \quad \text{is exact} \quad &\Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} \quad \text{is a gradient field} \\ M\,dx + N\,dy \;=\; df \quad &\Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} \;=\; \nabla f\ . \end{aligned}$$

In this language, the criterion (2) and its partial converse (4) become the

Exactness Criterion. *Assume M and N are continuously differentiable in a region D of the plane. Then in this region,*

$$M\,dx + N\,dy \quad \textit{exact} \quad \Rightarrow \quad M_y = N_x \;; \tag{11}$$

$$\textit{if } D \textit{ is the whole } xy\textit{-plane}, \quad M_y = N_x \quad \Rightarrow \quad M\,dx + N\,dy \quad \textit{exact.} \tag{12}$$

If the exactness criterion shows that $M\,dx + N\,dy$ is exact, then the function $f(x, y)$ may be found by either of the two methods previously described.

Exercises: Section 4C.

V3. Two-dimensional Flux

In this section and the next we give a different way of looking at Green's theorem which both shows its significance for flow fields and allows us to give an intuitive physical meaning for this rather mysterious equality between integrals.

We have seen that if $\mathbf{F}$ is a force field and C a directed curve, then

$$\text{(1)} \qquad \text{work done by } \mathbf{F} \text{ along } C \;=\; \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

In words, we are integrating $\mathbf{F}\cdot\mathbf{T}$, the *tangential component* of $\mathbf{F}$, along the curve C. In component notation, if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then the above reads

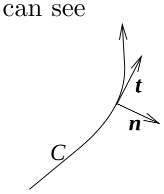
$$\text{(2)} \qquad \text{work} \;=\; \int_C M\,dx + N\,dy \;=\; \int_{t_0}^{t_1}\left(M\frac{dx}{dt} + N\frac{dy}{dt}\right)dt\,.$$

Analogously now, we may integrate $\mathbf{F}\cdot\mathbf{n}$, the *normal component* of $\mathbf{F}$ along C. To describe this, suppose the curve C is parametrized by the arclength s, increasing in the positive direction on C. The position vector for this parametrization and its corresponding tangent vector are given respectively by

$$\mathbf{r}(s) \;=\; x(s)\,\mathbf{i} + y(s)\,\mathbf{j}\,, \qquad\qquad \mathbf{t}(s) \;=\; \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j}\,;$$

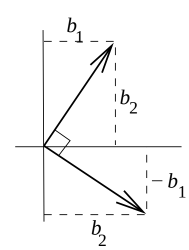
where we have used $\mathbf{t}$ instead of $\mathbf{T}$ since it is a unit vector— its length is 1, as one can see by dividing through by ds on both sides of

$$ds = \sqrt{(dx)^2 + (dy)^2}\,.$$



The unit normal vector $\mathbf{n}$ is the one shown in the picture, obtained by rotating $\mathbf{t}$ clockwise through a right angle.

> Unfortunately, this direction is opposite to the one customarily used in kinematics, where $\mathbf{t}$ and $\mathbf{n}$ form a right-handed coordinate system for motion along C. The choice of $\mathbf{n}$ depends therefore on the context of the problem; the choice we have given is the most natural for applying Green's theorem to flow problems.



The usual formula for rotating a vector clockwise by 90^o (see the figure) shows that

$$\text{(3)} \qquad \mathbf{n}(s) \;=\; \frac{dy}{ds}\,\mathbf{i} - \frac{dx}{ds}\,\mathbf{j}\,.$$

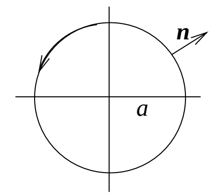
The line integral over C of the normal component $\mathbf{F}\cdot\mathbf{n}$ of the vector field $\mathbf{F}$ is called the **flux of F across** C. In symbols,

$$\text{(4)} \qquad \text{flux of } \mathbf{F} \text{ across } C \;=\; \int_C \mathbf{F}\cdot\mathbf{n}\,ds \;=\; \int_C\left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$$

In the notation of differentials, using (3) we write $\mathbf{n}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}$, so that

$$\text{(5)} \qquad \text{flux of } F \text{ across } C \;=\; \int_C M\,dy - N\,dx \;=\; \int_C\left(M\frac{dy}{dt} - N\frac{dx}{dt}\right)dt\,,$$

where $x(t)$, $y(t)$ is any parametrization of C. We will need both (4) and (5).



Example 1. Calculate the flux of the field $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$ across a circle of radius a and center at the origin, by a) using (4); b) using (5).

Solutions. a) The field is directed radially outward, so that $\mathbf{F}$ and $\mathbf{n}$ have the same direction. (As usual, the circle is directed counterclockwise, which means that $\mathbf{n}$ points outward.) Therefore, at each point of the circle,

$$\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{a}\ .$$

Therefore, by (4), we get

$$\text{flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C \frac{1}{a}\,ds = 2\pi\ .$$

b) We can also get the same result by straightforward computation using a parametrization of the circle: $x = \cos t$, $y = \sin t$. Using this and (5) above,

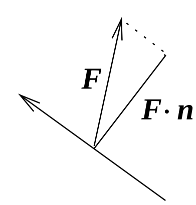
$$\text{flux} = \oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2}\,dt = 2\pi\ .$$

The natural physical interpretation for flux calls for thinking of $\mathbf{F}$ as representing a two-dimensional flow field (see section V1). Then the line integral represents the *rate with respect to time at which mass is being transported across* C. (We think of the flow as taking place in a shallow tank of unit depth. The convention about $\mathbf{n}$ makes this mass-transport rate positive if the flow is from left to right as you face in the positive direction on C, and negative in the other case.)

To see this, we follow the same procedure that was used to interpret the tangential integral in a force field as work.

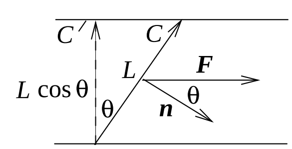
The essential step to see is that if $\mathbf{F}$ is a constant vector field representing a flow, and C is a directed line segment of length L, then

(6) $$\text{mass-transport rate across } C = (\mathbf{F}\cdot\mathbf{n})\,L$$



To see this, resolve the flow field into its components parallel to C and perpendicular to C. The component parallel to C contributes nothing to the flow rate *across* C, while the component perpendicular to C is $\mathbf{F}\cdot\mathbf{n}$.

Another way to see (6) is illustrated at the right. Letting C' be as shown, we see by conservation of mass that



$$\begin{aligned}\text{mass-transport rate across } C &= \text{mass-transport rate across } C' \\ &= |\mathbf{F}|(L\cos\theta) \\ &= (\mathbf{F}\cdot\mathbf{n})\,L\ .\end{aligned}$$

Once we have this, we follow the same procedure used to define work as a line integral. We divide up the curve and apply (6) to each of the approximating line segments, the k-th segment being of length approximately Δs_k. Thus

$$\text{mass-transport rate across } k\text{-th line segment} \;\approx\; (\mathbf{F}_k \cdot \mathbf{n}_k)\Delta s_k \; .$$

Adding these up and passing to the limit as the subdivision of the curve gets finer and finer then gives

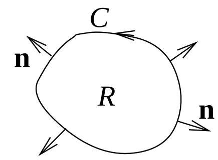
$$\text{mass-transport rate across } C \;=\; \int_C \mathbf{F} \cdot \mathbf{n}\, ds \; .$$

This interpretation shows why we call the line integral the *flux* of $\mathbf{F}$ across C. This terminology however is used even when $\mathbf{F}$ no longer represents a two-dimensional flow field. We speak of the flux of an electromagnetic field, for example.

Referring back to Example 1, the field $\mathbf{F} \;=\; \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2+y^2}$ discussed there represents a flow stemming from a single source of strength 2π at the origin; thus the flux across each circle centered at the origin should also be 2π, regardless of the radius of the circle. This is what we found by actual calculation.

Exercises: Section 4E

V4. Green's Theorem in Normal Form



1. Green's theorem for flux.

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ represent a two-dimensional flow field, and C a simple closed curve, positively oriented, with interior R.

According to the previous section,

$$\text{flux of } F \text{ across } C \;=\; \oint_C M\,dy - N\,dx\;. \tag{1}$$

Notice that since the normal vector points outwards, away from R, the flux is positive where the flow is out of R; flow into R counts as negative flux.

We now apply Green's theorem to the line integral in (1); first we write the integral in standard form (dx first, then dy):

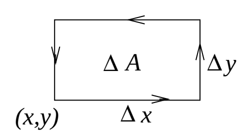
$$\oint_C M\,dy - N\,dx \;=\; \oint_C -N\,dx + M\,dy \;=\; \iint_R \big(M_x - (-N)_y\big)\,dA\;.$$

This gives us **Green's theorem in the normal form**

$$\oint_C M\,dy - N\,dx \;=\; \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA\;. \tag{2}$$

Mathematically this is the same theorem as the tangential form of Green's theorem — all we have done is to juggle the symbols M and N around, changing the sign of one of them. What is different is the physical interpretation. The left side represents the flux of $\mathbf{F}$ across the closed curve C. What does the right side represent?

2. The two-dimensional divergence.



Once again, let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. We give a name to and a notation for the integrand of the double integral on the right of (2):

$$\operatorname{div}\mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}, \qquad \text{the } \textbf{divergence} \text{ of } \mathbf{F}\;. \tag{3}$$

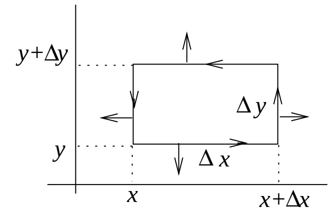
Evidently div $\mathbf{F}$ is a scalar function of two variables. To get at its physical meaning, look at the small rectangle pictured. If $\mathbf{F}$ is continuously differentiable, then div $\mathbf{F}$ is a continuous function, which is therefore approximately constant if the rectangle is small enough. We apply (2) to the rectangle; the double integral is approximated by a product, since the integrand is approximately constant:

$$\text{flux across sides of rectangle} \;\approx\; \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta A\;, \qquad \Delta A \;=\; \text{area of rectangle.} \tag{4}$$

Because of the importance of this approximate relation, we give a more direct derivation of it which doesn't use Green's theorem. The reasoning which follows is widely used in mathematical modeling of physical problems.

Consider the small rectangle shown; we calculate approximately the flux over each side.

$$\begin{aligned} \text{flux across top} &\approx (\mathbf{F}(x, y+\Delta y)\cdot \mathbf{j}\,)\,\Delta x &&= N(x, y+\Delta y)\,\Delta x \\ \text{flux across bottom} &\approx (\mathbf{F}(x,y)\cdot -\mathbf{j}\,)\Delta x &&= -N(x,y)\Delta x\ ; \end{aligned}$$



adding these up,

$$\begin{array}{l}\text{total flux across}\\ \text{top and bottom}\end{array} \quad\approx\quad \big(N(x, y+\Delta y) - N(x,y)\big)\Delta x \quad\approx\quad \left(\frac{\partial N}{\partial y}\,\Delta y\right)\Delta x.$$

By similar reasoning applied to the two sides,

$$\begin{array}{l}\text{total flux across}\\ \text{left and right sides}\end{array} \quad\approx\quad \Big(M(x+\Delta x, y) - M(x,y)\Big)\,\Delta y \quad\approx\quad \left(\frac{\partial M}{\partial x}\,\Delta x\right)\Delta y\ .$$

Adding up the flux over the four sides, we get (4) again:

$$\begin{array}{l}\text{total flux over four}\\ \text{sides of the rectangle}\end{array} \quad\approx\quad \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y\ .$$

Continuing our search for a physical meaning for the divergence, if the total flux over the sides of the small rectangle is positive, this means there is a net flow *out* of the rectangle. According to conservation of matter, the only way this can happen is if there is a *source* adding fluid directly to the rectangle. If the flow is taking place in a shallow tank of uniform depth, such a source can be visualized as someone standing over the tank, pouring fluid directly into the rectangle. Similarly, a net flow *into* the rectangle implies there is a *sink* withdrawing fluid from the rectangle. It is best to think of such a sink as a "negative source". The net rate (positive or negative) at which fluid is added directly to the rectangle from above may be called the "source rate" for the rectangle. Thus, since matter is conserved,

$$\text{flux over sides of rectangle} \quad=\quad \text{source rate for the rectangle;}$$

combining this with (4) shows that

$$\text{source rate for the rectangle} \quad\approx\quad \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta A. \tag{5}$$

We now divide by ΔA and pass to the limit, getting by definition

$$\text{the source rate at } (x,y) \quad=\quad \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \;=\; \operatorname{div}\mathbf{F}\ . \tag{6}$$

The definition of the double integral as the limit of a sum shows in the usual way now that

$$\text{source rate for } R \quad=\quad \iint_R \operatorname{div}\mathbf{F}\,dA\ . \tag{7}$$

These two relations (6) and (7) interpret the divergence physically, for a flow field, and they interpret also Green's theorem in the normal form:

$$\begin{aligned} \text{total flux across } C &\quad=\quad \text{source rate for } R \\ \oint_C M\,dy - N\,dx &\quad=\quad \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \end{aligned}$$

Since Green's theorem is a mathematical theorem, one might think we have "proved" the law of conservation of matter. This is not so, since this law was needed for our interpretation of div **F** as the source rate at (x, y).

We give side-by-side the two forms of Green's theorem, first in the vector form, then in the differential form used when calculations are to be done.

Tangential form	**Normal form**
$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{curl } \mathbf{F}\, dA$	$\oint_C \mathbf{F}\cdot \mathbf{n} ds = \iint_R \text{div } \mathbf{F}\, dA$
$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$	$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$
work by **F** around C	flux of **F** across C — source rate for R

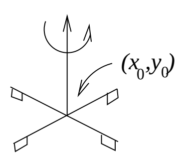
3. An interpretation for curl F.

We will start by looking at the two dimensional curl in the xy-plane. Our interpretation will be that the curl at a point represents twice the angular velocity of a small paddle wheel at that point. At the very end we will indicate how to extend this interpretation to 3 dimensions.

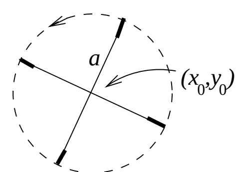
The function curl **F** can be thought of as measuring the tendency of **F** to produce rotation. Interpreting **F** either as a force field or a velocity field, **F** will make a suitable test object placed at a point P_0 spin about a vertical axis (i.e., one in the **k**-direction), and the angular velocity of the spin will be proportional to $(\text{curl } \mathbf{F})_0$.

To see this for the velocity field **v** of a flowing liquid, place a paddle wheel of radius a so its center is at (x_0, y_0), and its axis is vertical. We ask how rapidly the flow spins the wheel.

If the wheel had only one blade, the velocity of the blade would be $\mathbf{F}\cdot\mathbf{t}$, the component of the flow velocity vector **F** perpendicular to the blade, i.e., tangent to the circle of radius a traced out by the blade.



paddlewheel



top view

Since $\mathbf{F}\cdot\mathbf{t}$ is not constant along this circle, if the wheel had only one blade it would spin around at an uneven rate. But if the wheel has many blades, this unevenness will be averaged out, and it will spin around at approximately the *average value* of the tangential velocity $\mathbf{F}\cdot\mathbf{t}$ over the circle. Like the average value of any function defined along a curve, this average tangential velocity can be found by integrating $\mathbf{F}\cdot\mathbf{t}$ over the circle, and dividing by the length of the circle. Thus,

$$\begin{aligned}
\text{speed of blade} &= \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot\mathbf{t}\,ds \quad = \frac{1}{2\pi a}\oint_C \mathbf{F}\cdot d\mathbf{r} \\
&= \frac{1}{2\pi a}\iint_R (\text{curl } \mathbf{F})_0\, dx\,dy, \qquad \text{by Green's theorem,} \\
&\approx \frac{1}{2\pi a}(\text{curl } \mathbf{F})_0 \pi a^2 \ , \qquad (8)
\end{aligned}$$

where $(\text{curl } \mathbf{F})_0$ is the value of the function curl $\mathbf{F}$ at (x_0, y_0). The justification for the last approximation is that if the circle formed by the paddlewheel is small, then curl $\mathbf{F}$ has approximately the value $(\text{curl } \mathbf{F})_0$ over the interior R of the circle, so that multiplying this constant value by the area πa^2 of R should give approximately the value of the double integral.

From (8) we get for the tangential speed of the paddlewheel:

$$\text{tangential speed} \approx \frac{a}{2}(\text{curl } \mathbf{F})_0 \ . \tag{9}$$

We can get rid of the a by using the angular velocity ω_0 of the paddlewheel; since the tangential speed is $a\omega_0$, (9) becomes

$$\omega_0 \approx \frac{1}{2}(\text{curl } \mathbf{F})_0 \ . \tag{10}$$

As the radius of the paddlewheel gets smaller, the approximation becomes more exact, and passing to the limit as $a \to 0$, we conclude that, for a two-dimensional velocity field $\mathbf{F}$,

$$\boxed{\text{curl } \mathbf{F} = \textit{twice the angular velocity of an infinitesimal paddlewheel at } (x, y) \ .} \tag{11}$$

The curl thus measures the "vorticity" of the fluid flow — its tendency to produce rotation.

A consideration of curl $\mathbf{F}$ for a force field would be similar, interpreting $\mathbf{F}$ as exerting a torque on a spinnable object — a little dumbbell with two unit masses for a gravitational field, or with two unit positive charges for an electrostatic force field.

Example 1. Calculate and interpret curl $\mathbf{F}$ for (a) $x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\omega(-y\,\mathbf{i} + x\,\mathbf{j})$

Solution. (a) curl $\mathbf{F} = 0$; this makes sense since the field is radially outward and radially symmetric, there is no favored angular direction in which the paddlewheel could spin.

(b) curl $\mathbf{F} = 2\omega$ at every point. Since this field represents a fluid rotating about the origin with constant angular velocity ω (see section V1), it is at least clear that curl $\mathbf{F}$ should be 2ω at the origin; it's not so clear that it should have this same value everywhere, but it is true.

Extension to Three Dimensions. To extend this interpretation to three dimensions note that any component of the flow of $\mathbf{F}$ in the $\mathbf{k}$ direction will not have any effect on a paddle wheel in the xy-plane. In fact, for any plane with normal $\mathbf{n}$ the component of $\mathbf{F}$ in the direction of $\mathbf{n}$ has no effect on a paddle wheel in the plane. This leads to the following interpretation of the three dimensional curl:

For any plane with unit normal $\mathbf{n}$, $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ is two times the angular velocity of a small paddle wheel in the plane.

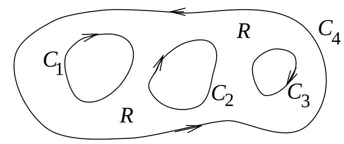
We could force through a proof along the lines of the 2D proof above. Once we learn Stokes Theorem we can make a much simpler argument.

Exercises: Section 4F

V5. Simply-Connected Regions

1. The Extended Green's Theorem.

In the work on Green's theorem so far, it has been assumed that the region R has as its boundary a single simple closed curve. But this isn't necessary. Suppose the region has a boundary composed of several simple closed curves, like the ones pictured. We suppose these boundary curves $C_1, \ldots, C_m$ all lie within the domain where $\mathbf{F}$ is continuously differentiable. Most importantly, all the curves must be directed so that the normal $\mathbf{n}$ points *away* from R.

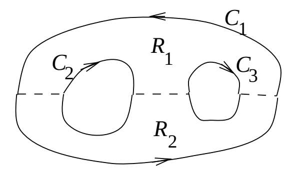


Extended Green's Theorem *With the curve orientations as shown,*

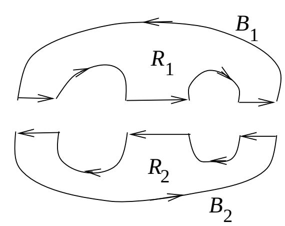
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \ldots + \int_{C_m} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_R \operatorname{curl} \mathbf{F}\, dA \tag{1}$$

In other words, Green's theorem also applies to regions with several boundary curves, provided that we take the line integral over the complete boundary, with each part of the boundary oriented so the normal $\mathbf{n}$ points outside R.

Proof. We use subdivision; the idea is adequately conveyed by an example. Consider a region with three boundary curves as shown. The three cuts illustrated divide up R into two regions R_1 and R_2, each bounded by a single simple closed curve, and Green's theorem in the usual form can be applied to each piece. Letting B_1 and B_2 be the boundary curves shown, we have therefore



$$\oint_{B_1} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{R_1} \operatorname{curl} \mathbf{F}\, dA \qquad \oint_{B_2} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{R_2} \operatorname{curl} \mathbf{F}\, dA \tag{2}$$

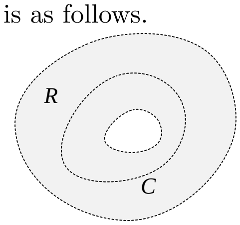


Add these two equations together. The right sides add up to the right side of (1). The left sides add up to the left side of (1) (for $m = 2$), since over each of the three cuts, there are two line integrals taken in opposite directions, which therefore cancel each other out. $\square$

2. Simply-connected and multiply-connected regions.

Though Green's theorem is still valid for a region with "holes" like the ones we just considered, the relation $\operatorname{curl} \mathbf{F} = 0 \;\Rightarrow\; \mathbf{F} = \nabla f$ is not. The reason for this is as follows.

We are trying to show that $\quad \operatorname{curl} \mathbf{F} \;=\; 0 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; 0$ for any closed curve lying in R. We expect to be able to use Green's theorem. But if the region has a hole, like the one pictured, we cannot apply Green's theorem to the curve C because the interior of C is not entirely contained in R.



To see what a delicate affair this is, consider the earlier Example 2 in Section V2. The field $\mathbf{G}$ there satisfies $\operatorname{curl} \mathbf{G} = 0$ everywhere but the origin. The region R is the xy-plane with $(0, 0)$ removed. But $\mathbf{G}$ is not a gradient field, because $\oint_C \mathbf{G} \cdot d\mathbf{r} \;\neq\; 0$ around a circle C surrounding the origin.

> This is clearer if we use Green's theorem in normal form (Section V4). If the flow field satisfies div $\mathbf{F} = 0$ everywhere except at one point, that doesn't guarantee that the flux through every closed curve will be 0. For the spot where div $\mathbf{F}$ is undefined might be a source, through which fluid is being added to the flow.

In order to be able to prove under reasonable hypotheses that curl $\mathbf{F} = 0 \Rightarrow \mathbf{F} = \nabla f$, we define our troubles away by assuming that R is the sort of region where the difficulties described above cannot occur—i.e., we assume that R has no holes; such regions are called *simply-connected*.

Definition. A two-dimensional region D of the plane consisting of one connected piece is called **simply-connected** if it has this property: whenever a simple closed curve C lies entirely in D, then its interior also lies entirely in D.

As examples: the xy-plane, the right-half plane where $x \geq 0$, and the unit circle with its interior are all simply-connected regions. But the xy-plane minus the origin is not simply-connected, since any circle surrounding the origin lies in D, yet its interior does not.

As indicated, one can think of a simply-connected region as one without "holes". Regions with holes are said to be *multiply-connected*, or *not simply-connected*.

Theorem. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a simply-connected region D of the xy-plane. Then in D,

(3) $$\text{curl } F = 0 \quad \Rightarrow \quad \mathbf{F} = \nabla f, \qquad \text{for some } f(x,y); \qquad \text{in terms of components,}$$

(3′) $$M_y = N_x \quad \Rightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} = \nabla f, \qquad \text{for some } f(x,y).$$

Proof. Since a field is a gradient field if its line integral around any closed path is 0, it suffices to show

(4) $$\text{curl } \mathbf{F} = 0 \quad \Rightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \qquad \text{for every closed curve } C \text{ in } D.$$

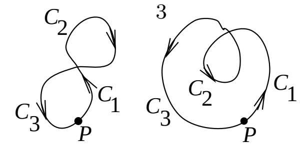
We prove (4) in two steps.

Assume first that C is a simple closed curve; let R be its interior. Then since D is simply-connected, R will lie entirely inside D. Therefore $\mathbf{F}$ will be continuously differentiable in R, and we can use Green's theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_R \text{curl } \mathbf{F}\, dxdy \;=\; 0\ .$$

Next consider the general case, where C is closed but not simple—i.e., it intersects itself. Then C can be broken into smaller simple closed curves for which the above argument will be valid. A formal argument would be awkward to give, but the examples illustrate. In both cases, the path starts and ends at P, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}\ .$$



In both cases, C_2 is a simple closed path, and also $C_1 + C_3$ is a simple closed path. Since D is simply-connected, the interiors automatically lie in D, so that by the first part of the argument,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0 \qquad \text{and} \qquad \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

Adding these up, we get $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. □

The above argument works if C intersects itself a finite number of times. If C intersects itself infinitely often, we would have to resort to approximations to C; we skip this case.

We pause now to summarize compactly the central result, both in the language of vector fields and in the equivalent language of differentials.

Curl Theorem. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be a continuously differentiable vector field in a simply-connected region D of the xy-plane. Then the following four statements are equivalent — if any one is true for $\mathbf{F}$ in D, so are the other three:

1. $\displaystyle\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is path-independent 1.′ $\displaystyle\int_P^Q M\,dx + N\,dy$ is path-independent

for any two points P, Q in D;

2. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, 2.′ $\displaystyle\oint_C M\,dx + N\,dy = 0$,

for any simple closed curve C lying in D;

3. $\mathbf{F} = \nabla f$ for some f in D 3.′ $M\,dx + N\,dy = df$ for some f in D

4. curl $\mathbf{F} = 0$ in D 4.′ $M_y = N_x$ in D.

Remarks. We summarize below what still holds true even if one or more of the hypotheses doesn't hold: D is not simply-connected, or the field $\mathbf{F}$ is not differentiable everywhere in D.

1. Statements 1, 2, and 3 are equivalent even if $\mathbf{F}$ is only continuous; D need not be simply-connected..

2. Statements 1, 2, and 3 each implies 4, if $\mathbf{F}$ is continuously differentiable; D need not be simply-connected. (But 4 implies 1, 2, 3 only if D is simply-connected.)

Example 1. Is $\mathbf{F} = xy\,\mathbf{i} + x^2\,\mathbf{j}$ a gradient field?

Solution. We have curl $\mathbf{F} = x \neq 0$, so the theorem says it is not.

Example 2. Is $\dfrac{ydx - xdy}{y^2}$ an exact differential? If so, find all possible functions $f(x,y)$ for which it can be written df .

Solution. $M = 1/y$ and $N = -x/y^2$ are continuously differentiable wherever $y \neq 0$, i.e., in the two half-planes above and below the x-axis. These are both simply-connected. In each of them,

$$M_y = -1/y^2 = N_x$$

Thus in each half-plane the differential is exact, by the theorem, and we can calculate $f(x, y)$ by the standard methods in Sction V2. They give

$$f(x, y) \;=\; \frac{x}{y} + c$$

where c is an arbitrary constant. This constant need not be the same for the two regions, since they do not touch. Thus the most general function is

$$f(x, y) = \begin{cases} x/y \ + c, & y > 0 \\ x/y \ + c', & y < 0 \end{cases}; \qquad c, c' \ \text{ are arbitrary constants.}$$

Example 3. Let $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j}),\ r = \sqrt{x^2 + y^2}$. For which integers n is $\mathbf{F}$ conservative? For each such, find a corresponding $f(x, y)$ such that $\mathbf{F} = \nabla f$.

Solution. By the usual calculation, using the chain rule and the useful polar coordinate relations $r_x = x/r,\ r_y = y/r$, we find that curl $\mathbf{F} = 0$. There are two cases.

Case 1: $n \geq 0$. Then $\mathbf{F}$ is continuously differentiable in the whole xy-plane, which is simply-connected. Thus by the preceding theorem, $\mathbf{F}$ is conservative, and we can calculate $f(x, y)$ as in Section V2.

We use method 1 (line integration). The radial symmetry suggests using the ray C from $(0, 0)$ to (x_1, y_1) as the path of integration, with the parametrization

$$x = x_1 t, \quad y = y_1 t, \quad 0 \leq t \leq 1;$$

also, let

$$r_1 = \sqrt{x_1^2 + y_1^2}; \qquad \text{then} \qquad r^n = r_1^n t^n, \quad x\,dx + y\,dy = r_1^2 t\,dt$$

and we get, by method 1 for finding $f(x, y)$,

$$\begin{aligned} f(x_1.y_1) \;&=\; \int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \int_C r^n(x\,dx + y\,dy) \\ &=\; \int_0^1 r_1^{n+2} t^{n+1} dt \;=\; r_1^{n+2} \frac{t^{n+2}}{n+2}\bigg|_0^1 \;=\; \frac{r_1^{n+2}}{n+2}. \end{aligned} \tag{6}$$

so that

$$f(x, y) \;=\; \frac{r^{n+2}}{n+2}, \qquad \mathbf{F} \;=\; \nabla f, \quad n \geq 0\ . \tag{7}$$

Case 2: $n < 0$. The field $\mathbf{F}$ is not defined at $(0, 0)$, so that its domain, the xy-plane with $(0, 0)$ removed, is not simply-connected. So even though curl $\mathbf{F} = 0$ in this region, (3) is not immediately applicable.

Nonetheless, if $n \neq -2$, one can check by differentiation that (7) is still valid.

If $n = -2$, guessing, inspection, or method 2 give $f(x, y) = \ln r$.

We conclude that the field in all cases is a gradient field. Note in particular that the two force fields given in section V1, representing respectively (apart from a constant factor) the fields arising from a positive charge at $(0, 0)$ and a uniform positive charge along the z-axis, correspond to the respective cases $n = -3$ and $n = -2$, and are both gradient fields:

$$\frac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\frac{1}{r}\right) \qquad (n = -3: \text{ positive charge at } (0,0))$$

$$\frac{x\,\mathbf{i} + y\,\mathbf{j}}{r^2} = \nabla(\ln r) \qquad (n = -2: \text{ uniform + charge on } z\text{-axis}).$$

Exercises: Section 4G

V6. Multiply-connected Regions; Topology

In Section V5, we called a region D of the plane *simply-connected* if it had no holes in it. This is a typical example of what would be called in mathematics a *topological property*, that is, a property that can be described without using measurement. For a curve, such properties as having length 3, or being a circle, or a line, or a triangle — these are not topological properties, since they involve measurement, whereas the property of being closed, or of intersecting itself once, would be topological.

The important thing about topological properties is that they are preserved when the geometric figure is deformed continuously without adding or subtracting points, whereas non-topological properties change under such a deformation. For example, if you deform a circle, it will not stay a circle, but it will still remain a closed curve that does not intersect itself.

Topology is that branch of mathematics which studies topological properties of geometrical figures; it's a kind of geometry, but at the opposite pole from Euclidean geometry, which emphasizes measurement ("congruent triangles" "right angles", "circles"). Topology is a large and active branch of mathematics today, one which is attracting attention from other disciplines, like theoretical physics and molecular biology. Most students have never heard of it, because topological properties don't enter very often into the first few years of mathematics. However, they do right here, and in fact it was just in the study of the possible values of a line integral around a closed curve that the central ideas of modern topology first entered into mathematics, in the middle of the 1800's.

So let $\mathbf{F}$ be a continuously differentiable vector field in a multiply-connected — i.e., not simply-connected — region D of the xy-plane, and suppose curl $\mathbf{F} = 0$. What values can $\oint_C \mathbf{F} \cdot d\mathbf{r}$ have?

We begin by considering an earlier example (Section V2, Example 2) in greater detail, because it gives the key to the general case. Consider the vector field representing the electromagnetic field of a wire along the z-axis carrying a constant current:

$$\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2} ; \tag{1}$$

we know that curl $\mathbf{F} = 0$ (see the cited Example.)

To evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C does not pass through the origin, we use polar coordinates and t as parameter:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = r(t), \qquad \theta = \theta(t) . \tag{2}$$

As you move around C, the polar angle θ must vary continuously; thus we allow it to take on all values, and do not restrict it to lie in the interval $[0, 2\pi]$. Using (2) and the chain rule for several variables, we get

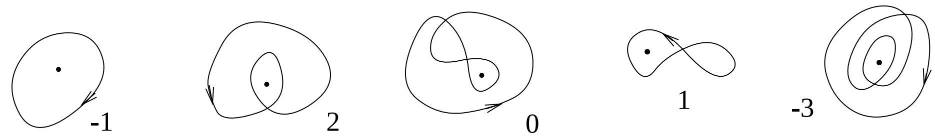
$$\begin{aligned} dx &= x_r dr + x_\theta d\theta = \cos\theta\, dr - r\sin\theta\, d\theta \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta ; \end{aligned}$$

We calculate the line integral as usual; everything cancels, and we get

$$\begin{aligned}
(3)\qquad \oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C \frac{-y\,dx + x\,dy}{r^2} = \oint_C d\theta = \int_{t_0}^{t_1} \frac{d\theta}{dt}\,dt = \theta(t_1) - \theta(t_0)\ . \\
(4)\qquad &= 2\pi n,
\end{aligned}$$

the net change in polar angle θ as you move around C — since C is closed, this must be an integer multiple n of 2π. It is called the **winding number** of C around the origin.

By way of illustration, the value of the winding number n is given for a few closed curves C below; the origin is indicated by a dot, and it's understood that as t runs from t_0 to t_1, the curve C is traced out just once, in the direction shown.

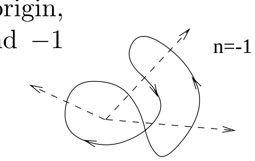


Intuitively, the winding number is the total number of times that C goes around the origin, counting $+1$ each time it goes around counterclockwise, and -1 when it goes around clockwise. The winding number of C around any other point not on C is defined the same way, by taking the point to be the origin of a polar coordinate system.

The winding number about the origin (or around any given point) is a topological property of the oriented curve C, since if C is deformed continuously without ever crossing the point, the winding numbers must also vary continuously, but the only way an integer can vary continuously is to always stay the same.

Here is a simple way of finding the winding number of C around the origin: the winding number is the total number of times that C crosses the positive x-axis, counting $+1$ each time C crosses from below, and -1 each time it crosses from above.

> Instead of the positive x-axis, you can use any directed ray from the origin, counting $+1$ when C crosses the ray in the countercloswise direction, and -1 when it crosses in the clockwise direction.



Example 1. For the curve shown, and the three choices of ray given, calculate the winding number using each in turn, and find $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Solution. Each ray gives $n = -1$ as the winding number. Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = -2\pi$, according to (4) above.

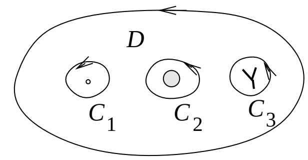
So far, we have shown by explicit calculation for the particular field $\mathbf{F}$ given by (1), the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the winding number of C around the origin. We now consider the general situation.

Let $\mathbf{F}$ be a vector field which is continuously differentiable in a region D, and assume that curl $\mathbf{F} = 0$ in D. We will show now that the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the topological properties of C, and not on its exact position or length.

To see this, we will assume that D consists of one connected region having k holes. The holes come from removing portions of the region — we might remove a point, a line segment, the interior of a circle, the letter Y, etc. We could not remove a circle, however, since then what was left would not be one connected piece.

Draw a simple closed curve C_1, directed counterclockwise, around the i-th hole, and let

$$A_i \;=\; \oint_{C_1} \mathbf{F}\cdot d\mathbf{r}.$$



Then for any closed curve C in the region, we claim that

(6) $$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; n_1A_1 + \ldots + n_kA_k, \qquad n_i \text{ integers,}$$

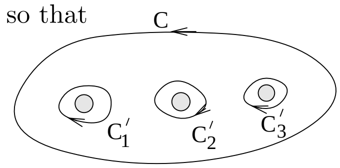
where n_i is the winding number of C around the i-th hole (i.e., the winding number of C around any point inside that i-th hole).

We shall indicate how the argument goes in a few cases; the general case would take us farther into topology than we are able to go at present. The essential tool is Green's theorem.

Example 2. Suppose C is a path like the one pictured ($k = 3$ in the picture, but this isn't significant.) Let C_i' be the curve C_i with its direction reversed. Then $\mathbf{F}$ is continuously differentiable in the region between C and the C_i', and also curl $\mathbf{F} = 0$. Therefore by the extended form of Green's theorem (Section V5), we get

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;+\; \oint_{C_1'} \mathbf{F}\cdot d\mathbf{r} \;+\; \ldots \;+\; \oint_{C_k'} \mathbf{F}\cdot d\mathbf{r} \;=\; 0, \qquad \text{so that}$$

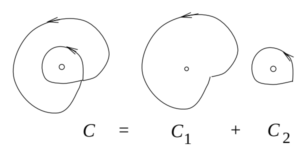
$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; A_1 \;+\ldots+\; A_k,$$



proving (6) in this case.

Example 3. If there is just one hole, Example 2 shows that for any simple closed curve going counterclockwise around the hole,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; A_1 \;.$$



(In the first example (4) discussed, $A_1 = 2\pi$.)

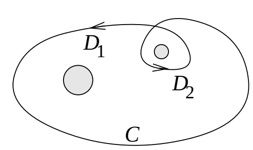
Suppose now that C is a closed curve going around twice, as in the illustration. Break C into the sum of two simple closed curves as shown, using a point where it crosses itself; then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \oint_{C_1} \mathbf{F}\cdot d\mathbf{r} \;+\; \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} \;=\; 2A_1.$$

Example 4. This example combines the two above. If C is as shown (here $k = 2$), then $C = D_1 + D_2$; let D_i' be D_i with the direction reversed. Then $D_1 + D_2'$ is a simple curve having only hole number 1 in its interior. Therefore

$$\oint_{D_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{D_2'} \mathbf{F}\cdot d\mathbf{r} \;=\; A_1, \qquad \oint_{D_2} \mathbf{F}\cdot d\mathbf{r} \;=\; A_2, \qquad \text{so that} \qquad \oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; A_1 + 2A_2.$$

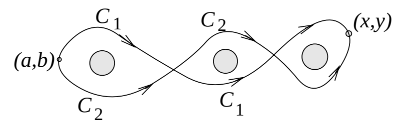
One final remark. If $\mathbf{F}$ is as above, with curl $\mathbf{F} = 0$, and we try to find a function $f(x,y)$ for which $\mathbf{F} = \Delta f$, by defining



(7) $$f(x,y) \;=\; \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}\;,$$

this won't work because the value of $f(x, y)$ depends not just on (x, y) but also on the path of integration used. However, equation (6) shows that two different values for the line integfral always differ by a number of the form $n_1A_1 + \ldots + n_kA_k$, because if C_1 and C_2 are two different choices of path from (a, b) to (x, y), then $C_1 + C_2'$ is a closed path, and thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \; + \; \int_{C_2'} \mathbf{F} \cdot d\mathbf{r} \; = \; n_1A_1 + \ldots + n_kA_k \; .$$



Here n_i is the winding number of the path $C_1 + C_2'$ around the i-th hole. Therefore

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \; = \; \oint_C \mathbf{F} \cdot d\mathbf{r} \; + n_1A_1 + \ldots + n_kA_k \; . \tag{8}$$

Now if (x, y) is moved a little, and the paths C_1 and C_2 are moved accordingly, the numbers $n_1, \ldots, n_k$ will not change since they are winding numbers. Thus, even though the function $f(x, y)$ defined by (7) is multiple-valued, according to (8) the different determinations of its value differ by a constant, and therefore all have the same derivative. Thus we can claim that the multiple-valued function $f(x, y)$ is differentiable, and

$$\mathbf{F} \; = \; \nabla f(x, y) \; .$$

Our conclusion is that, if curl $\mathbf{F} = 0$ in a multiply-connected region, even if the field is not conservative we can still view it as the gradient field of a multiple-valued function $f(x, y)$.

For example, going back to our first significant example, equation (3) shows that we may take $f(x, y) = \theta(x, y)$, the multiple-valued polar angle. No matter which determination of θ you pick, it is still true that for some n,

$$\begin{aligned} \theta \; &= \; \tan^{-1} \frac{y}{x} \; + \; 2n\pi, \qquad \text{and} \\ \nabla\theta \; &= \; \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2} \; . \end{aligned}$$

Exercises: Section 4H

V7. Laplace's Equation and Harmonic Functions

In this section, we will show how Green's theorem is closely connected with solutions to Laplace's partial differential equation in two dimensions:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \tag{1}$$

where $w(x, y)$ is some unknown function of two variables, assumed to be twice differentiable. Equation (1) models a variety of physical situations, as we discussed in Section P of these notes, and shall briefly review.

1. The Laplace operator and harmonic functions.

The two-dimensional **Laplace operator**, or *laplacian* as it is often called, is denoted by ∇^2 or *lap*, and defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \tag{2}$$

The notation ∇^2 comes from thinking of the operator as a sort of symbolic scalar product:

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

In terms of this operator, Laplace's equation (1) reads simply

$$\nabla^2 w = 0 . \tag{1'}$$

Notice that the laplacian is a linear operator, that is it satisfies the two rules

$$\nabla^2(u + v) = \nabla^2 u + \nabla^2 v , \qquad \nabla^2(cu) = c(\nabla^2 u), \tag{3}$$

for any two twice differentiable functions $u(x, y)$ and $v(x, y)$ and any constant c.

Definition. A function $w(x, y)$ which has continuous second partial derivatives and solves Laplace's equation (1) is called a **harmonic** function.

In the sequel, we will use the Greek letters ϕ and ψ to denote harmonic functions; functions which aren't assumed to be harmonic will be denoted by Roman letters f, g, u, v, etc.. According to the definition,

$$\phi(x, y) \text{ is harmonic } \Leftrightarrow \nabla^2 \phi = 0 . \tag{4}$$

By combining (4) with the rules (3) for using Laplace operator, we see

$$\textit{ϕ and ψ harmonic} \quad \Rightarrow \quad \textit{$\phi + \psi$ and $c\,\phi$ are harmonic (c constant).} \tag{5}$$

Examples of harmonic functions. Here are some examples of harmonic functions. The verifications are left to the Exercises.

A. Harmonic homogeneous polynomials[1] **in two variables.**
Degree 0: all constants c are harmonic.
Degree 1: all linear polynomials $ax + by$ are harmonic.
Degree 2: the quadratic polynomials $x^2 - y^2$ and xy are harmonic; all other harmonic homogeneous quadratic polynomials are linear combinations of these:

$$\phi(x, y) \;=\; a(x^2 - y^2) + bxy, \qquad (a, b \text{ constants}).$$

Degree n: the real and imaginary parts of the complex polynomial $(x+iy)^n$ are harmonic. (Check this against the above when $n = 2$.)

B. Functions with radial symmetry. Letting $r = \sqrt{x^2 + y^2}$, the function given by $\phi(r) \;=\; \ln r$ is harmonic, and its constant multiples $c \ln r$ are the only harmonic functions with radial symmetry, i.e., of the form $f(r)$.

C. Exponentially growing or decaying oscillations. For all k the functions $e^{kx} \sin ky$ and $e^{kx} \cos ky$ are harmonic.

In general, harmonic functions cannot be written down explicitly in terms of elementary functions. Nevertheless, we will be able to prove things about them, by using Green's theorem.

2. Harmonic functions and vector fields.

The relation between harmonic functions and vector fields rests on the simple identity

$$\text{div}\; \nabla f \;=\; \nabla^2 f \;, \tag{6}$$

which is easily verified, since

$$\text{div}\; (f_x\, \mathbf{i} + f_y\, \mathbf{j}\,) \;=\; \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \;+\; \frac{\partial}{\partial y}\frac{\partial f}{\partial y} \;=\; \nabla^2 f \;;$$

its truth is suggested symbolically by

$$\text{div}\; \nabla f \;=\; \nabla \cdot (\nabla\, f) \;=\; (\nabla \cdot \nabla)\, f \;=\; \nabla^2 f \;.$$

There is an important connection between harmonic functions and conservative fields which follows immediately from (6):

$$\textit{Let } \mathbf{F} \;=\; \nabla\, f. \quad \textit{Then div } \mathbf{F} = 0 \quad \Leftrightarrow \quad f \textit{ is harmonic.} \tag{7}$$

Another way to put this is to say: in a simply-connected region,

$$\textit{curl } \mathbf{F} = 0 \quad \textit{and} \quad \textit{div } \mathbf{F} = 0 \quad \Leftrightarrow \quad \mathbf{F} = \nabla\phi. \quad \textit{where } \phi \textit{ is harmonic.} \tag{7$'$}$$

[1] A homogeneous polynomial in several variables is one in which all the terms have the same total degree, like $x^2y + 2y^3$ or $x^5 - 6x^2y^3 + 4xy^4$.

This is just (7), combined with the criterion for gradient fields (Section V5, X).

In other words, from the vector field viewpoint, the theory of harmonic functions and Laplace's equation is the same as the theory of conservative vector fields with zero divergence. Where do such functions and fields occur?

One place is in heat flow problems. Imagine a thin uniform metal plate which is insulated on the faces so no heat can enter or escape on the faces, and imagine that some temperature distribution is maintained along the edge of the plate. Then when the temperature distribution on the plate has reached steady-state, it will be given by a harmonic function $\phi(x, y)$; namely, it must satisfy the heat equation (see Section P of these notes): $\phi_{xx} + \phi_{yy} \;=\; a^2\phi_t$, but $\phi_t = 0$ since the temperature is not changing with time, by assumption.

Harmonic functions also occur as the potential functions for two-dimensional gravitational, electrostatic, and electromagnetic fields, in regions of space which are respectively free of mass, static charge, or moving charges. (Here, "two-dimensional" means not that the fields lie in the xy-plane, but rather that as fields in three-space, the vectors all lie in horizontal planes, and the field looks the same no matter what horizontal plane it is viewed in. A typical example would be the field arising from a uniform mass or charge distribution on a set of vertical wires, or from uniform currents on vertical wires.)

3. Boundary-value problems.

As the example given above of a temperature distribution on a uniform insulated metal plate suggests, the typical problem in solving Laplace's equation would be to find a harmonic function satisfying given boundary conditions.

That is, we are given a region R of the xy-plane, bounded by a simple closed curve C. The problem is to find a function $\phi(x, y)$ which is defined and harmonic on R, and which takes on prescribed boundary values along the curve C.

The boundary values are commonly given in one of two ways:

(i) as the values of ϕ along C;

(ii) as the values of the *normal derivative* $\dfrac{\partial\phi}{\partial\eta}$ of ϕ along C.

To explain this last, the **normal derivative** is just the directional derivative in the direction of the (outward-pointing) unit normal vector $\mathbf{n}$:

$$(8) \qquad \frac{\partial\phi}{\partial\eta} \;=\; \frac{d\phi}{ds}\bigg|_{\mathbf{n}} \;=\; \nabla\,\phi\cdot\mathbf{n} \qquad \textit{(normal derivative)}$$

The *tangential derivative* is defined similarly, using the unit tangent vector $\mathbf{t}$ instead of $\mathbf{n}$.

For heat flow problems, boundary values of the first type (i) would be most common — you are maintaining a definite temperature distribution ϕ along C and want to know what the temperature will look like in R.

For conservative force field problems, with $\mathbf{F} = \nabla\phi$, one could also get boundary values of the second type (ii). For example, if you were given the field vector $\mathbf{F}$ at each point of C, then you would know $\nabla\phi\cdot\mathbf{n}$ and $\nabla\phi\cdot\mathbf{t}$ — the normal derivative and the tangential derivative — at each point of C. Knowing the tangential derivative however is equivalent to knowing ϕ itself on C, for

$$\left.\frac{d\phi}{ds}\right|_{\mathbf{t}} = \frac{d\phi(s)}{ds} \qquad (s = \text{arclength along } C)\ , \tag{9}$$

and therefore $\phi(s)$ can be obtained by integrating the tangential derivative. So, to prescribe **F** on the boundary is equivalent to prescribing both (i) and (ii) above for its potential function.

The basic problems are now these:

A. Existence. Does there exist a $\phi(x, y)$ harmonic in some region containing C and its interior R, and taking on the prescribed boundary values?

B. Uniqueness. If it exists, is there only one such $\phi(x, y)$?

C. Solving. If there is a unique $\phi(x, y)$, determine it by some explicit formula, or approximate it by some numerical method.

We shall now show how Green's theorem sheds some light on both the existence and the uniqueness questions.

4. Existence and uniqueness for harmonic functions.

In general, if the curve C is reasonable (sufficiently smooth, of finite length, and not too wiggly), the values of ϕ on the boundary can be prescribed more or less arbitrarily as long as they form a twice differentiable function on C. It can then be proved that the harmonic function ϕ taking on those boundary values will exist in the interior of C.

This is not so however for the second type of boundary condition, which cannot be prescribed arbitrarily, as the following theorem shows; its proof uses Green's theorem in the normal (flux) form.

Theorem 1. *If ϕ exists and is harmonic everywhere inside the closed curve C bounding the region R, then*

$$\oint_C \frac{\partial\phi}{\partial\eta}\, ds = 0\ . \tag{10}$$

Proof. We use (8), then Green's theorem in the normal form:

$$\oint_C \frac{\partial\phi}{\partial\eta}\, ds = \oint_C \nabla\phi\cdot\mathbf{n}\, ds = \iint_R \operatorname{div}\,(\nabla\phi)\, dA = 0;$$

the double integral is zero since ϕ is harmonic (cf. (7)). □

One can think of the theorem as a "non-existence" theorem, since it gives condition under which no harmonic ϕ can exist. For example, if C is the unit circle, and the normal derivative is prescribed to be 1 everywhere on C, then no harmonic ϕ can exist satisfying this condition, since the integral in (10) will have the value 2π, not 0.

As far as *uniqueness* goes, physical considerations suggest that if a harmonic function exists in R having given values on the boundary curve C, it should be unique. Namely, if the given temperature distribution is maintained on C, then the corresponding temperature distribution inside will approach a unique steady-state as $t \to \infty$.

> This argument however assumes that our model of heat flow is complete, i.e., that Laplace's equation is all that determines the heat flow. But maybe there are some other conditions we don't know about and it is these that make the solution unique.

We will prove the uniqueness of ϕ by a purely mathematical argument. It depends on the following theorem.

Theorem 2. Green's first identity. *If ϕ is harmonic in a region containing R, and $f(x,y)$ is continuously differentiable in R, then*

$$\oint_C f\,\frac{\partial\phi}{\partial\eta}\,ds \;=\; \iint_R \nabla f\cdot\nabla\phi\;dA\;. \tag{11}$$

Proof. As before, we use first (8), then Green's theorem in normal form:

$$\oint_C f\,\frac{\partial\phi}{\partial\eta}\,ds \;=\; \oint_C (f\,\nabla\phi)\cdot\mathbf{n}\,ds \;=\; \iint_R \operatorname{div}\,(f\,\nabla\phi)\,dA. \tag{12}$$

$$\begin{aligned}\operatorname{div}\,(f\,\nabla\phi) \;&=\; (f\phi_x)_x + (f\phi_y)_y \;=\; f_x\phi_x + f_y\phi_y + f(\phi_{xx}+\phi_{yy})\\ &=\; \nabla f\cdot\nabla\phi, \qquad \text{since}\;\; \phi_{xx}+\phi_{yy} \;=\; 0\;.\end{aligned}$$

Substituting this into the double integral in (12) gives us (11). □

The essential step in proving the uniquess of ϕ is to prove it when the prescribed boundary values are 0. We consider both types of boundary values.

Theorem 3. *Let ϕ be harmonic in a region containing R. Then*

$$\phi \;=\; 0 \;\; \text{on } C \quad\Rightarrow\quad \phi = 0 \;\; \text{on } R\;; \tag{13}$$

$$\frac{\partial\phi}{\partial\eta} \;=\; 0 \;\; \text{on } C \quad\Rightarrow\quad \phi = c \;\; \text{on } R \quad (c \text{ is a constant}). \tag{14}$$

Proof. We use Green's first identity (11), taking $f = \phi$. This gives

$$\oint_C \phi\,\frac{\partial\phi}{\partial\eta}\,ds \;=\; \iint_R |\nabla\phi|^2\,dA\;. \tag{15}$$

$$\begin{aligned}\phi = 0 \;\;\text{or}\;\; \frac{\partial\phi}{\partial\eta} \;=\; 0 \;\;\text{on } C \quad&\Rightarrow\quad \oint_C \phi\,\frac{\partial\phi}{\partial\eta}\,ds \;=\; 0\\ &\Rightarrow\quad \iint_R |\nabla\phi|^2\,dA \;=\; 0, \qquad \text{by (15);}\\ &\Rightarrow\quad |\nabla\phi|^2 \;=\; 0 \text{ everywhere in } R,\end{aligned}$$

since it is continuous and ≥ 0 everywhere, being a square;

$$
\begin{aligned}
&\Rightarrow \ \nabla\phi \ = \ 0 \quad \text{in } R, \ \text{since its magnitude is } 0; \\
&\Rightarrow \ \phi_x = 0 \ \text{ and } \ \phi_y = 0 \ \text{ in } R, \\
&\Rightarrow \ \phi \ = \ c \ \text{ in } R.
\end{aligned}
$$

This proves (14); it also proves (13), for in this case we know that since $\phi = 0$ on the boundary C, the constant c must be 0. $\square$

Corollary. Uniqueness Theorem.

Let ϕ and ψ be two functions harmonic in a region containing R.

$$\phi = \psi \ \text{ on } C \quad \Rightarrow \quad \phi = \psi \ \text{ on } R\,; \tag{16}$$

$$\frac{\partial\phi}{\partial\eta} = \frac{\partial\psi}{\partial\eta} \ \text{ on } C \quad \Rightarrow \quad \phi = \psi + c \ \text{ on } R, \qquad c \text{ constant}\,. \tag{17}$$

Proof. Consider the difference $\phi - \psi$. It is a harmonic function, by (5). The two hypotheses in (16) and (17) say respectively that

$$\phi - \psi = 0 \ \text{ on } C, \qquad \text{or} \qquad \frac{\partial(\phi-\psi)}{\partial\eta} = 0 \ \text{ on } C\,.$$

Therefore, by theorem 3, we conclude respectively that

$$\phi - \psi = 0 \ \text{ on } R, \qquad \text{or} \qquad \phi - \psi = c \ \text{ on } R;$$

these are respectively the conclusions of (16) and (17). □

Exercises: Section 4I

V8. Vector Fields in Space

Just as in Section V1 we considered vector fields in the plane, so now we consider vector fields in three-space. These are fields given by a vector function of the type

$$\mathbf{F}(x,y,z) \;=\; M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}\;. \tag{1}$$

Such a function assigns the vector $\mathbf{F}(x_0,y_0,z_0)$ to a point (x_0,y_0,z_0) where M, N, and P are all defined. We place the vector so its tail is at (x_0,y_0,z_0), and in this way get the vector field. Such a field in space looks a little like the interior of a haystack.

As before, we say $\mathbf{F}$ is *continuous* in some domain D of 3-space (we will usually use "domain" rather than "region" , when referring to a portion of 3-space) if M, N, and P are continuous in that domain. We say $\mathbf{F}$ is *continuously differentiable* in the domain D if all nine first partial derivatives

$$M_x,\; M_y,\; M_z;\quad N_x,\; N_y,\; N_z;\quad P_x,\; P_y,\; P_z$$

exist and are continuous in D.

Again as before, we give two physical interpretations for such a vector field.

The three-dimensional **force fields** of different sorts — gravitational, electrostatic, electromagnetic — all give rise to such a vector field: at the point (x_0,y_0,z_0) we place the vector having the direction and magnitude of the force which the field would exert on a unit test particle placed at the point.

The three-dimensional **flow fields** and **velocity fields** arising from the motion of a fluid in space are the other standard example. We assume the motion is steady-state (i.e., the direction and magnitude of the flow at any point does not change over time). We will call this a *three-dimensional flow*.

As before, we allow sources and sinks — places where fluid is being added to or removed from the flow. Obviously, we can no longer appeal to people standing overhead pouring fluid in at various points (they would have to be aliens in four-space), but we could think of thin pipes inserted into the domain at various points adding or removing fluid.

The velocity field of such a flow is defined just as it was previously: $\mathbf{v}(x,y,z)$ gives the direction and magnitude (speed) of the flow at (x,y,z).

The flow field $\mathbf{F} = \delta\mathbf{v}$, where $\delta(x,y,z)$ is the density, may be similarly interpreted:

$$\begin{aligned} \text{dir } \mathbf{F} \;&=\; \text{the direction of flow} \\ |\mathbf{F}| \;&=\; \text{mass transport rate (per unit area) at } (x,y,z) \text{ in the flow direction;} \end{aligned} \tag{2}$$

that is, $|\mathbf{F}|$ is the rate per unit area at which mass is transported across a small piece of plane perpendicular to the flow at the point (x,y,z)..

The derivation of this interpretation is exactly as in Sections V1 and V3, replacing the small line segment Δl by a small plane area ΔA perpendicular to the flow.

Example 1. Find the three-dimensional electrostatic force field $\mathbf{F}$ arising from a unit positive charge placed at the origin, given that in suitable units $\mathbf{F}$ is directed radially outward from the origin and has magnitude $1/\rho^2$, where ρ is the distance from the origin.

Solution. The vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ with tail at (x, y, z) is directed radially outward and has magnitude ρ. Therefore

$$\mathbf{F} \;=\; \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}\;, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

Example 2. a) Find the velocity field of a fluid rotating with constant angular velocity ω around the z-axis, in the direction given by the right-hand rule (right-hand fingers curl in direction of flow when thumb points in the $\mathbf{k}$-direction).

b) Find the analogous field if the flow is rotating about the y-axis.

Solution. a) The flow doesn't depend on z — it is really just a two-dimensional problem, whose solution is the same as before (section V1, Example 4):

$$\mathbf{F}(x, y, z) \;=\; \omega(-y\,\mathbf{i} + x\,\mathbf{j}\,)\;.$$

b) If the axis of flow is the y-axis, the flow will have no $\mathbf{j}$-component and will not depend on y. However, by the right-hand rule, the flow in the xz-plane is *clockwise*, when the positive x and z axes are drawn so as to give a right-handed system. Thus

$$\mathbf{F}(x, y, z) \;=\; \omega(z\,\mathbf{i} - x\,\mathbf{k}\,)\;.$$

Example 3. Find the three-dimensional flow field of a gas streaming radially outward with constant velocity from a source at the origin of constant strength.

Solution. This is like the corresponding two-dimensional problem (section V1, Example 3), except that the area of a sphere increases like the *square* of its radius. Therefore, to maintain constant velocity, the density of flow must decrease like $1/\rho^2$ as you go out from the origin; letting δ be the density and c_i be constants, we get

$$\mathbf{F}(x, y, z) \;=\; \delta\mathbf{v} \;=\; \frac{c_1}{\rho^2}\,\frac{c_2(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,)}{\rho} \;=\; \frac{c(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,)}{\rho^3}\;.$$

Notice that in the three-dimensional case, this field is the same as the one in Example 1 above, with the magnitude falling off like $1/\rho^2$. For the two-dimensional case, the analogue of a point fluid source at the origin is not a point charge at the origin, but a uniform charge along a vertical wire; both give the field whose magnitude falls off like $1/r$.

Exercises: Section 6A

V9. Surface Integrals

Surface integrals are a natural generalization of line integrals: instead of integrating over a curve, we integrate over a surface in 3-space. Such integrals are important in any of the subjects that deal with continuous media (solids, fluids, gases), as well as subjects that deal with force fields, like electromagnetic or gravitational fields.

Though most of our work will be spent seeing how surface integrals can be calculated and what they are used for, we first want to indicate briefly how they are defined. The surface integral of the (continuous) function $f(x,y,z)$ over the surface S is denoted by

$$\iint_S f(x,y,z)\,dS \ . \tag{1}$$

You can think of dS as the area of an infinitesimal piece of the surface S. To define the integral (1), we subdivide the surface S into small pieces having area ΔS_i, pick a point (x_i, y_i, z_i) in the i-th piece, and form the Riemann sum

$$\sum f(x_i, y_i, z_i)\Delta S_i \ . \tag{2}$$

As the subdivision of S gets finer and finer, the corresponding sums (2) approach a limit which does not depend on the choice of the points or how the surface was subdivided. The surface integral (1) is defined to be this limit. (The surface has to be smooth and not infinite in extent, and the subdivisions have to be made reasonably, otherwise the limit may not exist, or it may not be unique.)

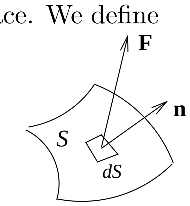
1. The surface integral for flux.

The most important type of surface integral is the one which calculates the flux of a vector field across S. Earlier, we calculated the flux of a plane vector field $\mathbf{F}(x,y)$ across a directed curve in the xy-plane. What we are doing now is the analog of this in space.

We assume that S is *oriented*: this means that S has two sides and one of them has been designated to be the *positive side*. At each point of S there are two unit normal vectors, pointing in opposite directions; the *positively directed* unit normal vector, denoted by $\mathbf{n}$, is the one standing with its base (i.e., tail) on the positive side. If S is a closed surface, like a sphere or cube — that is, a surface with no boundaries, so that it completely encloses a portion of 3-space — then by convention it is oriented so that the outer side is the positive one, i.e., so that $\mathbf{n}$ always points towards the outside of S.

Let $\mathbf{F}(x,y,z)$ be a continuous vector field in space, and S an oriented surface. We define

$$\textsf{\textbf{flux of } } F \textsf{\textbf{ through }} S \ = \ \iint_S (\mathbf{F}\cdot\mathbf{n})\,dS \ = \ \iint_S \mathbf{F}\cdot d\mathbf{S} \ ; \tag{3}$$



the two integrals are the same, but the second is written using the common and suggestive abbreviation $d\mathbf{S} = \mathbf{n}\,dS$.

If $\mathbf{F}$ represents the velocity field for the flow of an incompressible fluid of density 1, then $\mathbf{F}\cdot\mathbf{n}$ represents the component of the velocity in the positive perpendicular direction to the surface, and $\mathbf{F}\cdot\mathbf{n}\,dS$ represents the flow rate across the little infinitesimal piece of surface having area dS. The integral in (3) adds up these flows across the pieces of surface, so that we may interpret (3) as saying

$$\text{(4)} \qquad \textit{flux of } F \textit{ through } S \;=\; \textit{net flow rate across } S,$$

where we count flow in the direction of $\mathbf{n}$ as positive, flow in the opposite direction as negative. More generally, if the fluid has varying density, then the right side of (4) is the net mass transport rate of fluid across S (per unit area, per time unit).

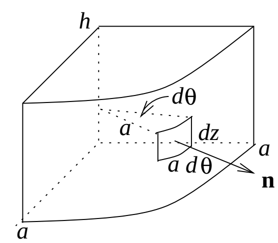
If $\mathbf{F}$ is a force field, then nothing is physically flowing, and one just uses the term "flux" to denote the surface integral, as in (3).

2. Flux through a cylinder and sphere.

We now show how to calculate the flux integral, beginning with two surfaces where $\mathbf{n}$ and dS are easy to calculate — the cylinder and the sphere.

Example 1. Find the flux of $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ outward through the portion of the cylinder $x^2+y^2=a^2$ in the first octant and below the plane $z=h$.

Solution. The piece of cylinder is pictured. The word "outward" suggests that we orient the cylinder so that $\mathbf{n}$ points outward, i.e., away from the z-axis. Since by inspection $\mathbf{n}$ is radially outward and horizontal,



$$\text{(5)} \qquad \mathbf{n} \;=\; \frac{x\,\mathbf{i}+y\,\mathbf{j}}{a}\;.$$

(This is the outward normal to the circle $x^2+y^2=a^2$ in the xy-plane; $\mathbf{n}$ has no z-component since it is horizontal. We divide by a to make its length 1.)

To get dS, the infinitesimal element of surface area, we use cylindrical coordinates to parametrize the cylinder:

$$\text{(6)} \qquad x = a\cos\theta, \qquad y = a\sin\theta \qquad z = z\;.$$

As the parameters θ and z vary, the whole cylinder is traced out ; the piece we want satisfies $0\le\theta\le\pi/2,\ 0\le z\le h$. The natural way to subdivide the cylinder is to use little pieces of curved rectangle like the one shown, bounded by two horizontal circles and two vertical lines on the surface. Its area dS is the product of its height and width:

$$\text{(7)} \qquad dS = dz\cdot a\,d\theta\;.$$

Having obtained $\mathbf{n}$ and dS, the rest of the work is routine. We express the integrand of our surface integral (3) in terms of z and θ:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n}\,dS &= \frac{zx+xy}{a}\cdot a\,dz\,d\theta\;, && \text{by (5) and (7);}\\ &= (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta, && \text{using (6).}\end{aligned}$$

This last step is essential, since the dz and $d\theta$ tell us the surface integral will be calculated in terms of z and θ, and therefore the integrand must use these variables also. We can now calculate the flux through S:

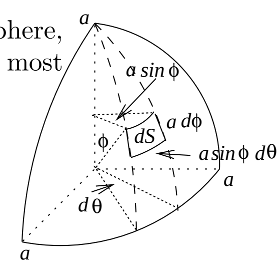
$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{\pi/2}\int_0^h (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta \\
\text{inner integral} &= \frac{ah^2}{2}\cos\theta + a^2 h\sin\theta\cos\theta \\
\text{outer integral} &= \left[\frac{ah^2}{2}\sin\theta + a^2 h\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \;=\; \frac{ah}{2}(a+h)\ .
\end{aligned}$$

Example 2. Find the flux of $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$ outward through that part of the sphere $x^2+y^2+z^2=a^2$ lying in the first octant $(x,y,z,\geq 0)$.

Solution. Once again, we begin by finding $\mathbf{n}$ and dS for the sphere. We take the outside of the sphere as the positive side, so $\mathbf{n}$ points radially outward from the origin; we see by inspection therefore that

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}\ , \tag{8}$$

where we have divided by a to make $\mathbf{n}$ a unit vector.



To do the integration, we use spherical coordinates ρ,ϕ,θ. On the surface of the sphere, $\rho = a$, so the coordinates are just the two angles ϕ and θ. The area element dS is most easily found using the volume element:

$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = dS\cdot d\rho \quad=\quad \text{area}\cdot\text{thickness}$$

so that dividing by the thickness $d\rho$ and setting $\rho = a$, we get

$$dS = a^2\sin\phi\,d\phi\,d\theta. \tag{9}$$

Finally since the area element dS is expressed in terms of ϕ and θ, the integration will be done using these variables, which means we need to express x,y,z in terms of ϕ and θ. We use the formulas expressing Cartesian in terms of spherical coordinates (setting $\rho = a$ since (x,y,z) is on the sphere):

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta, \qquad z = a\cos\phi\ . \tag{10}$$

We can now calculate the flux integral (3). By (8) and (9), the integrand is

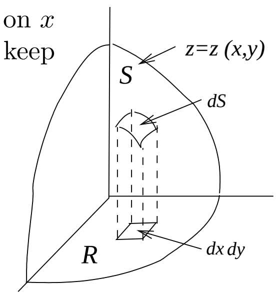
$$\mathbf{F}\cdot\mathbf{n}\,dS = \frac{1}{a}\,(x^2z + y^2z + z^2z)\cdot a^2\sin\phi\,d\phi\,d\theta\ .$$

Using (10), and noting that $x^2+y^2+z^2=a^2$, the integral becomes

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\;dS &= a^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta \\
&= a^4\frac{\pi}{2}\frac{1}{2}\,\sin^2\phi\Big]_0^{\pi/2} \;=\; \frac{\pi a^4}{4}\ .
\end{aligned}$$

3. Flux through general surfaces.

For a general surface, we will use xyz-coordinates. It turns out that here it is simpler to calculate the infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ directly, rather than calculate $\mathbf{n}$ and dS separately and multiply them, as we did in the previous section. Below are the two standard forms for the equation of a surface, and the corresponding expressions for $d\mathbf{S}$. In the first we use z both for the dependent variable and the function which gives its dependence on x and y; you can use $f(x, y)$ for the function if you prefer, but that's one more letter to keep track of.

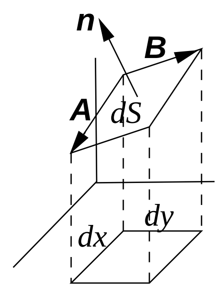


(11a) $\quad z = z(x, y), \qquad d\mathbf{S} = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k})\,dx\,dy \qquad$ ($\mathbf{n}$ points "up")

(11b) $\quad F(x, y, z) = c, \qquad d\mathbf{S} = \pm\dfrac{\nabla F}{F_z}\,dx\,dy \qquad$ (choose the right sign);

Derivation of formulas for $d\mathbf{S}$.

Refer to the pictures at the right. The surface S lies over its projection R, a region in the xy-plane. We divide up R into infinitesimal rectangles having area $dx\,dy$ and sides parallel to the xy-axes — one of these is shown. Over it lies a piece dS of the surface, which is approximately a parallelogram, since its sides are approximately parallel.

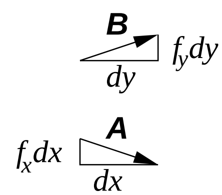


The infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ we are looking for has

direction: perpendicular to the surface, in the "up" direction;
magnitude: the area dS of the infinitesimal parallelogram.

This shows our infinitesimal vector is the cross-product

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B}$$



where $\mathbf{A}$ and $\mathbf{B}$ are the two infinitesimal vectors forming adjacent sides of the parallelogram. To calculate these vectors, from the definition of the partial derivative, we have

$\mathbf{A}$ lies over the vector $dx\,\mathbf{i}$ and has slope f_x in the $\mathbf{i}$ direction, so $\mathbf{A} = dx\,\mathbf{i} + f_x\,dx\,\mathbf{k}$;
$\mathbf{B}$ lies over the vector $dy\,\mathbf{j}$ and has slope f_y in the $\mathbf{j}$ direction, so $\mathbf{B} = dy\,\mathbf{j} + f_y\,dy\,\mathbf{k}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k})\,dx\,dy\ ,$$

which is (11a).

To get (11b) from (11a), , our surface is given by

$$F(x, y, z) = c, \qquad z = z(x, y) \tag{12}$$

where the right-hand equation is the result of solving $F(x, y, z) = c$ for z in terms of the independent variables x and y. We differentiate the left-hand equation in (12) with respect to the independent variables x and y, using the chain rule and remembering that $z = z(x, y)$:

$$F(x, y, z) = c \quad \Rightarrow \quad F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad F_x + F_z\frac{\partial z}{\partial x} = 0$$

from which we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \text{and similarly,} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \ .$$

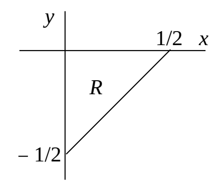
Therefore by (11a),

$$d\mathbf{S} \ = \ \left(-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + 1\right) dx\,dy \ = \ \left(\frac{F_x}{F_z}\,\mathbf{i} + \frac{F_y}{F_z}\,\mathbf{j} + 1\right) dx\,dy \ = \ \frac{\nabla F}{F_z}\,dx\,dy \ ,$$

which is (11b).

Example 3. The portion of the plane $2x - 2y + z = 1$ lying in the first octant forms a triangle S. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through S; take the positive side of S as the one where the normal points "up".

Solution. Writing the plane in the form $z = 1 - 2x + 2y$, we get using (11a),



$$\begin{aligned} d\mathbf{S} \ &= \ (2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy \ , \qquad \text{so} \\ \iint_S \mathbf{F}\cdot d\mathbf{S} \ &= \ \iint_S (2x - 2y + z)\,dy\,dx \\ &= \ \iint_R \big(2x - 2y + (1 - 2x + 2y)\big)\,dy\,dx \ , \end{aligned}$$

where R is the region in the xy-plane over which S lies. (Note that since the integration is to be in terms of x and y, we had to express z in terms of x and y for this last step.) To see what R is explicitly, the plane intersects the three coordinate axes respectively at $x = 1/2,\ y = -1/2,\ z = 1$. So R is the region pictured; our integral has integrand 1, so its value is the area of R, which is $1/8$.

Remark. When we write $z = f(x, y)$ or $z = z(x, y)$, we are agreeing to parametrize our surface using x and y as parameters. Thus the flux integral will be reduced to a double integral over a region R in the xy-plane, involving only x and y. Therefore you must *get rid of z by using the relation $z = z(x, y)$* after you have calculated the flux integral using (11a). Then determine the region R (the projection of S onto the xy-plane), and supply the limits for the iterated integral over R.

Example 4. Set up a double integral in the xy-plane which gives the flux of the field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through that portion of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ lying in the first octant; take $\mathbf{n}$ in the "up" direction.

Solution. Using (11b), we have $d\,\mathbf{S} = \dfrac{\langle 8x, 2y, 8z\rangle}{8z}\,dx\,dy$. Therefore

$$\iint_S \mathbf{F}\cdot d\,\mathbf{S} = \iint_S \frac{8x^2 + 2y^2 + 8z^2}{8z}\,dx\,dy = \iint_S \frac{1}{z}\,dx\,dy = \iint_R \frac{dx\,dy}{\sqrt{1 - x^2 - (y/2)^2}} \ ,$$

where R is the portion of the ellipse $4x^2 + y^2 = 4$ lying the the first quadrant.

The double integral would be most simply evaluated by making the change of variable $u = y/2$, which would convert it to a double integral over a quarter circle in the xu-plane easily evaluated by a change to polar coordinates.

4. General surface integrals.* The surface integral $\iint_S f(x,y,z)\,dS$ that we introduced at the beginning can be used to calculate things other than flux.

a) **Surface area.** We let the function $f(x,y,z)=1$. Then the area of $S\ =\ \iint_S dS$.

b) **Mass, moments, charge.** If S is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by $\delta(x,y,z)$, then

$$\text{mass of } S\ =\ \iint_S \delta(x,y,z)\,dS, \tag{13}$$

$$x\text{-component of center of mass}\ =\ \overline{x}\ =\ \frac{1}{\text{mass } S}\iint_S x\cdot\delta\,dS \tag{14}$$

with the y- and z-components of the center of mass defined similarly. If $\delta(x,y,z)$ represents an electric charge density, then the surface integral (13) will give the total charge on S.

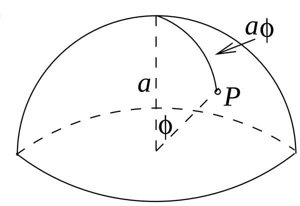
c) **Average value.** The average value of a function $f(x,y,z)$ over the surface S can be calculated by a surface integral:

$$\text{average value of } f \text{ on } S\ =\ \frac{1}{\text{area } S}\iint_S f(x,y,z)\,dS\ . \tag{15}$$

Calculating general surface integrals; finding dS.

To evaluate general surface integrals we need to know dS for the surface. For a sphere or cylinder, we can use the methods in section 2 of this chapter.

Example 5. Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius a.)



Solution. — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at $z=a$ on the z-axis. The distance of the point (a,ϕ,θ) from $(a,0,0)$ is $a\phi$, measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere S. Integrating, and using (9), we get

$$\iint_S a\phi\,dS\ =\ \int_0^{2\pi}\int_0^{\pi/2} a\phi a^2\sin\phi\,d\phi\,d\theta\ =\ 2\pi a^3\int_0^{\pi/2}\phi\,\sin\phi\,d\phi\ =\ 2\pi a^3\ .$$

(The last integral used integration by parts.) Since the area of $S=2\pi a^2$, we get using (15) the striking answer: average distance $=a$.

For more general surfaces given in xyz-coordinates, since $d\mathbf{S}=\mathbf{n}\,dS$, the area element dS is the magnitude of $d\mathbf{S}$. Using (11a) and (11b), this tells us

$$z=z(x,y), \qquad dS=\sqrt{z_x^2+z_y^2+1}\;dx\,dy \tag{16a}$$

$$F(x,y,z)=c, \qquad dS=\frac{|\nabla F|}{|F_z|}\;dx\,dy \tag{16b}$$

Example 6. The area of the piece S of $z=xy$ lying over the unit circle R in the xy-plane is calculated by (a) above and (16a) to be:

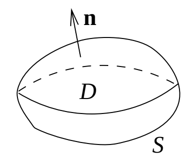
$$\iint_S dS=\iint_R\sqrt{y^2+x^2+1}\;dx\,dy=\int_0^{2\pi}\int_0^1\sqrt{r^2+1}\,r\,dr\,d\theta=2\pi\cdot\frac{1}{3}(r^2+1)^{3/2}\Big]_0^1=\frac{2\pi}{3}(2\sqrt{2}-1).$$

Exercises: Section 6B

V10. The Divergence Theorem

1. Introduction; statement of the theorem.

The divergence theorem is about closed surfaces, so let's start there. By a **closed** surface S we will mean a surface consisting of one connected piece which doesn't intersect itself, and which completely encloses a single finite region D of space called its *interior*. The closed surface S is then said to be the *boundary* of D; we include S in D. A sphere, cube, and torus (an inflated bicycle inner tube) are all examples of closed surfaces. On the other hand, these are not closed surfaces: a plane, a sphere with one point removed, a tin can whose cross-section looks like a figure-8 (it intersects itself), an infinite cylinder.



A closed surface always has two sides, and it has a natural positive direction — the one for which $\mathbf{n}$ points away from the interior, i.e., points toward the outside. We shall always understand that the closed surface has been oriented this way, unless otherwise specified.

We now generalize to 3-space the normal form of Green's theorem (Section V4).

Definition. Let $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field differentiable in some region D. By the **divergence** of $\mathbf{F}$ we mean the scalar function $\operatorname{div} \mathbf{F}$ of three variables defined in D by

$$\text{(1)} \qquad \operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} .$$

The divergence theorem. *Let S be a positively-oriented closed surface with interior D, and let $\mathbf{F}$ be a vector field continuously differentiable in a domain contatining D. Then*

$$\text{(2)} \qquad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

We write dV on the right side, rather than $dx\,dy\,dz$ since the triple integral is often calculated in other coordinate systems, particularly spherical coordinates. The theorem is sometimes called **Gauss' theorem**.

Physically, the divergence theorem is interpreted just like the normal form for Green's theorem. Think of $\mathbf{F}$ as a three-dimensional flow field. Look first at the left side of (2). The surface integral represents the mass transport rate across the closed surface S, with flow out of S considered as positive, flow into S as negative.

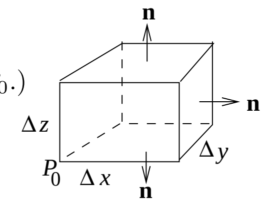
Look now at the right side of (2). In what follows, we will show that the value of div $\mathbf{F}$ at (x, y, z) can be interpreted as the **source rate** at (x, y, z): the rate at which fluid is being added to the flow at this point. (Negative rate means fluid is being removed from the flow.) The integral on the right of (2) thus represents the *source rate for D*. So what the divergence theorem says is:

$$\text{(3)} \qquad \textit{flux across } S = \textit{source rate for } D\ ;$$

i.e., the net flow outward across S is the same as the rate at which fluid is being produced (or added to the flow) inside S.

To complete the argument for (3) we still have to show that

(3) $$\text{div } \mathbf{F} \;=\; \textit{source rate at } (x,y,z)\ .$$



To see this, let $P_0 : (x_0, y_0, z_0)$ be a point inside the region D where $\mathbf{F}$ is defined. (To simplify, we denote by $(\text{div } \mathbf{F})_0, (\partial M/\partial x)_0$, etc., the value of these functions at P_0.)

Consider a little rectangular box, with edges $\Delta x, \Delta y, \Delta z$ parallel to the coordinate axes, and one corner at P_0. We take $\mathbf{n}$ to be always pointing outwards, as usual; thus on top of the box $\mathbf{n} = \mathbf{k}$, but on the bottom face, $\mathbf{n} = -\mathbf{k}$.

The flux across the top face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0 + \Delta z) \cdot \mathbf{k}\ \Delta x \Delta y \;=\; P(x_0, y_0, z_0 + \Delta z)\ \Delta x \Delta y,$$

while the flux across the bottom face in the $\mathbf{n}$ direction is approximately

$$\mathbf{F}(x_0, y_0, z_0) \cdot -\mathbf{k}\ \Delta x \Delta y \;=\; -P(x_0, y_0, z_0)\ \Delta x \Delta y\ .$$

So the net flux across the two faces combined is approximately

$$[P(x_0, y_0, z_0 + \Delta z) - P(x_0, y_0, z_0)]\ \Delta x \Delta y \;=\; \left(\frac{\Delta P}{\Delta z}\right) \Delta x \Delta y \Delta z\ .$$

Since the difference quotient is approximately equal to the partial derivative, we get the first line below; the reasoning for the following two lines is analogous:

$$\begin{aligned}
\textit{net flux across top and bottom} &\approx \left(\frac{\partial P}{\partial z}\right)_0 \Delta x \Delta y \Delta z; \\
\textit{net flux across two side faces} &\approx \left(\frac{\partial N}{\partial y}\right)_0 \Delta x \Delta y \Delta z; \\
\textit{net flux across front and back} &\approx \left(\frac{\partial M}{\partial x}\right)_0 \Delta x \Delta y \Delta z;
\end{aligned}$$

Adding up these three net fluxes, and using (3), we see that

$$\begin{aligned}
\textit{source rate for box} &= \textit{net flux across faces of box} \\
&\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)_0 \Delta x \Delta y \Delta z\ .
\end{aligned}$$

Using this, we get the interpretation for div $\mathbf{F}$ we are seeking:

$$\textit{source rate at } P_0 \;=\; \lim_{\text{box}\to 0} \frac{\textit{source rate for box}}{\textit{volume of box}} \;=\; (\text{div } \mathbf{F})_0\ .$$

Example 1. Verify the theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the sphere $\rho = a$.

Solution. For the sphere, $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; thus $\mathbf{F} \cdot \mathbf{n} = a$, and $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi a^3$.

On the other side, div $\mathbf{F} = 3$, $\iiint_D 3\,dV = 3\cdot\frac{4}{3}\pi a^3$; thus the two integrals are equal. □

Example 2. Use the divergence theorem to evaluate the flux of $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the sphere $\rho = a$.

Solution. Here div $\mathbf{F} = 3(x^2+y^2+z^2) = 3\rho^2$. Therefore by (2),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} \;=\; 3\iiint_D \rho^2\,dV \;=\; 3\int_0^a \rho^2\cdot 4\pi\rho^2\,d\rho \;=\; \frac{12\pi a^5}{5}\;;$$

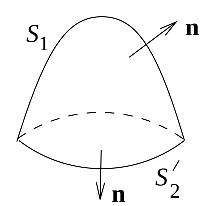
we did the triple integration by dividing up the sphere into thin concentric spheres, having volume $dV = 4\pi\rho^2\,d\rho$.

Example 3. Let S_1 be that portion of the surface of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, and let S_2 be the part of the xy-plane lying inside the unit circle, directed so the normal $\mathbf{n}$ points upwards. Take $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$; evaluate the flux of $\mathbf{F}$ across S_1 by using the divergence theorem to relate it to the flux across S_2.

Solution. We see immediately that div $\mathbf{F} = 0$. Therefore, if we let S_2' be the same surface as S_2, but oppositely oriented (so $\mathbf{n}$ points downwards), the surface $S_1 + S_2'$ is a closed surface, with $\mathbf{n}$ pointing outwards everywhere. Hence by the divergence theorem,

$$\iint_{S_1+S_2'} \mathbf{F}\cdot d\mathbf{S} \;=\; 0 \;=\; \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$$

Therefore, since we have $\mathbf{n} = \mathbf{k}$ on S_2,



$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} \;=\; \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} \;=\; \iint_{S_2} \mathbf{F}\cdot\mathbf{k}\;dS \;=\; \iint_{S_2} xy\,dx\,dy$$
$$=\; 0,$$

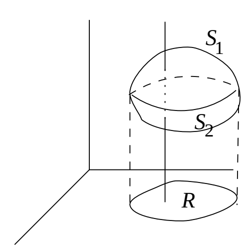
by integrating in polar coordinates (or by symmetry).

2. Proof of the divergence theorem.

We give an argument assuming first that the vector field $\mathbf{F}$ has only a $\mathbf{k}$-component: $\mathbf{F} = P(x,y,z)\,\mathbf{k}$. The theorem then says

$$\text{(4)} \qquad \iint_S P\,\mathbf{k}\cdot\mathbf{n}\,dS \;=\; \iiint_D \frac{\partial P}{\partial z}\,dV\ .$$

The closed surface S projects into a region R in the xy-plane. We assume S is vertically simple, i.e., that each vertical line over the interior of R intersects S just twice. (S can have vertical sides, however — a cylinder would be an example.) S is then described by two equations:



(5) $\quad z = g(x,y)$ *(lower surface);* $\qquad z = h(x,y)$ *(upper surface)*

The strategy of the proof of (4) will be to reduce each side of (4) to a double integral over R; the two double integrals will then turn out to be the same.

We do this first for the triple integral on the right of (4). Evaluating it by iteration, we get as the first step in the iteration,

$$\begin{aligned} \iiint_D \frac{\partial P}{\partial z}\, dV &= \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\, dz\, dx\, dy \\ &= \iint_R \big(P(x,y,h) - P(x,y,g)\big)\, dx\, dy \end{aligned} \tag{6}$$

To calculate the surface integral on the left of (4), we use the formula for the surface area element $d\mathbf{S}$ given in V9, (13):

$$d\mathbf{S} \;=\; \pm(-z_x\,\mathbf{i} - z_y\,\mathbf{j} + k)\, dx\, dy,$$

where we use the + sign if the normal vector to S has a positive k-component, i.e., points generally upwards (as on the upper surface here), and the − sign if it points generally downwards (as it does for the lower surface here).

This gives for the flux of the field $P\,\mathbf{k}$ across the upper surface S_2, on which $z = h(x,y)$,

$$\iint_{S_2} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,z)\, dx\, dy \;=\; \iint_R P(x,y,h(x,y))\, dx\, dy\ ,$$

while for the flux across the lower surface S_1, where $z = g(x,y)$ and we use the − sign as described above, we get

$$\iint_{S_1} P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R -P(x,y,z)\, dx\, dy \;=\; \iint_R -P(x,y,g(x,y))\, dx\, dy\ ;$$

adding up the two fluxes to get the total flux across S, we have

$$\iint_S P\,\mathbf{k}\cdot d\mathbf{S} \;=\; \iint_R P(x,y,h)\, dx\, dy \;-\; \iint_R P(x,y,g)\, dx\, dy$$

which is the same as the double integral in (6). This proves (4). □

In the same way, if $\mathbf{F} = M(x,y,z)\ \mathbf{i}$ and the surface is simple in the $\mathbf{i}$ direction, we can prove

$$\iint_S M\,\mathbf{i}\cdot\mathbf{n}\, dS \;=\; \iiint_D \frac{\partial M}{\partial x}\, dV \tag{4$'$}$$

while if $\mathbf{F} = N(x,y,z)\ \mathbf{j}$ and the surface is simple in the $\mathbf{j}$ direction,

$$\iint_S N\,\mathbf{j}\cdot\mathbf{n}\, dS \;=\; \iiint_D \frac{\partial N}{\partial y}\, dV\ . \tag{4$''$}$$

Finally, for a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and a closed surface S which is simple in all three directions, we have only to add up (4), (4′), and (4″). and we get the divergence theorem.

If the domain D is not bounded by a closed surface which is simple in all three directions, it can usually be divided up into smaller domains D_i which are bounded by such surfaces S_i. Adding these up gives the divergence theorem for D and S, since the surface integrals over the new faces introduced by cutting up D each occur twice, with the opposite normal vectors $\mathbf{n}$, so that they cancel out; after addition, one ends up just with the surface integral over the original S.

Exercises: Section 6C

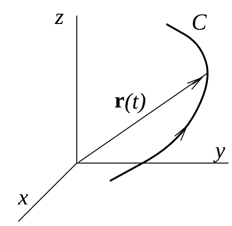
V11. Line Integrals in Space

1. Curves in space.

In order to generalize to three-space our earlier work with line integrals in the plane, we begin by recalling the relevant facts about parametrized space curves.

In 3-space, a vector function of one variable is given as

$$\mathbf{r}(t) \;=\; x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\ . \tag{1}$$

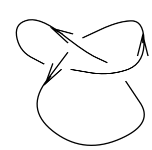


It is called *continuous* or *differentiable* or *continuously differentiable* if respectively $x(t), y(t)$, and $z(t)$ all have the corresponding property. By placing the vector so that its tail is at the origin, its head moves along a curve C as t varies. This curve can be described therefore either by its position vector function (1), or by the three parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t)\ . \tag{2}$$

The curves we will deal with will be finite, connected, and piecewise smooth; this means that they have finite length, they consist of one piece, and they can be subdivided into a finite number of smaller pieces, each of which is given as the position vector of a continuously differentiable function (i.e., one whose derivative is continuous).

In addition, the curves will be *oriented*, or *directed*, meaning that an arrow has been placed on them to indicate which direction is considered to be the positive one. The curve is called *closed* if a point P moving on it always in the positive direction ultimately returns to its starting position, as in the accompanying picture.



The derivative of $\mathbf{r}(t)$ is defined in terms of components by

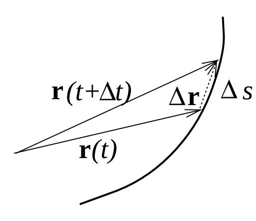
$$\frac{d\mathbf{r}}{dt} \;=\; \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\ . \tag{3}$$

If the parameter t represents time, we can think of $d\mathbf{r}/dt$ as the *velocity vector* $\mathbf{v}$. If we let s denote the arclength along C, measured from some fixed starting point in the positive direction, then in terms of s the magnitude and direction of $\mathbf{v}$ are given by

$$|\mathbf{v}| \;=\; \left|\frac{ds}{dt}\right|, \qquad \text{dir } \mathbf{v} \;=\; \begin{cases} \mathbf{t}, & \text{if } ds/dt > 0; \\ -\mathbf{t}, & \text{if } ds/dt < 0\ . \end{cases} \tag{4}$$

Here $\mathbf{t}$ is the unit tangent vector (pointing in the positive direction on C :

$$\mathbf{t} \;=\; \frac{d\mathbf{r}}{ds} \;=\; \frac{d\mathbf{r}/dt}{ds/dt}\ . \tag{5}$$



You can see from the picture that $\mathbf{t}$ is a unit vector, since

$$\left|\frac{d\mathbf{r}}{ds}\right| \;=\; \lim_{\Delta s \to 0} \left|\frac{\Delta \mathbf{r}}{\Delta s}\right| \;=\; 1\ .$$

2. Line integrals in space. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field in space, assumed continuous.

We define the *line integral of the tangential component of* $\mathbf{F}$ along an oriented curve C in space in the same way as for the plane. We approximate C by an inscribed sequence of directed line segments $\Delta\mathbf{r}_k$, form the approximating sum, then pass to the limit:

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \lim_{k\to\infty}\sum_k \mathbf{r}_k\cdot\Delta\mathbf{r}_k\ .$$

The line integral is calculated just like the one in two dimensions:

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{t_0}^{t_1}\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt\ , \tag{6}$$

if C is given by the position vector function $\mathbf{r}(t)$, $\quad t_0 \le t \le t_1$. Using x, y, z-components, one would write (6) as

$$\int_C M\,dx + N\,dy + P\,dz \;=\; \int_{t_0}^{t_1}\left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt \tag{6$'$}$$

In particular, if the parameter is the arclength s, then (6) becomes (since $\mathbf{t} = d\mathbf{r}/ds$)

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{s_0}^{s_1}\mathbf{F}\cdot\mathbf{t}\,ds\ , \tag{7}$$

which shows that the line integral is the integral along C of the tangential component of $\mathbf{F}$. As in two dimensions, this line integral represents the work done by the field $\mathbf{F}$ carrying a unit point mass (or charge) along the curve C.

Example 1. Find the work done by the electrostatic force field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ in carrying a positive unit point charge from $(1,1,1)$ to $(2,4,8)$ along
a) a line segment b) the twisted cubic curve $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

Solution. a) The line segment is given parametrically by

$$x-1=t,\quad y-1=3t,\quad z-1=7t,\qquad 0\le t\le 1\ .$$

$$\begin{aligned}\int_C y\,dx + z\,dy + x\,dz &= \int_0^1 (3t+1)\,dt + (7t+1)\cdot 3\,dt + (t+1)\cdot 7\,dt, \quad \text{using } (6')\\ &= \int_0^1 (31t+11)\,dt \;=\; \frac{31}{2}t^2 + 11t\Big]_0^1 \;=\; \frac{31}{2} + 11 = 26.5\ .\end{aligned}$$

b) Here the curve is given by $x = t,\ y = t^2,\ z = t^3, \quad 1 \le t \le 2$. For this curve, the line integral is

$$\begin{aligned}\int_1^2 t^2dt + t^3\cdot 2t\,dt + t\cdot 3t^2dt &= \int_1^2 (t^2 + 3t^3 + 2t^4)dt\\ &= \frac{t^3}{3} + \frac{3t^4}{4} + \frac{2t^5}{5}\Big]_1^2 \;\approx\; 25.18\ .\end{aligned}$$

The different results for the two paths shows that for this field, the line integral between two points depends on the path.

3. Gradient fields and path-independence.

The two-dimensional theory developed for line integrals in the plane generalizes easily to three-space. For the part where no new ideas are involved, we will be brief, just stating the results, and in places sketching the proofs.

Definition. Let $\mathbf{F}$ be a continuous vector field in a region D of space. The line integral $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is called **path-independent** if, for any two points P and Q in the region D, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a directed curve C lying in D and running from P to Q depends only on the two endpoints, and not on C.

An equivalent formulation is (the proof of equivalence is the same as before):

$$(8) \qquad \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path independent } \Leftrightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed curve } C \text{ in } D$$

Definition Let $f(x, y, z)$ be continuously differentiable in a region D. The vector field

$$(9) \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is called the **gradient field** of f in D. Any field of the form ∇f is called a gradient field.

Theorem. First fundamental theorem of calculus for line integrals. *If $f(x, y, z)$ is continuously differentiable in a region D, then for any two points P_1, P_2 lying in D,*

$$(10) \qquad \int_{P_1}^{P_2} \nabla f \cdot d\mathbf{r} = f(P_2) - f(P_1),$$

where the integral is taken along any curve C lying in D and running from P_1 to P_2. In particular, the line integral is path-independent.

The proof is exactly the same as before — use the chain rule to reduce it to the first fundamental theorem of calculus for functions of one variable.

There is also an analogue of the second fundamental theorem of calculus, the one where we first integrate, then differentiate.

Theorem. Second fundamental theorem of calculus for line integrals.

Let $\mathbf{F}(x, y, z)$ be continuous and $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ path-independent in a region D; and define

$$(11) \qquad \begin{aligned} f(x, y, z) &= \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}; \qquad \textit{then} \\ \nabla f &= \mathbf{F} \qquad \textit{in } D. \end{aligned}$$

Note that since the integral is path-independent, no C need be specified in (11). The theorem is proved in your book for line integrals in the plane. The proof for line integrals in space is analogous.

Just as before, these two theorems produce the three equivalent statements: in D,

$$\text{(12)} \qquad \mathbf{F} = \nabla f \quad \Leftrightarrow \quad \int_P^Q \mathbf{F} \cdot d\mathbf{r} \ \text{ path-independent} \quad \Leftrightarrow \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed } C$$

As in the two-dimensional case, if $\mathbf{F}$ is thought of as a force field, then the gradient force fields are called *conservative* fields, since the work done going around any closed path is zero (i.e., energy is conserved). If $\mathbf{F} = \nabla f$, then f is the called the (mathematical) *potential function* for $\mathbf{F}$; the physical potential function is defined to be $-f$.

Example 2. Let $f(x, y, z) = (x + y^2)z$. Calculate $\mathbf{F} = \nabla f$, and find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the helix $x = \cos t, y = \sin t, z = t, 0 \le t \le \pi$.

Solution. By differentiating, $\mathbf{F} = z\,\mathbf{i} + 2yz\,\mathbf{j} + (x + y^2)\,\mathbf{k}$. The curve C runs from $(1, 0, 0)$ to $(-1, 0, \pi)$. Therefore by (10),

$$\int_C \mathbf{F} \cdot d\mathbf{r} \;=\; (x + y^2)z \Big]_{(1,0,0)}^{(-1,0,\pi)} \;=\; -\pi - 0 \;=\; -\pi.$$

No direct calculation of the line integral is needed, notice.

Exercises: Section 6D

V12. Gradient Fields in Space

1. The criterion for gradient fields. The curl in space.

We seek now to generalize to space our earlier criterion (Section V2) for gradient fields in the plane.

Criterion for a Gradient Field. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be continuously differentiable. Then

$$(1)\quad \mathbf{F} = \nabla f \quad \text{for some } f(x,y,z) \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\,, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}\,, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}\,.$$

Proof. Since $\mathbf{F} = \nabla f$, when written out this says

$$(2)\qquad M = \frac{\partial f}{\partial x}, \qquad N = \frac{\partial f}{\partial y}. \qquad P = \frac{\partial f}{\partial z}\,; \qquad \text{therefore}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}\,.$$

The two mixed partial derivatives are equal since they are continuous, by the hypothesis that $\mathbf{F}$ is continuously differentiable.

The other two equalities in (1) are proved similarly. □

Though the criterion looks more complicated to remember and to check than the one in two dimensions, which involves just a single equation, it is not difficult to learn and apply. For theoretical purposes, it can be expressed more elegantly by using the three-dimensional vector curl $\mathbf{F}$.

Definition. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be differentiable. We define **curl F** by

$$(3)\qquad \text{curl}\,\mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}$$

$$(3')\qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad \text{(symbolic notation; } \partial_x = \frac{\partial}{\partial x}\text{, etc.)}$$

$$(3'')\qquad = \nabla \times \mathbf{F}, \qquad \text{where } \nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\,.$$

The equation (3) is the definition. The other two lines give symbolic ways of writing and of remembering the right side of (3). Neither the first nor second row of the determinant contains the sort of thing you are allowed to put into a determinant; however, if you "evaluate" it using the Laplace expansion by the first row, what you get is the right side of (3). Similarly, to evaluate the symbolic cross-product in (3″), we use the determinant (3′). In doing these, by the "product" of $\dfrac{\partial}{\partial x}$ and M we mean $\dfrac{\partial M}{\partial x}$.

By using the vector field curl $\mathbf{F}$, our criterion (1) becomes

$$(1') \qquad \mathbf{F} = \nabla f \quad \Rightarrow \quad \text{curl}\, \mathbf{F} = \mathbf{0}\ .$$

In dealing with a plane vector field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, we gave the name curl $\mathbf{F}$ to the *scalar* function $N_x - M_y$, whereas for a vector field $\mathbf{F}$ in space, curl $\mathbf{F}$ is a *vector* function. However, if we think of the two-dimensional field $\mathbf{F}$ as a field in space (i.e., one with zero $\mathbf{k}$-component and not depending on z), then using definition (3) you can compute that

$$\text{curl}\, \mathbf{F} = (N_x - M_y)\,\mathbf{k}\ .$$

Thus curl $\mathbf{F}$ has only a $\mathbf{k}$-component, so if we are dealing just with two-dimensional fields, it is natural to give the name curl $\mathbf{F}$ just to this $\mathbf{k}$-component. This is not a universally accepted terminology, however; some call it the "scalar curl", others don't use any name at all for $N_x - M_y$.

Naturally, the question arises as to whether the converse of (1′) is true — if curl $\mathbf{F} = \mathbf{0}$, is $\mathbf{F}$ a gradient field? As in two dimensionas, this requires some sort of restriction on the domain, and we will return to this point after we have studied Stokes' theorem. For now we will assume the domain is the whole three-space, in which case it is true:

Theorem. *If* $\mathbf{F}$ *is continuously differentiable for all* x, y, z,

$$(4) \qquad \textit{curl}\, \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f, \qquad \textit{for some differentiable } f(x,y,z).$$

We will prove this later. If $\mathbf{F}$ is a gradient field, we can calculate the corresponding (mathematical) potential function $f(x,y,z)$ by the three-dimensional analogue of either of the two methods described before (Section V2). We illustrate with an example.

Example 1. For what value(s), if any, of c will $\mathbf{F} = y\,\mathbf{i} + (x + cyz)\,\mathbf{j} + (y^2 + z^2)\,\mathbf{k}$ be a conservative (i.e., gradient) field? For each such c, find a corresponding potential function $f(x,y,z)$.

Solution. Using (1) and (4), we calculate the relevant partial derivatives:

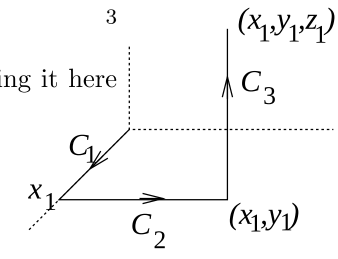
$$M_y = 1,\ N_x = 1; \quad N_z = cy,\ P_y = 2y; \quad M_z = 0,\ P_x = 0$$

Thus all three equations in (1) are satisfied $\Leftrightarrow$ $c = 2$. For this value of c, we now find $f(x,y,z)$ by two methods.

Method 1. We use the second fundamental theorem (Section V11, (11)), taking $(0,0,0)$ as a convenient lower limit for the integral, and using the subscript 1 on the upper limit to avoid confusion with the variables of integration. This gives

$$(5) \qquad f(x_1,y_1,z_1) = \int_{(0,0,0)}^{(x_1,y_1,z_1)} y\,dx + (x + 2yz)\,dy + (y^2 + z^2)\,dz$$

Since the integral is path-independent for the choice $c = 2$, we can use any path. The usual choice is the path illustrated, consisting of three line segments C_1, C_2 and C_3. The



parametrizations for them are (don't write these out yourself — we are only doing it here this first time to make it clear how the line integral is being calculated):

$$\begin{array}{lll} C_1: & x = x,\ y = 0,\ z = 0; & \text{thus}\ \ dx = dx,\ dy = 0,\ dz = 0; \\ C_2: & x = x_1,\ y = y,\ z = 0; & \text{thus}\ \ dx = 0,\ dy = dy,\ dz = 0; \\ C_3: & x = x_1,\ y = y_1,\ z = z; & \text{thus}\ \ dx = 0,\ dy = 0,\ dz = dz\ . \end{array}$$

Using these, we calculate the line integral (5) over each of the C_i in turn:

$$\begin{aligned} \int_{C_1+C_2+C_3} y\,dx + (x+2yz)\,dy + (y^2+z^2)\,dz \ &= \ \int_0^{x_1} 0\cdot dx \ + \ \int_0^{y_1}(x_1 + 2y\cdot 0)\,dy \ + \ \int_0^{z_1}(y_1^2+z^2)\,dz \\ &= \ 0 + x_1y_1 + (y_1^2 z_1 + \frac{1}{3}z_1^3)\ . \end{aligned}$$

Dropping subscripts, we have therefore by (5),

$$f(x,y,z) \ = \ xy + y^2z + \frac{1}{3}z^3 + c, \tag{6}$$

where we have added an arbitrary constant of integration to compensate for our arbitary choice of $(0,0,0)$ as the lower limit of integration — a different choice would have added a constant to the right side of (6).

The work should always be checked; from (6) one sees easily that $\nabla f \ = \ \mathbf{F}$, the field we started with.

Method 2. This requires no line integrals, but the work must be carried out systematically, otherwise you'll get lost in a mess of equations.

We are looking for an $f(x,y,z)$ such that $(f_x, f_y, f_z) \ = \ (y,\ x+2yz,\ y^2+z^2)$. This is equivalent to the three equations

$$f_x \ = \ y, \qquad f_y \ = \ x + 2yz, \qquad f_z \ = \ y^2+z^2\ . \tag{7}$$

From the first equation, integrating with respect to x (holding y and z fixed), we get

$$\begin{array}{lrcll}
(8) & f(x,y,z) & = & xy + g(y,z), & g \text{ is an arbitrary function} \\
& \dfrac{\partial f}{\partial y} & = & x + \dfrac{\partial g}{\partial y}, & \text{from (8)} \\
& & = & x + 2yz & \text{from (7), second equation; comparing,} \\
& \dfrac{\partial g}{\partial y} & = & 2yz. & \text{Integrating with respect to } y, \\
& g(y,z) & = & y^2z + h(z), & h \text{ is an arbitrary function; thus} \\
(9) & f(x,y,z) & = & xy + y^2z + h(z), & \text{from the preceding and (8)} \\
& \dfrac{\partial f}{\partial z} & = & y^2 + h'(z) & \\
& & = & y^2 + z^2, & \text{from (7), third equation; comparing,} \\
& h'(z) & = & z^2, & \\
& h(z) & = & \dfrac{1}{3}z^3 + c; & \text{finally, by (9)} \\
& f(x,y,z) & = & xy + y^2z + \dfrac{1}{3}z^3 + c & \text{as in Method 1.}
\end{array}$$

2. Exact differentials

Just as we did in the two-dimensional case, we translate the previous ideas into the language of differentials.

The formal expression

$$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz \tag{10}$$

which appears as the integrand in our line integrals is called a **differential**. If $f(x,y,z)$ is a differentiable function, then its **total differential** (or just *differential*) is defined to be

$$df \;=\; \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{11}$$

The differential (10) is said to be **exact**, in some domain D where M, N and P are defined, if it is the total differential of some differentiable function $f(x,y,z)$ in this domain, that is, if there exists an $f(x,y,z)$ in D such that

$$M \;=\; \frac{\partial f}{\partial x}, \quad N \;=\; \frac{\partial f}{\partial y}, \quad P \;=\; \frac{\partial f}{\partial z}\ . \tag{12}$$

Criterion for exact differentials. *Let D be a domain in which M, N, P are continuously differentiable. Then in D,*

$$M\,dx + N\,dy + P\,dz \quad \text{is exact} \quad \Rightarrow \quad P_y = N_z,\quad M_z = P_x,\quad N_x = M_y\ ; \tag{13}$$

if D is all of 3-space, then the converse is true:

$$P_y = N_z,\quad M_z = P_x,\quad N_x = M_y \quad \Rightarrow \quad M\,dx + N\,dy + P\,dz \quad \text{is exact.} \tag{14}$$

If the test in this criterion shows that the differential (10) is exact, the function $f(x,y,z)$ may be found be either method 1 or method 2. The converse (14) is true under weaker hypotheses about D, which we will come back to after we have taken up Stokes' Theorem.

Exercises: Section 6E

V13. Stokes' Theorem

1. Introduction; statement of the theorem.

The normal form of Green's theorem generalizes in 3-space to the divergence theorem. What is the generalization to space of the tangential form of Green's theorem? It says

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_R \operatorname{curl} \mathbf{F}\, dA \tag{1}$$

where C is a simple closed curve enclosing the plane region R.

Since the left side represents work done going around a closed curve in the plane, its natural generalization to space would be the integral $\oint \mathbf{F} \cdot d\mathbf{r}$ representing work done going around a closed curve in 3-space.

In trying to generalize the right-hand side of (1), the space curve C can only be the boundary of some piece of surface S — which of course will no longer be a piece of a plane. So it is natural to look for a generalization of the form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S (\text{something derived from } \mathbf{F})\, dS$$

The surface integral on the right should have these properties:

a) If $\operatorname{curl} \mathbf{F} = 0$ in 3-space, then the surface integral should be 0; (for $\mathbf{F}$ is then a gradient field, by V12, (4), so the line integral is 0, by V11, (12)).

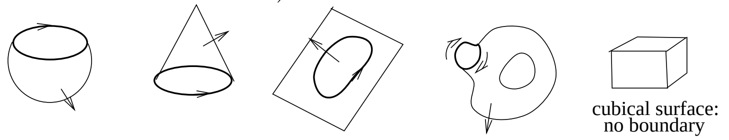
b) If C is in the xy-plane with S as its interior, and the field $\mathbf{F}$ does not depend on z and has only a $\mathbf{k}$-component, the right-hand side should be $\displaystyle\iint_S \operatorname{curl} \mathbf{F}\, dS$.

These things suggest that the theorem we are looking for in space is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \qquad\qquad \textbf{Stokes' theorem} \tag{2}$$

For the hypotheses, first of all C should be a closed curve, since it is the boundary of S, and it should be oriented, since we have to calculate a line integral over it.

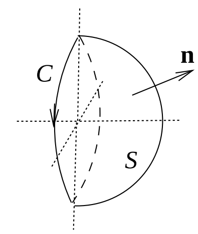
S is an oriented surface, since we have to calculate the flux of curl F through it. This means that S is two-sided, and one of the sides designated as positive; then the unit normal $\mathbf{n}$ is the one whose base is on the positive side. (There is no "standard" choice for positive side, since the surface S is not closed.)



It is important that C and S be *compatibly* oriented. By this we mean that the right-hand rule applies: when you walk in the positive direction on C, keeping S to your left, then your head should point in the direction of $\mathbf{n}$. The pictures give some examples.

The field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ should have continuous first partial derivatives, so that we will be able to integrate curl F. For the same reason, the piece of surface S should be piecewise smooth and should be finite— i.e., not go off to infinity in any direction, and have finite area.

2. Examples and discussion.



Example 1. Verify the equality in Stokes' theorem when S is the half of the unit sphere centered at the origin on which $y \geq 0$, oriented so $\mathbf{n}$ makes an acute angle with the positive y-axis; take $\mathbf{F} = y\,\mathbf{i} + 2x\,\mathbf{j} + x\,\mathbf{k}$.

Solution. The picture illustrates C and S. Notice how C must be directed to make its orientation compatible with that of S.

We turn to the line integral first. C is a circle in the xz-plane, traced out *clockwise* in the plane. We select a parametrization and calculate:

$$x = \cos t, \qquad y = 0, \qquad z = -\sin t, \qquad\qquad 0 \leq t \leq 2\pi \ .$$

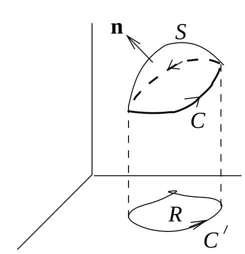
$$\oint_C y\,dx + 2x\,dy + x\,dz \ = \ \oint_C x\,dz \ = \ \int_0^{2\pi} -\cos^2 t\,dt \ = \ \left[-\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} \ = \ -\pi \ .$$

For the surface S, we see by inspection that $\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; this is a unit vector since $x^2 + y^2 + z^2 = 1$ on S. We calculate

$$\text{curl}\,\mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2x & x \end{vmatrix} \ = \ -\mathbf{j} + \mathbf{k}\,; \qquad (\text{curl}\,\mathbf{F}) \cdot \mathbf{n} = -y + z$$

Integrating in spherical coordinates, we have $y = \sin\phi\sin\theta$, $z = \cos\phi$, $dS = \sin\phi\,d\phi\,d\theta$, since $\rho = 1$ on S; therefore

$$\begin{aligned} \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} \ &= \ \iint_S (-y + z)\,dS \\ &= \ \int_0^\pi \int_0^\pi (-\sin\phi\sin\theta + \cos\phi)\sin\phi\,d\phi\,d\theta; \\ \text{inner integral} \ &= \ \sin\theta\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + \frac{1}{2}\sin^2\phi\bigg]_0^\pi \ = \ \frac{\pi}{2}\sin\theta \\ \text{outer integral} \ &= -\frac{\pi}{2}\cos\theta\bigg]_0^\pi \ = \ -\pi\ , \qquad \text{which checks.} \end{aligned}$$



Example 2. Suppose $\mathbf{F} = x^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and S is given as the graph of some function $z = g(x,y)$, oriented so $\mathbf{n}$ points upwards.

Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ $ *area of* R, where C is the boundary of S, compatibly oriented, and R is the projection of S onto the xy-plane.

Solution. We have $\text{curl}\,\mathbf{F} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & x & z^2 \end{vmatrix} \ = \mathbf{k}$. By Stokes' theorem, (cf. V9, (12))

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ \iint_S \mathbf{k} \cdot \mathbf{n}\,dS \ = \ \iint_R \mathbf{n} \cdot \mathbf{k}\,\frac{dA}{|\mathbf{n} \cdot \mathbf{k}|},$$

since $\mathbf{n}\cdot\mathbf{k} > 0$, $\quad |\mathbf{n}\cdot\mathbf{k}| = \mathbf{n}\cdot\mathbf{k}$; therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_R dA \;=\; \text{area of } R\,.$$

The relation of Stokes' theorem to Green's theorem.

Suppose $\mathbf{F}$ is a vector field in space, having the form $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$, and C is a simple closed curve in the xy-plane, oriented positively (so the interior is on your left as you walk upright in the positive direction). Let S be its interior, compatibly oriented — this means that the unit normal $\mathbf{n}$ to S is the vector $\mathbf{k}$, and $dS = dA$.

Then we get by the usual determinant method curl $\mathbf{F} = (N_x - M_y)\,\mathbf{k}$; since $\mathbf{n} = \mathbf{k}$, Stokes theorem becomes

$$\oint \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS \;=\; \iint_R (N_x - M_y)\,dA\;,$$

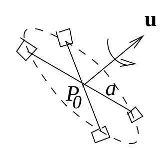
which is Green's theorem in the plane.

The same is true for other choices of the two variables; the most interesting one is $F = M(x,z)\,\mathbf{i} + P(x,z)\,\mathbf{k}$, where C is a simple closed curve in the xz-plane. If careful attention is paid to the choice of normal vector and the orientations, once again Stokes' theorem becomes just Green's theorem for the xz-plane. (See the Exercises.)

Interpretation of curl F.

Suppose now that $\mathbf{F}$ represents the velocity vector field for a three-dimensional fluid flow. Drawing on the interpretation we gave for the two-dimensional curl in Section V4, we can give the analog for 3-space.

The essential step is to interpret the $\mathbf{u}$-component of $(\text{curl}\,\mathbf{F})_0$ at a point P_0, , where $\mathbf{u}$ is a given unit vector, placed so its tail is at P_0.



Put a little paddlewheel of radius a in the flow so that its center is at P_0 and its axis points in the direction $\mathbf{u}$. Then by applying Stokes' theorem to a little circle C of radius a and center at P_0, lying in the plane through P_0 and having normal direction $\mathbf{u}$, we get just as in Section V4 (p. 4) that

$$\begin{aligned}\text{angular velocity of the paddlewheel} \;&=\; \frac{1}{2\pi a^2}\oint_C \mathbf{F}\cdot d\mathbf{r}\;;\\ &=\; \frac{1}{2\pi a^2}\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{u}\,dS,\end{aligned}$$

by Stokes' theorem, S being the circular disc having C as boundary;

$$\approx\; \frac{1}{2\pi a^2}(\text{curl}\,\mathbf{F})_0\cdot\mathbf{u}\;(\pi a^2),$$

since curl $\mathbf{F}\cdot\mathbf{u}$ is approximately constant on S if a is small, and S has area πa^2; passing to the limit as $a \to 0$, the approximation becomes an equality:

$$\text{angular velocity of the paddlewheel} \;=\; \frac{1}{2}\,(\text{curl}\,\mathbf{F})\cdot\mathbf{u}\;.$$

The preceding interprets $(\text{curl } \mathbf{F})_0 \cdot \mathbf{u}$ for us. Since it has its maximum value when $\mathbf{u}$ has the direction of $(\text{curl } \mathbf{F})_0$, we conclude

$$\begin{aligned} \text{direction of } (\text{curl } \mathbf{F})_0 &= \text{axial direction in which wheel spins fastest} \\ \text{magnitude of } (\text{curl } \mathbf{F})_0 &= \text{twice this maximum angular velocity.} \end{aligned}$$

3. Proof of Stokes' Theorem.

We will prove Stokes' theorem for a vector field of the form $P(x, y, z)\,\mathbf{k}$. That is, we will show, with the usual notations,

$$\oint_C P(x,y,z)\,dz = \iint_S \text{curl } (P\,\mathbf{k}) \cdot \mathbf{n}\,dS\ . \tag{3}$$

We assume S is given as the graph of $z = f(x, y)$ over a region R of the xy-plane; we let C be the boundary of S, and C' the boundary of R. We take $\mathbf{n}$ on S to be pointing generally upwards, so that $|\mathbf{n} \cdot \mathbf{k}| = \mathbf{n} \cdot \mathbf{k}$.

To prove (3), we turn the left side into a line integral around C', and the right side into a double integral over R, both in the xy-plane. Then we show that these two integrals are equal by Green's theorem.

To calculate the line integrals around C and C', we parametrize these curves. Let

$$C' :\ x = x(t),\ y = y(t), \qquad\qquad t_0 \le t \le t_1$$

be a parametrization of the curve C' in the xy-plane; then

$$C :\ x = x(t),\ y = y(t),\ z = f(x(t), y(t)), \quad t_0 \le t \le t_1$$

gives a corresponding parametrization of the space curve C lying over it, since C lies on the surface $z = f(x, y)$.

Attacking the line integral first, we claim that

$$\oint_C P(x,y,z)\,dz = \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy)\ . \tag{4}$$

This looks reasonable purely formally, since we get the right side by substituting into the left side the expressions for z and dz in terms of x and y: $z = f(x, y)$, $\quad dz = f_x dx + f_y dy$. To justify it more carefully, we use the parametrizations given above for C and C' to calculate the line integrals.

$$\begin{aligned} \oint_C P(x,y,z)\,dz &= \int_{t_0}^{t_1} \big(P(x(t), y(t), z(t)\big)\,\frac{dz}{dt}\,dt \\ &= \int_{t_0}^{t_1} \big(P(x(t), y(t), z(t)\big)\,(f_x \frac{dx}{dt} + f_y \frac{dy}{dt})\,dt, \text{ by the chain rule} \\ &= \oint_{C'} P(x,y,f(x,y))(f_x dx + f_y dy), \quad \text{the right side of (4).} \end{aligned}$$

We now calculate the surface integral on the right side of (3), using x and y as the variables. In the calculation, we must distinguish carefully between such expressions as $P_1(x, y, f)$ and $\frac{\partial}{\partial x}P(x, y, f)$. The first of these means: calculate the partial derivative with respect to the first variable x, treating x, y, z as independent; then substitute $f(x, y)$ for z. The second means: calculate the partial with respect to x, after making the substitution $z = f(x, y)$; the answer is

$$\frac{\partial}{\partial x}P(x, y, f) \;=\; P_1(x, y, f) + P_3(x, y, f)\, f_x\ .$$

(We use P_1 rather than P_x since the latter would be ambiguous — when you use numerical subscripts, everyone understands that the variables are being treated as independent.)

With this out of the way, the calculation of the surface integral is routine, using the standard procedure of an integral over a surface having the form $z = f(x, y)$ given in Section V9. We get

$$d\mathbf{S} \;=\; (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy, \qquad \text{by V9, (13);}$$

$$\text{curl } (P(x, y, z)\ \mathbf{k}\,) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & P \end{vmatrix} \;=\; P_2(x, y, z)\,\mathbf{i} - P_1(x, y, z)\,\mathbf{j}$$

$$\iint_S \text{curl } (P(x, y, z)\ \mathbf{k}\,)\cdot d\mathbf{S} \;=\; \iint_S \big(-P_2(x, y, z)f_x + P_1(x, y, z)f_y\big)\,dx\,dy$$

$$(5) \qquad \;=\; \iint_R \big(-P_2(x, y, f)f_x + P_1(x, y, f)f_y\big)\,dx\,dy$$

We have now turned the line integral into an integral around C' and the surface integral into a double integral over R. As the final step, we show that the right sides of (4) and (5) are equal by using Green's theorem:

$$\oint_{C'} U\,dx + V\,dy \;=\; \iint_R (V_x - U_y)\,dx\,dy\ .$$

(We have to state it using U and V rather than M and N, or P and Q, since in three-space we have been using these letters for the components of the general three-dimensional field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$.) To substitute into the two sides of Green's theorem, we need four functions:

$$\begin{aligned} V &= P(x, y, f(x, y))f_y, &\quad \text{so} \quad V_x &= (P_1 + P_3 f_x)f_y + P(x, y, f)f_{yx} \\ U &= P(x, y, f(x, y))f_x, &\quad \text{so} \quad U_y &= (P_2 + P_3 f_y)f_x + P(x, y, f)f_{xy} \end{aligned}$$

Therefore, since $f_{xy} = f_{yx}$, four terms cancel, and the right side of Green's theorem becomes

$$V_x - U_y \;=\; P_1(x, y, f)f_y - P_2(x, y, f)f_x,$$

which is precisely the integrand on the right side of (5). This completes the proof of Stokes' theorem when $\mathbf{F} = P(x, y, z)\,\mathbf{k}$.

In the same way, if $\mathbf{F} = M(x, y, z)\,\mathbf{i}$ and the surface is $x = g(y, z)$, we can reduce Stokes' theorem to Green's theorem in the yz-plane.

If $\mathbf{F} = N(x, y, z)\,\mathbf{j}$ and $y = h(x, z)$ is the surface, we can reduce Stokes' theorem to Green's theorem in the xz-plane.

Since a general field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ can be viewed as a sum of three fields, each of a special type for which Stokes' theorem is proved, we can add up the three Stokes' theorem equations of the form (3) to get Stokes' theorem for a general vector field.

A difficulty arises if the surface cannot be projected in a 1-1 way onto each of three coordinate planes in turn, so as to express it in the three forms needed above:

$$z = f(x, y), \qquad x = g(y, z), \qquad y = h(x, z)\ .$$

In this case, it can usually be divided up into smaller pieces which can be so expressed (if some of these are parallel to one of the coordinate planes, small modifications must be made in the argument). Stokes' theorem can then be applied to each piece of surface, then the separate equalities can be added up to get Stokes' theorem for the whole surface (in the addition, line integrals over the cut-lines cancel out, since they occur twice for each cut, in opposite directions). This completes the argument, *manus undulans*, for Stokes' theorem.

Exercises: Section 6F

V14. Some Topological Questions

We consider once again the criterion for a gradient field. We know that

$$\mathbf{F} = \nabla f \quad \Rightarrow \quad \text{curl } \mathbf{F} = \mathbf{0}\ , \tag{1}$$

and inquire about the converse. It is natural to try to prove that

$$\text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f \tag{2}$$

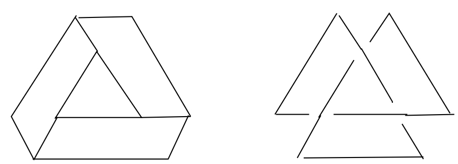
by using Stokes' theorem: if curl $\mathbf{F} = \mathbf{0}$, then for any closed curve C in space,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \ = \ 0. \tag{3}$$

The difficulty is that we are given C, but not S. So we have to ask:

Question. Let D be a region of space in which $\mathbf{F}$ is continuously differentiable. Given a closed curve lying in D; is it the boundary of some two-sided surface S lying inside D?

We explain the "two-sided" condition. Some surfaces are only one-sided: if you start painting them, you can use only one color, if you don't allow abrupt color changes. An example is S below, formed giving three half-twists to a long strip of paper before joining the ends together.



Surface S　　　　Boundary C (trefoil knot)

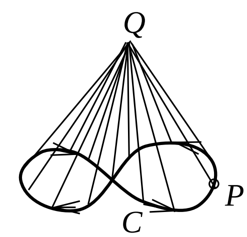
S has only one side. This means that it cannot be oriented: there is no continuous choice for the normal vector $\mathbf{n}$ over this surface. (If you start with a given $\mathbf{n}$ and make it vary continuously, when you return to the same spot after having gone all the way around, you will end up with $-\mathbf{n}$, the oppositely pointing vector.) For such surfaces, it makes no sense to speak of "the flux through S", because there is no consistent way of deciding on the positive direction for flow through the surface. Stokes' theorem does not apply to such surfaces.

To see what practical difficulties this causes, even when the domain is all of 3-space, consider the curve C in the above picture. It's called the trefoil knot. We know it is the boundary of the one-sided surface S, but this is no good for equation (3), which requires that we find a *two-sided* surface which has C for boundary.

> There are such surfaces; try to find one. It should be smooth and not cross itself. If successful, consider yourself a brown-belt topologist.

The preceding gives some ideas about the difficulties involved in finding a two-sided surface whose boundary is a closed curve C when the curve is knotted, i.e., cannot be continuously deformed into a circle without crossing itself at some point during the deformation. It is by no means clear that such a two-sided surface exists in general.

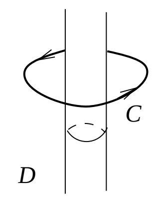
There are two ways out of the dilemma.



1. If we allow the surface to cross itself, and allow it to be not smooth along some lines, we can easily find such a two-sided surface whose boundary is a given closed curve C. The procedure is simple. Pick some fixed point Q not on the curve C, and join it to a point P on the curve (see the figure). Then as P moves around C, the line segment QP traces out a surface whose boundary is C. It will not be smooth at Q, and it will cross itself along a certain number of lines, but it's easy to see that this is a two-sided surface.

The point now is that Stokes' theorem can still be applied to such a surface: just use subdivision. Divide up the surface into skinny "triangles", each having one vertex at Q, and include among the edges of these triangles the lines where the surface crosses itself. Apply Stokes' theorem to each triangle, and add up the resulting equations.

2. Though the above is good enough for our purposes, it's an amazing fact that for any C there is always a smooth two-sided surface which doesn't cross itself, and whose boundary is C. (This was first proved around 1930 by van Kampen.)



The above at least answers our question affirmatively when D is all of 3-space. Suppose however that it isn't. If for instance D is the exterior of the cylinder $x^2 + y^2 = 1$, then it is intuitively clear that a circle C around the outside of this cylinder isn't the boundary of any finite surface lying entirely inside D.

A class of domains for which it is true however are the *simply-connected* ones.

Definition. A domain D in 3-space is **simply-connected** if each closed curve in it can be shrunk to a point without ever getting outside of D during the shrinking.

For example, 3-space itself is simply-connected, as is the interior or the exterior of a sphere. However the interior of a torus (a bagel, for instance) is not simply-connected, since any circle in it going around the hole cannot be shrunk to a point while staying inside the torus.

If D is simply-connected, then any closed curve C is the boundary of a two-sided surface (which may cross itself) lying entirely inside D. We can't prove this here, but it gives us the tool we need to establish the converse to the criterion for gradient fields in 3-space.

Theorem. *Let D be a simply-connected region in 3-space, and suppose that the vector field $\mathbf{F}$ is continuously differentiable in D. Then in D,*

$$\text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f\ . \tag{5}$$

Proof. According to the two fundamental theorems of calculus for line integrals (section V11.3), it is enough to prove that $\oint_C \mathbf{F} \cdot d\mathbf{r} \ = \ 0$ for every closed curve C lying in D.

Since D is simply-connected, given such a curve C, we can find a two-sided surface S lying entirely in D and having C as its boundary. Applying Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{R} \ = \ \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \ = \ 0,$$

which shows that $\mathbf{F}$ is conservative, and hence that $\mathbf{F}$ is a gradient field. $\square$

Summarizing, we can say that *if D is simply-connected, the following statements are equivalent—if one is true, so are the other two:*

$$(6) \qquad \mathbf{F} = \nabla f \quad \Leftrightarrow \quad \text{curl } \mathbf{F} = \mathbf{0} \quad \Leftrightarrow \quad \int_P^Q \mathbf{F} \cdot d\mathbf{r} \ \textit{ is path independent.}$$

Concluding remarks about Stokes' theorem.

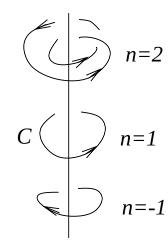
Just as problems of sources and sinks lead one to consider Green's theorem in the plane for regions which are not simply-connected, it is important to consider such domains in connection with Stokes' theorem.

For example, if we put a closed loop of wire in space, the exterior of this loop — the region consisting of 3-space with the wire removed — is not simply-connected. If the wire carries current, the resulting electromagnetic force field $\mathbf{F}$ will satisfy curl $\mathbf{F} = \mathbf{0}$, but $\mathbf{F}$ will not be conservative. In particular, the value of $\oint \mathbf{F} \cdot d\mathbf{r}$ around a closed path which links with the loop will *not* in general be zero, (which explains why you can get power from a wire carrying current, even though the curl of its electromagnetic field is zero).

As an example, consider the vector field in 3-space

$$\mathbf{F} \ = \ \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2}, \qquad\qquad r = \sqrt{x^2+y^2}\ .$$

The domain of definition is xyz-space, with the z-axis removed (since the z-axis is where $r = 0$). Just as before, (Section V2,p.2), we can calculate that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ if C is a suitably directed circle lying in a plane $z = z_0$ and centered on the z-axis.



Exercise. By using Stokes' theorem for a surface with more than one boundary curve, show informally that for the field above, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any closed curve C going once around the z-axis, oriented so when that when the right thumb points in the direction $\mathbf{k}$, the fingers curl in the positive direction on C. Then show that if C goes around n times, the value of the integral will be $2n\pi$.

Suppose now the wire is a closed curve that is *knotted*, i.e., it cannot be continuously deformed to a circle, without crossing itself at some point in the deformation. Let D be the exterior of the wire loop, and consider the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for a vector field $\mathbf{F}$ defined in D and having curl $\mathbf{F} = \mathbf{0}$. If one closed curve C_1 can be deformed into another closed curve C_2 without leaving D (i.e., without crossing the wire), then by using Stokes' theorem for surfaces with two boundary curves, we conclude

$$(6) \qquad \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} \ = \ \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}\ .$$

More generally, two closed curves C_1 and C_2 are called *homologous*, written $C_1 \sim C_2$, if C_1 and C_2' (this means C_2 with its direction reversed) form the complete boundary of some surface lying entirely in D. Then by an extended form of Stokes' theorem, (6) will be true whenever $C_1 \sim C_2$. Thus the problem of determining the possible values for the line integral is reduced to the purely topological problem of finding a set of closed curves in D, no two of which are homologous, but such that every other closed curve is homologous to one of the curves in the set. For any particular knot in 3-space, such a set can be determined by an algorithm, but if one asks for general results relating the appearance of the knot to the number of such basic curves that will be needed, one gets into unsolved problems of topology.

In another vein, the theorems of Green, Stokes, and Gauss (as the divergence theorem is often called) all relate an integral over the interior of some closed curve or surface with an integral over its boundary. There is a much more general result — the generalized Stokes' theorem — which connects an integral over an n-dimensional hypersurface with an integral taken over its $n-1$-dimensional boundary. Green's and Stokes' theorems are the case $n = 2$ of this result, while the divergence theorem is closely related to the case $n = 3$ in 3-space. Just as the theorems we have studied are the key to an understanding of geometry and analysis in the plane and space, so this theorem is central to an understanding of n-dimensional space, and more general sorts of spaces called "n-dimensional manifolds".

Exercises: Section 6G

V15. Relation to Physics

The three theorems we have studied: the divergence theorem and Stokes' theorem in space, and Green's theorem in the plane (which is really just a special case of Stokes' theorem) are widely used in physics and continuum mechanics, in the study of fields, potentials, heat flow, wave motion in liquids, gases, and solids, and thermodynamics, to name some of the uses. Often partial differential equations which model some physical situation are derived using the vector integral calculus theorems. This section is devoted to a brief account of where you will first meet the theorems: in electromagnetic theory.

1. Symbolic notation: the del operator

To have a compact notation, wide use is made of the symbolic operator "del" (some call it "nabla"):

$$\nabla \;=\; \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k} \tag{1}$$

Recall that the "product" of $\dfrac{\partial}{\partial x}$ and the function $M(x,y,z)$ is understood to be $\dfrac{\partial M}{\partial x}$. Then we have

$$\text{grad } f \;=\; \nabla f \;=\; \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \tag{2}$$

The divergence is sort of a symbolic scalar product: if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$,

$$\text{div } \mathbf{F} \;=\; \nabla\cdot\mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \tag{3}$$

while the curl, as we have noted, as a symbolic cross-product:

$$\text{curl } \mathbf{F} \;=\; \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}\;.$$

Notice how this notation reminds you that $\nabla\cdot\mathbf{F}$ is a scalar function, while $\nabla\times\mathbf{F}$ is a vector function.

We may also speak of the Laplace operator (also called the "Laplacian"), defined by

$$\text{lap } f \;=\; \nabla^2 f \;=\; (\nabla\cdot\nabla)\,f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\;. \tag{5}$$

Thus, Laplace's equation may be written: $\nabla^2 f = 0$. (This is for example the equation satisfied by the potential function for an electrostatic field, in any region of space where there are no charges; or for a gravitational field, in a region of space where there are no masses.)

In this notation, the divergence theorem and Stokes' theorem are respectively

$$(6) \qquad \iint_S \mathbf{F}\cdot d\mathbf{S} \;=\; \iiint_D \nabla\cdot\mathbf{F}\,dV \qquad\qquad \oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

Two important relations involving the symbolic operator are:

$$(7) \qquad \text{curl}\,(\text{grad}\,f) \;=\; \mathbf{0} \qquad\qquad \text{div curl}\,\mathbf{F} \;=\; 0$$

$$(7') \qquad \nabla\times\nabla f \;=\; \mathbf{0} \qquad\qquad \nabla\cdot\nabla\times\mathbf{F} \;=\; 0$$

The first we have proved (it was part of the criterion for gradient fields); the second is an easy exercise. Note however how the symbolic notation suggests the answer, since we know that for any vector $\mathbf{A}$, we have

$$\mathbf{A}\times\mathbf{A} \;=\; \mathbf{0}, \qquad\qquad \mathbf{A}\cdot\mathbf{A}\times\mathbf{F} \;=\; 0\,,$$

and $(7')$ says this is true for the symbolic vector ∇ as well.

2. Application to Maxwell's equations.

Each of Maxwell's equations in electromagnetic theory can be written in two equivalent forms: a differential form which involves only partial derivatives, and an integrated form involving line, surface, and other multiple integrals.

In a sense we have already seen this with our criterion for conservative fields; we assume $\mathbf{F}$ is continuously differentiable in all of 3-space. Then the integrated form of the criterion is on the left, and the differential form is on the right:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \quad \text{for all closed } C \quad\Leftrightarrow\quad \text{curl}\,\mathbf{F} = \mathbf{0} \quad \text{for all } x, y, z\,.$$

And we know that it is Stokes' theorem which provides the bridge between these two equivalent forms of the criterion.

The situation with respect to Maxwell's equations is similar. We consider here two of them, as typical.

Gauss-Coulomb Law. Let $\mathbf{E}$ be an electrostatic field, arising from a distribution in space of positive and negative electric charge. Then the Gauss-Coulomb Law may be written in either of the two forms

$$(8) \qquad \nabla\cdot\mathbf{E} \;=\; 4\pi\rho, \qquad \rho \;=\; \text{charge density}; \qquad \textit{(differential form)}$$

$$(8') \qquad \iint_S \mathbf{E}\cdot d\mathbf{S} \;=\; 4\pi Q, \qquad Q \;=\; \text{total net charge inside } S. \qquad \textit{(integrated form)}$$

These are two equivalent statements of the same physical law. The integrated form is perhaps a little easier to understand, since the left hand side is the flux of $\mathbf{E}$ through S, which is a more intuitive idea than div $\mathbf{E}$. On the other hand, quite a lot of technique is required actually to calculate the flux, whereas very little is needed to calculate the divergence.

Neither (8) nor $(8')$ is mathematics — both are empirically established laws of physics. But their *equivalence* is a purely mathematical statement that can be proved by using the divergence theorem.

Proof that (8) $\Rightarrow$ (8′)**.**

Let D be the interior of the closed surface S. Then

$$\begin{aligned}
\iint_S \mathbf{E}\cdot d\mathbf{S} &= \iiint_D \nabla\cdot\mathbf{E}\,dV && \text{by the divergence theorem;}\\
&= 4\pi\iiint_D \rho\,dV && \text{by (8)}\\
&= 4\pi Q\ , && \text{by definition of } \rho \text{ and } Q.
\end{aligned}$$

Proof that (8′) $\Rightarrow$ (8)**.**

We reason by contraposition: that is, we show that if (8) is false, then (8′) must also be false.

If (8) is false, this means that we can find some point $P_0 : (x_0, y_0, z_0)$ where $\mathbf{E}$ is defined and such that $\nabla\cdot\mathbf{E} \neq 4\pi\rho$ at P_0; we write this inequality as

$$\nabla\cdot\mathbf{E} - 4\pi\rho \neq 0, \quad \text{at } P_0.$$

Say the quantity on the left is positive at P_0. Then by continuity, it is also positive in the interior of a small sphere S_0 centered at P_0; call this interior B_0. Then

$$\iiint_{B_0} (\nabla\cdot\mathbf{E} - 4\pi\rho)\,dV > 0\ ,$$

which we write

$$\iiint_{B_0} \nabla\cdot\mathbf{E}\,dV > 4\pi\iiint_{B_0} \rho\,dV.$$

The integral on the right gives the total net charge Q_0 inside S_0; applying the divergence theorem to the integral on the left, we get

$$\iint_{S_0} \mathbf{E}\cdot d\mathbf{S} > 4\pi Q_0$$

which shows that (8′) is also false, since it fails for S_0. □

Faraday's Law A changing magnetic field $\mathbf{B}(x,y,z;t)$ produces an electric field $\mathbf{E}$. The relation between the two fields is given by Faraday's law, which can be stated (in suitable units) in two equivalent forms (c is the velocity of light):

(9) $$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad \textit{differential form}$$

(9′) $$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{c}\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S} \qquad \textit{integrated form}$$

As before, it is the integrated form which is more intuitive, though harder to calculate. The line integral on the left is called the *electromotive force* around the closed loop C; Faraday's law (9′) relates it to the magnetic flux through any surface S spanning the loop C.

A few comments on the two forms. The derivative in (9) is taken by just differentiating each component of $\mathbf{B}$ with respect to the time t. It is a partial derivative, since the components of $\mathbf{B}$ are also functions of x, y, z. In (9′) on the other hand we have an ordinary derivative, since after the integration, the flux is a function of t alone.

It is understood in physics that on S the positive direction for flux and the positive direction on C must be compatibly chosen.

The magnetic flux through S is the same for all surfaces S spanning the loop C. (This is a consequence of the physical law $\nabla\cdot\mathbf{B} = 0$.) As a result, one speaks simply of "the flux through the loop C", meaning the flux through any surface spanning C, i.e. having C as its boundary.

Once again, though (9) and (9′) both express the same physical law, the equivalence between them is a mathematical statement; to prove it we use Stokes' theorem.

Proof that (9) $\Rightarrow$ (9′)**,**

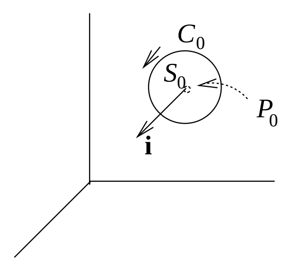
$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{E}\cdot d\mathbf{S}, &&\text{by Stokes' theorem,}\\ &= -\frac{1}{c}\iint_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}, &&\text{by (9)}\\ &= -\frac{1}{c}\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S}\ ,\end{aligned}$$

if $\mathbf{B}$ has a continuous derivative and S is smooth, and finite in extent and in area. (This last equality is fairly subtle, and is taken up in theoretical advanced calculus courses.) □

Proof that (9′) $\Rightarrow$ (9). We show that if (9) is false, then (9′) is false:

If (9) is false, this means that at some point P_0, $\nabla\times\mathbf{E} \ne -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$; we write this

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \ne \mathbf{0}. \tag{10}$$



This means that at least one of the components of this vector is not 0 at P_0; say it is the $\mathbf{i}$-component, and it's positive. Then by continuity it will remain positive in a small ball around P_0. Inside this little ball, draw a little disc S_0 as shown with center at P_0, having normal vector $\mathbf{i}$; orient its circular boundary C_0 compatibly.

Since the vector on the left in (10) has a positive $\mathbf{i}$-component on S_0,

$$\iint_{S_0}\left(\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} > 0,$$

which we may write

$$\iint_{S_0}\nabla\times\mathbf{E}\cdot d\mathbf{S} > -\frac{1}{c}\iint_{S_0}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S};$$

applying Stokes' theorem to the left-hand side, and interchanging the order of differentiation and integration on the right (this is valid under the reasonable hypotheses we stated before), we get

$$\oint_C \mathbf{E} \cdot d\mathbf{r} \;>\; -\frac{1}{c}\,\frac{d}{dt}\int\!\!\int_S \mathbf{B} \cdot d\mathbf{S},$$

which shows the integrated form $(9')$ is false for this little circle and disc, and therefore not true in general. $\square$

3. Harmonic functions in space.

A harmonic function in space is by definition a function $f(x, y, z)$ which satisfies Laplace's equation $\nabla^2 f = 0$, or written out (see (5)):

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \;=\; 0\ .$$

For example, the potential function for an electrostatic field $\mathbf{E}$ is harmonic in any region of space which is free of electrostatic charge. Similarly, the potential function for a gravitational field $\mathbf{F}$ is harmonic in any region where there is no mass. These statements are mathematical consequences of physical laws, and therefore are also physical laws — i.e., experimental facts, not mathematical facts.

To see why the potential function for $\mathbf{E}$ is harmonic, suppose we are in a simply-connected region of space where there is no charge. We then have

$$\begin{array}{rrcll} \textit{Gauss-Coulomb law} & \nabla\cdot\mathbf{E} & = & 0 & \text{since } \rho = 0 \text{ in the region} \\ \textit{Faraday's law} & \nabla\times\mathbf{E} & = & 0; & \end{array}$$

the second equation is valid since the field arises from a distribution of static electric charges — there is no changing magnetic field. Faraday's law shows that $\mathbf{E}$ is conservative, so that it has a (mathematical) potential function $f(x, y, z)$; the physical potential function would be $-f(x, y, z)$. By the Gauss-Coulomb law, noting that $\mathbf{E} = \nabla f$, we get

$$\nabla\cdot\mathbf{E} = \nabla\cdot\nabla f = 0, \qquad \text{or} \qquad \nabla^2 f \;=\; 0,$$

showing that $f(x, y, z)$ is a harmonic function.

Because harmonic functions can represent potential functions, there is great interest in finding harmonic functions in a region D of space. Typically, one prescribes the values that $f(x, y, z)$ should have on the boundary of D, and then searches analytically (or numerically by computer) for the values of $f(x, y, z)$ inside D. In this work, the divergence theorem gives an important theoretical tool; some of the Exercises use it to explore the situation a little further.

In general, this aspect of the subject properly belongs to the realm of partial differential equations, i.e., to Differential Equations and Advanced Calculus courses: see you there, maybe.

Exercises: Section 6H

1. Vectors and Matrices

1A. Vectors

Definition. *A* **direction** *is just a unit vector. The* **direction of A** *is defined by*

$$\text{dir } \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \qquad (\mathbf{A} \neq \mathbf{0});$$

it is the unit vector lying along **A** *and pointed like* **A** *(not like* $-\mathbf{A}$*).*

1A-1 Find the magnitude and direction (see the definition above) of the vectors

a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$ b) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ c) $3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

1A-2 For what value(s) of c will $\frac{1}{5}\mathbf{i} - \frac{1}{5}\mathbf{j} + c\mathbf{k}$ be a unit vector?

1A-3 a) If $P = (1, 3, -1)$ and $Q = (0, 1, 1)$, find $\mathbf{A} = PQ$, $|\mathbf{A}|$, and dir $\mathbf{A}$.

b) A vector $\mathbf{A}$ has magnitude 6 and direction $(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})/3$. If its tail is at $(-2, 0, 1)$, where is its head?

1A-4 a) Let P and Q be two points in space, and X the midpoint of the line segment PQ. Let O be an arbitrary fixed point; show that as vectors, $OX = \frac{1}{2}(OP + OQ)$.

b) With the notation of part (a), assume that X divides the line segment PQ in the ratio $r : s$, where $r + s = 1$. Derive an expression for OX in terms of OP and OQ.

1A-5 What are the $\mathbf{i}\,\mathbf{j}$-components of a plane vector $\mathbf{A}$ of length 3, if it makes an angle of 30^o with $\mathbf{i}$ and 60^o with $\mathbf{j}$. Is the second condition redundant?

1A-6 A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the $\mathbf{i}\,\mathbf{j}$-components).

1A-7 Let $\mathbf{A} = a\mathbf{i} + b\mathbf{j}$ be a plane vector; find in terms of a and b the vectors $\mathbf{A}'$ and $\mathbf{A}''$ resulting from rotating $\mathbf{A}$ by 90^o a) clockwise b) counterclockwise.

(Hint: make $\mathbf{A}$ the diagonal of a rectangle with sides on the x and y-axes, and rotate the whole rectangle.)

c) Let $\mathbf{i}' = (3\mathbf{i} + 4\mathbf{j})/5$. Show that $\mathbf{i}'$ is a unit vector, and use the first part of the exercise to find a vector $\mathbf{j}'$ such that $\mathbf{i}', \mathbf{j}'$ forms a right-handed coordinate system.

1A-8 The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a space vector, represented as an origin vector, and let α, β, and γ be the three angles ($\leq \pi$) that $\mathbf{A}$ makes respectively with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

a) Show that dir $\mathbf{A} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$. (The three coefficients are called the *direction cosines* of $\mathbf{A}$.)

b) Express the direction cosines of $\mathbf{A}$ in terms of a, b, c; find the direction cosines of the vector $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

c) Prove that three numbers t, u, v are the direction cosines of a vector in space if and only if they satisfy $t^2 + u^2 + v^2 = 1$.

1A-9 Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides $\mathbf{A}$ and $\mathbf{B}$.)

1A-10 Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

1A-11 Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show $X = Y$.)

1A-12* Label the four vertices of a parallelogram in counterclockwise order as OPQR. Prove that the line segment from O to the midpoint of PQ intersects the diagonal PR in a point X that is 1/3 of the way from P to R.

(Let $\mathbf{A} = \text{OP}$, and $\mathbf{B} = \text{OR}$; express everything in terms of $\mathbf{A}$ and $\mathbf{B}$.)

1A-13* a) Take a triangle PQR in the plane; prove that as vectors $PQ + QR + RP = \mathbf{0}$.

b) Continuing part a), let $\mathbf{A}$ be a vector the same length as PQ, but perpendicular to it, and pointing outside the triangle. Using similar vectors $\mathbf{B}$ and $\mathbf{C}$ for the other two sides, prove that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. (This only takes one sentence, and no computation.)

1A-14* Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn't cross itself (i.e., one whose interior is a single region); label its vertices $P_1, P_2, \ldots, P_n$ as you walk around it.

1A-15* Let $P_1, \ldots, P_n$ be the vertices of a regular n-gon in the plane, and O its center; show without computation or coordinates that $OP_1 + OP_2 + \ldots + OP_n = \mathbf{0}$,

a) if n is even; b) if n is odd.

1B. Dot Product

1B-1 Find the angle between the vectors

a) $\mathbf{i} - \mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ b) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

1B-2 Tell for what values of c the vectors $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ will

a) be orthogonal b) form an acute angle

1B-3 Using vectors, find the angle between a longest diagonal PQ of a cube, and

a) a diagonal PR of one of its faces; b) an edge PS of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

1B-4 Three points in space are $P : (a, 1, -1)$, $Q : (0, 1, 1)$, $R : (a, -1, 3)$. For what value(s) of a will PQR be

a) a right angle b) an acute angle ?

1B-5 Find the component of the force $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in

a) the direction $\dfrac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}$ b) the direction of the vector $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.

1B-6 Let O be the origin, c a given number, and $\mathbf{u}$ a given direction (i.e., a unit vector). Describe geometrically the locus of all points P in space that satisfy the vector equation

$$OP \cdot \mathbf{u} \;=\; c|OP|\;.$$

In particular, tell for what value(s) of c the locus will be (Hint: divide through by $|OP|$):

a) a plane b) a ray (i.e., a half-line) c) a point

1B-7 a) Verify that $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ and $\mathbf{j}' = \dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ are perpendicular unit vectors that form a right-handed coordinate system

b) Express the vector $\mathbf{A} = 2\,\mathbf{i} - 3\,\mathbf{j}$ in the $\mathbf{i}'\mathbf{j}'$-system by using the dot product.

c) Do b) a different way, by solving for $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{i}'$ and $\mathbf{j}'$ and then substituting into the expression for $\mathbf{A}$.

1B-8 The vectors $\mathbf{i}' = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$, $\mathbf{j}' = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$, and $\mathbf{k}' = \dfrac{\mathbf{i}+\mathbf{j}-2\,\mathbf{k}}{\sqrt{6}}$ are three mutually perpendicular unit vectors that form a right-handed coordinate system.

a) Verify this. b) Express $\mathbf{A} = 2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ in this system (cf. 1B-7b)

1B-9 Let $\mathbf{A}$ and $\mathbf{B}$ be two plane vectors, neither one of which is a multiple of the other. Express $\mathbf{B}$ as the sum of two vectors, one a multiple of $\mathbf{A}$, and the other perpendicular to $\mathbf{A}$; give the answer in terms of $\mathbf{A}$ and $\mathbf{B}$.

(Hint: let $\mathbf{u} = \text{dir } \mathbf{A}$; what's the $\mathbf{u}$-component of $\mathbf{B}$?)

1B-10 Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

1B-11 Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

1B-12 Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

1B-13 Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

(Hint: consider two unit vectors making angles θ_1 and θ_2 with the positive x-axis.)

1B-14 Prove the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$ by using the algebraic laws for the dot product and its geometric interpretation.

1B-15* The **Cauchy-Schwarz inequality**

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or space; under what circumstances does equality hold?

$$(*) \qquad |\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}||\mathbf{B}|\;.$$

b) If the vectors are plane vectors, write out what this inequality says in terms of $\mathbf{i}\,\mathbf{j}$-components.

c) Give a different argument for the inequality (*) as follows (this argument generalizes to n-dimensional space):

i) for all values of t, we have $(\mathbf{A}+t\mathbf{B})\cdot(\mathbf{A}+t\mathbf{B})\geq 0$;

ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in t ;

iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

1C. Determinants

1C-1 Calculate the value of the determinants a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

1C-2 Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

a) the first row b) the first column

1C-3 Find the area of the plane triangle whose vertices lie at

a) $(0,0),(1,2),(1,-1)$; b) $(1,2),(1,-1),(2,3)$.

1C-4 Show that $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1-x_2)(x_2-x_3)(x_3-x_1)$.

(This type of determinant is called a **Vandermonde** determinant.)

1C-5 a) Show that the value of a 2×2 determinant is unchanged if you add to the second row a scalar multiple of the first row.

b) Same question, with "row" replaced by "column".

1C-6 Use a Laplace expansion and Exercise 5a to show the value of a 3×3 determinant is unchanged if you add to the second row a scalar multiple of the third row.

1C-7 Let (x_1, y_1) and (x_2, y_2) both range over all unit vectors.

Find the maximum value of the function $f(x_1,x_2,y_1,y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

1C-8* The base of a parallelepiped is a parallelogram whose edges are the vectors **b** and **c**, while its third edge is the vector **a**. (All three vectors have their tail at the same vertex; one calls them "coterminal".)

a) Show that the volume of the parallelepiped **abc** is $\pm\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

b) Show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) =$ the determinant whose rows are respectively the components of the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$.

(These two parts prove (3), the volume interpretation of a 3×3 determinant.

1C-9 Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices $(0,0,0),(0,-1,2),(0,1,-1),(1,2,1)$. (The volume of a tetrahedron is $\frac{1}{3}$(base)(height).)

1C-10 Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

1D. Cross Product

1D-1 Find $\mathbf{A}\times\mathbf{B}$ if

a) $\mathbf{A}=\mathbf{i}-2\mathbf{j}+\mathbf{k}, \quad \mathbf{B}=2\mathbf{i}-\mathbf{j}-\mathbf{k}$ b) $\mathbf{A}=2\mathbf{i}-3\mathbf{k},\ \mathbf{B}=\mathbf{i}+\mathbf{j}-\mathbf{k}$.

1D-2 Find the area of the triangle in space having its vertices at the points

$$P:(2,0,1),\ Q:(3,1,0),\ R:(-1,1,-1).$$

1D-3 Two vectors $\mathbf{i}'$ and $\mathbf{j}'$ of a right-handed coordinate system are to have the directions respectively of the vectors $\mathbf{A}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{B}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Find all three vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$.

1D-4 Verify that the cross product $\times$ does not in general satisfy the associative law, by showing that for the particular vectors $\mathbf{i},\mathbf{i},\mathbf{j}$, we have $(\mathbf{i}\times\mathbf{i})\times\mathbf{j}\neq\mathbf{i}\times(\mathbf{i}\times\mathbf{j})$.

1D-5 What can you conclude about $\mathbf{A}$ and $\mathbf{B}$

a) if $|\mathbf{A}\times\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$; b) if $|\mathbf{A}\times\mathbf{B}|=\mathbf{A}\cdot\mathbf{B}$.

1D-6 Take three faces of a unit cube having a common vertex P; each face has a diagonal ending at P; what is the volume of the parallelepiped having these three diagonals as coterminous edges?

1D-7 Find the volume of the tetrahedron having vertices at the four points

$$P:(1,0,1),\ Q:(-1,1,2),\ R:(0,0,2),\ S:(3,1,-1).$$

Hint: volume of tetrahedron $=\frac{1}{6}$(volume of parallelepiped with same 3 coterminous edges)

1D-8 Prove that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

1D-9 Show that the area of a triangle in the xy-plane having vertices at (x_i,y_i), for $i=1,2,3$, is given by the determinant $\dfrac{1}{2}\text{abs}\left(\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\right)$. (Here abs means take absolute value.) Do this two ways:

a) by relating the area of the triangle to the volume of a certain parallelepiped

b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the 2×2 determinant that would normally be used to calculate the area.

1E. Equations of Lines and Planes

1E-1 Find the equations of the following planes:

a) through $(2, 0, -1)$ and perpendicular to $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
b) through the origin, $(1, 1, 0)$, and $(2, -1, 3)$
c) through $(1, 0, 1)$, $(2, -1, 2)$, $(-1, 3, 2)$
d) through the points on the x, y and z-axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
e) through $(1, 0, 1)$ and $(0, 1, 1)$ and parallel to $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

1E-2 Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.

1E-3 Find in parametric form the equations for

a) the line through $(1, 0, -1)$ and parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
b) the line through $(2, -1, -1)$ and perpendicular to the plane $x - y + 2z = 3$
c) all lines passing through $(1, 1, 1)$ and lying in the plane $x + 2y - z = 2$

1E-4 Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

1E-5 The line passing through $(1, 1, -1)$ and perpendicular to the plane $x + 2y - z = 3$ intersects the plane $2x - y + z = 1$ at what point?

1E-6 Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

(Hint: Let $\mathbf{n}$ be the unit normal to the plane. and P be a point on the plane; consider the component of OP in the direction $\mathbf{n}$.)

1E-7* Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

(Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren't, it could be shortened).)

1F. Matrix Algebra

1F-1* Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$. Compute

a) $B + C$, $B - C$, $2B - 3C$.

b) AB, AC, BA, CA, BC^T, CB^T

c) $A(B + C)$, $AB + AC$; $(B + C)A$, $BA + CA$

1F-2* Let A be an arbitrary $m \times n$ matrix, and let I_k be the identity matrix of size k. Verify that $I_m A = A$ and $AI_n = A$.

1F-3 Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1F-4* Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix $AB - BA$:

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$

.

1F-5 a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3. b) Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1F-6* Let A, A', B, B' be 2×2 matrices, and O the 2×2 zero matrix. Express in terms of these five matrices the product of the 4×4 matrices $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$.

1F-7* Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Show there are no values of a and b such that $AB - BA = I_2$.

1F-8 a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$ what is the 3×3 matrix A?

b)* If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix},$ what is A?

1F-9 A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that: each row of A is a row vector of length 1, *and* every pair of different rows contain orthogonal vectors.

1F-10* Suppose A is a 2×2 orthogonal matrix, whose first entry is $a_{11} = \cos\theta$. Fill in the rest of A. (There are four possibilities. Use Exercise 9.)

1F-11* a) Show that for any matrices A and B having the same dimensions, the identity $(A + B)^T = A^T + B^T$.

b) Show that for any matrices A and B such that AB is defined, then the identity $(AB)^T = B^T A^T$ holds.

1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-1* $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$.

1G-2* a) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; b) $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

1G-3 $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$ by finding A^{-1}.

1G-4 Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, \quad x_2 + x_3 = y_2 \quad -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i.

1G-5 Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.

1G-6* **Another calculation of the inverse matrix.**

If we know A^{-1}, we can solve the system $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by writing $\mathbf{x} = A^{-1}\mathbf{y}$. But conversely, if we can solve by some other method (elimination, say) for $\mathbf{x}$ in terms of $\mathbf{y}$, getting $\mathbf{x} = B\mathbf{y}$, then the matrix $B = A^{-1}$, and we will have found A^{-1}.

This is a good method if A is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

a) Let $A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$; find A^{-1} by solving $\begin{aligned} -x_1 + x_2 + 3x_3 &= y_1 \\ 2x_2 - x_3 &= y_2 \\ x_3 &= y_3 \end{aligned}$ for the x_i in terms of the y_i (start from the bottom and proceed upwards).

b) Calculate A^{-1} by the method given in the notes.

1G-7* Consider the rotation matrix $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ corresponding to rotation of the x and y axes through the angle θ. Calculate A_θ^{-1} by the adjoint matrix method, and explain why your answer looks the way it does.

1G-8* a) Show: A is an orthogonal matrix (cf. Exercise 1F-9) if and only if $A^{-1} = A^T$.

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if A and B are $n \times n$ orthogonal matrices, so is AB.

1G-9* a) Let A be a 3×3 matrix such that $|A| \neq 0$. The notes construct a right-inverse A^{-1}, that is, a matrix such that $A \cdot A^{-1} = I$. Show that every such matrix A also has a left inverse B (i.e., a matrix such that $BA = I$.)

(Hint: Consider the equation $A^T(A^T)^{-1} = I$; cf. Exercise 1F-11.)

b) Deduce that $B = A^{-1}$ by a one-line argument.

(This shows that the right inverse A^{-1} is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be I also.)

1G-10* Let A and B be two $n \times n$ matrices. Suppose that $B = P^{-1}AP$ for some invertible $n \times n$ matrix P. Show that $B^n = P^{-1}A^nP$. If $B = I_n$, what is A?

1G-11* Repeat Exercise 6a and 6b above, doing it this time for the general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assuming $|A| \neq 0$.

1H. Cramer's Rule; Theorems about Square Systems

1H-1 Use Cramer's rule to solve for x in the following:

$$\text{(a)}\quad \begin{aligned} 3x - y + z &= 1 \\ -x + 2y + z &= 2 \\ x - y + z &= -3 \end{aligned}\ , \qquad\qquad \text{(b)}\quad \begin{aligned} x - y + z &= 0 \\ x - z &= 1 \\ -x + y + z &= 2 \end{aligned}.$$

1H-2 Using Cramer's rule, give another proof that if A is an $n \times n$ matrix whose determinant is non-zero, then the equations $A\mathbf{x} = 0$ have only the trivial solution.

1H-3 a) For what c-value(s) will $\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ -x_1 + cx_2 + 2x_3 &= 0 \end{aligned}$ have a non-trivial solution?

b) For what c-value(s) will $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$ have a non-trivial solution? (Write it as a system of homogeneous equations.)

c) For each value of c in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)* For each value of c in part (b), find a non-trivial solution to the corresponding system.

1H-4* Find all solutions to the homogeneous system $\begin{aligned} x - 2y + z &= 0 \\ x + y - z &= 0 \\ 3x - 3y + z &= 0 \end{aligned}$; use the method suggested in Exercise 3c above.

1H-5 Suppose that for the system $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. Assume that $a_1 \neq 0$. Show that the system is consistent (i.e., has solutions) if and only if $c_2 = \dfrac{a_2}{a_1} c_1$.

1H-6* Suppose that $\mathbf{x}_1$ is a particular solution of the system $A\mathbf{x} = \mathbf{b}$.

a) Show that if $\mathbf{x}_0$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$.

b) Show that if one takes the set of solutions to $A\mathbf{x} = \mathbf{0}$ and adds $\mathbf{x}_1$ to each such solution then one obtains the set of solutions to $A\mathbf{x} = \mathbf{b}$.

1H-7 Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants a and b so that the function

$$f(x) = a \cos x + b \sin x$$

has prescribed values at two given x-values: $f(x_1) = y_1,\ f(x_2) = y_2$.

a) Show this is possible in one and only one way, if we assume that $x_2 \neq x_1 + n\pi$, for every integer n.

b) If $x_2 = x_1 + n\pi$ for some integer n, when can a and b be found?

1H-8* The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

$$\frac{ax+b}{(x-r_1)(x-r_2)} = \frac{c}{x-r_1} + \frac{d}{x-r_2}, \qquad (a, b, r_1, r_2 \text{ given; } c, d \text{ to be found});$$

what are the linear equations which determine the constants c and d? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots r_i, $i = 1, 2, 3$. Evaluate the determinant by using column operations to get zeros in the top row.)

1I. Vector Functions and Parametric Equations

1I-1 The point P moves with constant speed v in the direction of the constant vector $a\,\mathbf{i} + b\,\mathbf{j}$. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $\mathbf{r}(t)$?

1I-2 A point moves *clockwise* with constant angular velocity ω on the circle of radius a centered at the origin. What is its position vector function $\mathbf{r}(t)$, if at time $t = 0$ it is at
(a) $(a, 0)$ (b) $(0, a)$

1I-3 Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.

a) $\mathbf{r} = 2\cos^2 t\,\mathbf{i} + \sin^2 t\,\mathbf{j}$ b) $\mathbf{r} = \cos 2t\,\mathbf{i} + \cos t\,\mathbf{j}$ c) $\mathbf{r} = (t^2+1)\,\mathbf{i} + t^3\,\mathbf{j}$
d) $\mathbf{r} = \tan t\,\mathbf{i} + \sec t\,\mathbf{j}$

1I-4 A roll of plastic tape of outer radius a is held in a fixed position while the tape is being unwound counterclockwise. The end P of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin O, and the end P to be initially at $(a, 0)$, write parametric equations for the motion of P.
(Use vectors; express the position vector OP as a vector function of one variable.)

1I-5 A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.
(Use vectors; express the position vector OP as a vector function of one variable.)

1I-6 A bow-and-arrow hunter walks toward the origin along the positive x-axis, with unit speed; at time 0 he is at $x = 10$. His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity $\sqrt{5}$ in the first quadrant along the line $y = 2x$; at time 0 it is at the origin.

a) Write down the vector function $\mathbf{A}(t)$ for the arrow at time t.

b) The hunter shoots (and misses) when closest to the rabbit; when is that?

1I-7 The cycloid is the curve traced out by a fixed point P on a circle of radius a which rolls along the x-axis in the positive direction, starting when P is at the origin O. Find the

vector function OP; use as variable the angle θ through which the circle has rolled.
(Hint: begin by expressing OP as the sum of three simpler vector functions.)

1J. Differentiation of Vector Functions

1J-1 1. For each of the following vector functions of time, calculate the velocity, speed $|ds/dt|$, unit tangent vector (in the direction of velocity), and acceleration.

a) $e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ b) $t^2\,\mathbf{i} + t^3\,\mathbf{j}$ c) $(1-2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2+2t^2)\,\mathbf{k}$

1J-2 Let $OP = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$ be the position vector for a motion.

a) Calculate $\mathbf{v}$, $|ds/dt|$, and $\mathbf{T}$.

b) At what point in the speed greatest? smallest?

c) Find the xy-equation of the curve along which the point P is moving, and describe it geometrically.

1J-3 Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt}\,\mathbf{r}\cdot\mathbf{s} \;=\; \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\;,$$

by calculating with components, letting $\mathbf{r} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{s} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$.

1J-4 Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\,.$$

Show that the velocity vector $\mathbf{v}$ is always perpendicular to $\mathbf{r}$ two different ways:

a) using the x, y, z-coordinates

b) without coordinates (use the formula in **1J-3**, which is valid also in space).

c) Prove the converse: if $\mathbf{r}$ and $\mathbf{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin.

1J-5 a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**.)

b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point P moves with constant speed.

1J-6 For the helical motion $r(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$,

a) calculate $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, $|ds/dt|$

b) show that $\mathbf{v}$ and $\mathbf{a}$ are perpendicular; explain using **1J-5**

1J-7 a) Suppose you have a differentiable vector function $\mathbf{r}(t)$. How can you tell if the parameter t is the arclength s (measured from some point in the direction of increasing t) without actually having to calculate s explicitly?

b) How should a be chosen so that t is the arclength if $\mathbf{r}(t) = (x_0+at)\,\mathbf{i} + (y_0+at)\,\mathbf{j}$?

c) How should a and b be chosen so that t is the arclength in the helical motion described in Exercise **1J-6**?

1J-8 a) Prove the formula $\dfrac{d}{dt}\,u(t)\mathbf{r}(t) \;=\; \dfrac{du}{dt}\,\mathbf{r}(t) + u(t)\dfrac{d\mathbf{r}}{dt}$.

(You may assume the vectors are in the plane; calculate with the components.)

b) Let $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$, the exponential spiral. Use part (a) to find the speed of this motion.

1J-9 A point P is moving in space, with position vector

$$\mathbf{r} = OP = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k}$$

a) Show it moves on the surface of a sphere.
b) Show its speed is constant.
c) Show the acceleration is directed toward the origin.
d) Show it moves in a plane through the origin.
e) Describe the path of the point.

1J-10 The **positive curvature** κ of the vector function $\mathbf{r}(t)$ is defined by $\kappa \;=\; \left|\dfrac{d\mathbf{T}}{ds}\right|$.

a) Show that the helix of **1J-6** has constant curvature. (It is not necessary to calculate s explicitly; calculate $d\mathbf{T}/dt$ instead and relate it to κ by using the chain rule.)

b) What is this curvature if the helix is reduced to a circle in the xy-plane?

1K. Kepler's Second Law

1K-1 (Same as 1J-3). Prove the rule (1) in Notes K for differentiating the dot product of two plane vectors: do the calculation using an $\mathbf{i}\,\mathbf{j}$-coordinate system.
(Let $\mathbf{r}(t) = x_1(t)\,\mathbf{i} + y_1(t)\,\mathbf{j}$ and $\mathbf{s}(t) = x_2(t)\,\mathbf{i} + y_2(t)\,\mathbf{j}$.)

1K-2 Let $\mathbf{s}(t)$ be a vector function. Prove by using components that

$$\frac{d\mathbf{s}}{dt} = \mathbf{0} \quad\Rightarrow\quad \mathbf{s}(t) = \mathbf{K}, \qquad \text{where } \mathbf{K} \text{ is a constant vector.}$$

1K-3 In Notes K, by reversing the steps (5) – (8), prove the statement in the last paragraph. You will need the statement in exercise 1K-2.

2. Partial Differentiation

2A. Functions and Partial Derivatives

2A-1 Sketch five level curves for each of the following functions. Also, for a-d, sketch the portion of the graph of the function lying in the first octant; include in your sketch the traces of the graph in the three coordinate planes, if possible.

a) $1-x-y$ b) $\sqrt{x^2+y^2}$ c) x^2+y^2 d) $1-x^2-y^2$ e) x^2-y^2

2A-2 Calculate the first partial derivatives of each of the following functions:

a) $w = x^3y - 3xy^2 + 2y^2$ b) $z = \dfrac{x}{y}$ c) $\sin(3x+2y)$ d) e^{x^2y}

e) $z = x\ln(2x+y)$ f) $x^2z - 2yz^3$

2A-3 Verify that $f_{xy} = f_{yx}$ for each of the following:

a) x^my^n, (m, n positive integers)   b) $\dfrac{x}{x+y}$ c) $\cos(x^2+y)$

d) $f(x)g(y)$, for any differentiable f and g

2A-4 By using $f_{xy} = f_{yx}$, tell for what value of the constant a there exists a function $f(x,y)$ for which $f_x = axy + 3y^2$, $f_y = x^2 + 6xy$, and then using this value, find such a function by inspection.

2A-5 Show the following functions $w = f(x,y)$ satisfy the equation $w_{xx} + w_{yy} = 0$ (called the two-dimensional *Laplace equation*):

a) $w = e^{ax}\sin ay$ (a constant)   b) $w = \ln(x^2+y^2)$

2B. Tangent Plane; Linear Approximation

2B-1 Give the equation of the tangent plane to each of these surfaces at the point indicated.

a) $z = xy^2$, $(1,1,1)$ b) $w = y^2/x$, $(1,2,4)$

2B-2 a) Find the equation of the tangent plane to the cone $z = \sqrt{x^2+y^2}$ at the point $P_0 : (x_0, y_0, z_0)$ on the cone.

b) Write parametric equations for the ray from the origin passing through P_0, and using them, show the ray lies on both the cone and the tangent plane at P_0.

2B-3 Using the approximation formula, find the approximate change in the hypotenuse of a right triangle, if the legs, initially of length 3 and 4, are each increased by .010 .

2B-4 The combined resistance R of two wires in parallel, having resistances R_1 and R_2 respectively, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\ .$$

If the resistance in the wires are initially 1 and 2 ohms, with a possible error in each of $\pm.1$ ohm, what is the value of R, and by how much might this be in error? (Use the approximation formula.)

2B-5 Give the linearizations of each of the following functions at the indicated points:

a) $(x+y+2)^2$ at $(0,0)$; at $(1,2)$ b) $e^x\cos y$ at $(0,0)$; at $(0,\pi/2)$

2B-6 To determine the volume of a cylinder of radius around 2 and height around 3, about how accurately should the radius and height be measured for the error in the calculated volume not to exceed .1 ?

2B-7 a) If x and y are known to within .01, with what accuracy can the polar coordinates r and θ be calculated? Assume $x = 3,\ y = 4$.

b) At this point, are r and θ more sensitive to small changes in x or in y? Draw a picture showing x, y, r, θ and confirm your results by using geometric intuition.

2B-8* Two sides of a triangle are a and b, and θ is the included angle. The third side is c.

a) Give the approximation for Δc in terms of a, b, c, θ, and $\Delta a, \Delta b, \Delta\theta$.

b) If $a = 1,\ b = 2,\ \theta = \pi/3$, is c more sensitive to small changes in a or b?

2B-9 a) Around the point $(1, 0)$, is $w = x^2(y + 1)$ more sensitive to changes in x or in y?

b) What should the ratio of Δy to Δx be in order that small changes with this ratio produce *no* change in w, i.e., no first-order change — of course w will change a little, but like $(\Delta x)^2$, not like Δx.

2B-10* a) If $|a|$ is much larger than $|b|, |c|$, and $|d|$, to which entry is the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ most sensitive?

b) Given a 3×3 determinant, how would you determine to which entry the value of the determinant is most sensitive? (Consider the various Laplace expansions by the cofactors of a given row or column.)

2C. Differentials; Approximations

2C-1 Find the differential (dw or dz). Make the answer look as neat as possible.

a) $w = \ln(xyz)$ b) $w = x^3y^2z$ c) $z = \dfrac{x - y}{x + y}$ d) $w = \sin^{-1}\dfrac{u}{t}$ (use $\sqrt{t^2 - u^2}$)

2C-2 The dimensions of a rectangular box are 5, 10, and 20 cm., with a possible measurement error in each side of $\pm.1$ cm. Use differentials to find what possible error should be attached to its volume.

2C-3 Two sides of a triangle have lengths respectively a and b, with θ the included angle. Let A be the area of the triangle.

a) Express dA in terms of the variables and their differentials.

b) If $a = 1,\ b = 2,\ \theta = \pi/6$, to which variable is A most sensitive? least sensitive?

c) Using the values in (b), if the possible error in each value is .02, what is the possible error in A, to two decimal places?

2C-4 The pressure, volume, and temperature of an ideal gas confined to a container are related by the equation $PV = kT$, where k is a constant depending on the amount of gas and the units. Calculate dP two ways:

a) Express P in terms of V and T, and calculate dP as usual.

b) Calculate the differential of both sides of the equation, getting a "differential equation", and then solve it algebraically for dP.

c) Show the two answers agree.

2C-5 The following equations define w implicitly as a function of the other variables. Find dw in terms of all the variables by taking the differential of both sides and solving algebraically for dw.

a) $\frac{1}{w} = \frac{1}{t} + \frac{1}{u} + \frac{1}{v}$ b) $u^2 + 2v^2 + 3w^2 = 10$

2D. Gradient and Directional Derivative

2D-1 In each of the following, a function f, a point P, and a vector $\mathbf{A}$ are given. Calculate the gradient of f at the point, and the directional derivative $\left.\frac{df}{ds}\right|_{\mathbf{u}}$ at the point, in the direction $\mathbf{u}$ of the given vector $\mathbf{A}$.

a) $x^3 + 2y^3$; $(1,1)$, $\mathbf{i} - \mathbf{j}$ b) $w = \frac{xy}{z}$; $(2,-1,1)$, $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

c) $z = x\sin y + y\cos x$; $(0,\pi/2)$, $-3\mathbf{i} + 4\mathbf{j}$ d) $w = \ln(2t + 3u)$; $(-1,1)$, $4\mathbf{i} - 3\mathbf{j}$

e) $f(u,v,w) = (u + 2v + 3w)^2$; $(1,-1,1)$, $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

2D-2 For the following functions, each with a given point P,

(i) find the maximum and minimum values of $\left.\frac{dw}{ds}\right|_{\mathbf{u}}$, as $\mathbf{u}$ varies;

(ii) tell for which directions the maximum and minimum occur;

(iii) find the direction(s) $\mathbf{u}$ for which $\left.\frac{dw}{ds}\right|_{\mathbf{u}} = 0$.

a) $w = \ln(4x - 3y)$, $(1,1)$ b) $w = xy + yz + xz$, $(1,-1,2)$

c) $w = \sin^2(t - u)$, $(\pi/4, 0)$

2D-3 By viewing the following surfaces as a contour surface of a function $f(x,y,z)$, find its tangent plane at the given point.

a) $xy^2z^3 = 12$, $(3,2,1)$; b) the ellipsoid $x^2 + 4y^2 + 9z^2 = 14$, $(1,1,1)$

c) the cone $x^2 + y^2 - z^2 = 0$, (x_0, y_0, z_0) (simplify your answer)

2D-4 The function $T = \ln(x^2 + y^2)$ gives the temperature at each point in the plane (except $(0,0)$).

a) At the point $P : (1,2)$, in which direction should you go to get the most rapid increase in T?

b) At P, about how far should you go in the direction found in part (a) to get an increase of .20 in T?

c) At P, approximately how far should you go in the direction of $\mathbf{i} + \mathbf{j}$ to get an increase of about .12?

d) At P, in which direction(s) will the rate of change of temperature be 0?

2D-5 The function $T = x^2 + 2y^2 + 2z^2$ gives the temperature at each point in space.

a) What shape are the isotherms?.

b) At the point $P : (1,1,1)$, in which direction should you go to get the most rapid decrease in T?

c) At P, about how far should you go in the direction of part (b) to get a decrease of 1.2 in T?

d) At P, approximately how far should you go in the direction of $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ to get an increase of .10?

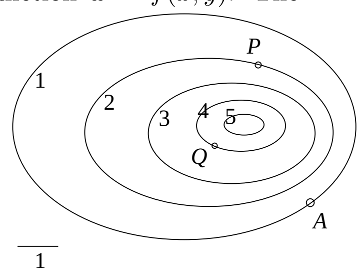
2D-6 Show that $\nabla(uv) = u\nabla v + v\nabla u$, and deduce that $\left.\dfrac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\dfrac{dv}{ds}\right|_{\mathbf{u}} + v\left.\dfrac{du}{ds}\right|_{\mathbf{u}}$.
(Assume that u and v are functions of two variables.)

2D-7 Suppose $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 2$, $\left.\dfrac{dw}{ds}\right|_{\mathbf{v}} = 1$ at P, where $\mathbf{u} = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$, $\mathbf{v} = \dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. Find $(\nabla w)_P$.

(This illustrates that the gradient can be calculated knowing the directional derivatives in any two non-parallel directions, not just the two standard directions $\mathbf{i}$ and $\mathbf{j}$.)

2D-8 The atmospheric pressure in a region of space near the origin is given by the formula $P = 30 + (x+1)(y+2)e^z$. Approximately where is the point closest to the origin at which the pressure is 31.1?

2D-9 The accompanying picture shows the level curves of a function $w = f(x,y)$. The value of w on each curve is marked. A unit distance is given.



a) Draw in the gradient vector at A.
b) Find a point B where $w = 3$ and $\partial w/\partial x = 0$.
c) Find a point C where $w = 3$ and $\partial w/\partial y = 0$.
d) At the point P estimate the value of $\partial w/\partial x$ and $\partial w/\partial y$.
e) At the point Q, estimate dw/ds in the direction of $\mathbf{i}+\mathbf{j}$
f) At the point Q, estimate dw/ds in the direction of $\mathbf{i}-\mathbf{j}$.
g) Approximately where is the gradient $\mathbf{0}$?

2E. Chain Rule

2E-1 In the following, find $\dfrac{dw}{dt}$ for the composite function $w = f(x(t), y(t), z(t))$ in two ways:

(i) use the chain rule, then express your answer in terms of t by using $x = x(t)$, etc.;
(ii) express the composite function f in terms of t, and differentiate.

a) $w = xyz$, $x = t$, $y = t^2$, $z = t^3$ b) $w = x^2 - y^2$, $x = \cos t$, $y = \sin t$
c) $w = \ln(u^2+v^2)$, $u = 2\cos t$, $v = 2\sin t$

2E-2 In each of these, information about the gradient of an unknown function $f(x,y)$ is given; x and y are in turn functions of t. Use the chain rule to find out additional information about the composite function $w = f\big(x(t), y(t)\big)$, without trying to determine f explicitly.

a) $\nabla w = 2\mathbf{i} + 3\mathbf{j}$ at $P: (1,0)$; $x = \cos t$, $y = \sin t$. Find the value of $\dfrac{dw}{dt}$ at $t = 0$.

b) $\nabla w = y\,\mathbf{i} + x\,\mathbf{j}$; $x = \cos t$, $y = \sin t$. Find $\dfrac{dw}{dt}$ and tell for what t-values it is zero.

c) $\nabla f = \langle 1, -1, 2\rangle$ at $(1,1,1)$. Let $x = t$, $y = t^2$, $z = t^3$; find $\dfrac{df}{dt}$ at $t = 1$.

d) $\nabla f = \langle 3x^2y, x^3+z, y\rangle$; $x = t$, $y = t^2$, $z = t^3$. Find $\dfrac{df}{dt}$.

2E-3 a) Use the chain rule for $f(u,v)$, where $u = u(t)$, $v = v(t)$, to prove the product rule

$$D(uv) = v\,Du + u\,Dv, \qquad \text{where } D = \frac{d}{dt}.$$

b) Using the chain rule for $f(u,v,w)$, derive a similar product rule for $\dfrac{d}{dt}(uvw)$, and use it to differentiate $te^{2t}\sin t$.

c)* Derive similarly a rule for the derivative $\dfrac{d}{dt}u^v$, and use it to differentiate $(\ln t)^t$.

2E-4 Let $w = f(x, y)$, and assume that $\nabla w = 2\,\mathbf{i} + 3\,\mathbf{j}$ at $(0, 1)$. If $x = u^2 - v^2,\ y = uv$, find $\dfrac{\partial w}{\partial u},\ \dfrac{\partial w}{\partial v}$ at $u = 1,\ v = 1$.

2E-5 Let $w = f(x, y)$, and suppose we change from rectangular to polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.

a) Show that $(w_x)^2 + (w_y)^2 = (w_r)^2 + \dfrac{1}{r^2}(w_\theta)^2$.

b) Suppose $\nabla w = 2\,\mathbf{i} - \mathbf{j}$ at the point $x = 1,\ y = 1$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ when $r = \sqrt{2},\ \theta = \pi/4$, and verify the relation in part (a), at the point.

2E-6 Let $w = f(x, y)$, and make the change of variables $x = u^2 - v^2,\ y = 2uv$. Show

$$(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}$$

2E-7 The Jacobian matrix for the change of variables $x = x(u, v),\ y = y(u, v)$ is defined to be $J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$. Let $\nabla f(x, y)$ be represented as the row vector $\langle f_x\ f_y \rangle$.
Show that

$$\nabla f\big(x(u, v), y(u, v)\big) \;=\; \nabla f(x, y) \cdot J \qquad \text{(matrix multiplication).}$$

2E-8 a) Let $w = f(y/x)$; i.e., w is the composite of the functions $w = f(u),\ u = y/x$. Show that w satisfies the PDE (partial differential equation) $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 0$.

b)* Let $w = f(x^2 - y^2)$; show that w satisfies the PDE $y\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial y} = 0$.

c)* Let $w = f(ax + by)$; show that w satisfies the PDE $b\dfrac{\partial w}{\partial x} - a\dfrac{\partial w}{\partial y} = 0$.

2F. Maximum-minimum Problems

2F-1 Find the point(s) on each of the following surfaces which is closest to the origin. (Hint: it's easier to minimize the square of the distance, rather than the distance itself.)

a) $xyz^2 = 1$ b) $x^2 - yz = 1$

2F-2 A rectangular produce box is to be made of cardboard; the sides of single thickness, the front and back of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what proportions for the sides will use the least cardboard?

2F-3* Consider all planes passing through the point (2,1,1) and such that the intercepts on the three coordinate axes are all positive. For which of these planes is the product of the three intercepts smallest? (Hint: take the plane in the form $z = ax + by + c$, where a and b are the independent variables.)

2F-4* Find the extremal point of $x^2 + 2xy + 4y^2 + 6$ and show it is a minimum point by completing the square.

2F-5 A drawer in a chest has an open top; the bottom and back are made of cheap wood costing \$1/sq. ft; the sides have to be thicker, and cost \$2/sq.ft., while the front costs \$4/sq.ft. for the better quality wood and finishing. The volume is to be 2.5 cu. ft. What dimensions will produce the drawer costing the least to manufacture?

2G. Least-squares Interpolation

2G-1 Find by the method of least squares the line which best fits the three data points given. Do it from scratch, using (2) in Notes LS and differentiation (use the chain rule). Sketch the line and the three points as a check.

a)* $(0,0),(0,2),(1,3)$ b)* $(0,0),(1,2),(2,1)$ c) $(1,1),(2,3),(3,2)$

2G-2* Show that the equations (4) for the method of least squares have a unique solution, unless all the x_i are equal. Explain geometrically why this exception occurs.

> Hint: use the fact that for all values of u, we have $\sum_1^n (x_i - u)^2 \geq 0$, since squares are always non-negative. Write the left side as a quadratic polynomial in u. Usually it has no roots. What does this imply about the coefficients? When does it have a root? (Answer these two questions by using the quadratic formula.)

2G-3* Use least squares to fit a second degree polynomial exactly through the points $(-1,-1),(0,0),(1,3)$ (see (6) and (7) in Notes LS).

2G-4 What linear equations in a, b, c does the method of least squares lead to, when you use it to fit a linear function $z = a + bx + cy$ to a set of data points (x_i, y_i, z_i), $i = 1, \ldots, n$?

2G-5* What equations are you led to for determining a when you try to fit the exponential curve $y = e^{ax}$ to two data points $(1, y_1), (2, y_2)$ by the method of least squares?

> The moral is: don't do it this way. In general to fit an exponential $y = ce^{ax}$ to a set of data points (x_i, y_i), take the log of both sides:
>
> $$\ln y \;=\; a\,x + \ln c$$
>
> This gives a linear function in the variables x and $\ln y$, whose coefficients a and $\ln c$ can be determined by applying the method of least squares to fit the data points $(x_i, \ln y_i)$.

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1 For each of the following functions, find the critical points, and classify them using the 2nd-derivative criterion.

a) $x^2 - xy - 2y^2 - 3x - 3y + 1$ b) $3x^2 + xy + y^2 - x - 2y + 4$ c) $2x^4 + y^2 - xy + 1$

d) $x^3 - 3xy + y^3$ e) $(x^3+1)(y^3+1)$

2H-2* In Notes LS, use the 2nd-derivative criterion to verify that the critical point (m_0, b_0) determining the regression (= least-squares) line $y = m_0 x + b_0$ really minimizes the function $D(m, b)$ giving the sum of the squares of the deviations. (You will need the inequality in 1B-15, for n-vectors $\mathbf{A} = \langle a_1, a_2, \ldots, a_n\rangle$, defining $|A| = \sqrt{\sum a_i^2}$ and $\mathbf{A} \cdot \mathbf{B} = \sum a_i b_i$.)

2H-3 Find the maximum and minimum of the function $f(x,y) = x^2 + y^2 + 2x + 4y - 1$ in the right half-plane R bounded by the diagonal line $y = -x$.

2H-4 Find the maximum and minimum points of the function $xy - x - y + 2$ on
a) the first quadrant b) the square $R:\ 0 \le x \le 2,\ 0 \le y \le 2$;
(study its values at the unique critical point and on the boundary lines) point.
c) use the data to guess the critical point type, and confirm it by the second derivative test.

2H-5 Find the maximum and minimum points of the function $f(x, y) = x + \sqrt{3}\,y + 2$ on the unit disc $R : x^2 + y^2 \le 1$.

2H-6 a) Two wires of length 4 are cut in the same way into three pieces, of length x, y and z; the four x, y pieces are used as the four sides of a rectangle; the two z pieces are bent at the middle and joined at the ends to make a square of side $z/2$. Find the rectangle and square made this way which together have the largest and the smallest total area.
Using the answer, tell what type the critical point is.
b) Confirm the critical point type by using the second derivative test.

2H-7 a) Find the maximum and minimum points of the function $2x^2 - 2xy + y^2 - 2x$ on the rectangle $R:\ 0 \le x \le 2;\ -1 \le y \le 2$; using this information, determine the type of the critical point.
b) Confirm the critical point type by using the second derivative test.

2I. Lagrange Multipliers

2I-1 A rectangular box is placed in the first octant so that one corner Q is at the origin and the three sides adjacent to Q lie in the coordinate planes. The corner P diagonally opposite Q lies on the surface $f(x, y, z) = c$. Using Lagrange multipliers, tell for which point P the box will have the largest volume, and tell how you know it gives a maximum point, if the surface is

a) the plane $x + 2y + 3z = 18$ b) the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$.

2I-2 Using Lagrange multipliers, tell which point P in the first octant and on the surface $x^3y^2z = 6\sqrt{3}$ is closest to the origin. (As usual, it is easier algebraically to minimize $|OP|^2$ rather than $|OP|$.)

2I-3 (Repeat of 2F-2, but this time use Lagrange multipliers.) A rectangular produce box is to be made of cardboard; the sides of single thickness, the ends of double thickness, and the bottom of triple thickness, with the top left open. Its volume is to be 1 cubic foot; what should be its proportions in order to use the least cardboard?

2I-4 In an open-top wooden drawer, the two sides and back cost \$2/sq.ft., the bottom \$1/sq.ft. and the front \$4/sq.ft. Using Lagrange multipliers, show that the following problems lead to the same set of three equations in λ, plus a different fourth equation, and they have the same solution.

a) Find the dimensions of the drawer with largest capacity that can be made for a total wood cost of \$72.

b) Find the dimensions of the most economical drawer having volume 24 cu. ft.

2J. Non-independent Variables

2J-1 Let $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

Calculate by direct substitution: a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ b) $\left(\dfrac{\partial w}{\partial z}\right)_y$.

2J-2 Calculate the two derivatives in 2J-1 by using
(i) the chain rule (differentiate $z = x^2 + y^2$ implicitly) (ii) differentials

2J-3 Let $w = x^3y - z^2t$ and $xy = zt$.

Using the chain rule calculate, in terms of x, y, z, t, the derivatives

a) $\left(\dfrac{\partial w}{\partial t}\right)_{x,z}$ b) $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$

2J-4 Repeat 2J-3, doing the calculation using differentials.

2J-5 Let $S = S(p, v, T)$ be the entropy of a gas, assumed to obey the ideal gas law: $pv = nRT$ (n, R constants). Give expressions in terms of the formal partial derivatives S_p, S_v, and S_T for

a) $\left(\dfrac{\partial S}{\partial p}\right)_v$ b) $\left(\dfrac{\partial S}{\partial T}\right)_v$

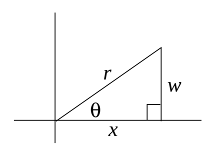
2J-6 If $w = u^3 - uv^2$, $u = xy$, $v = u + x$, find $\left(\dfrac{\partial w}{\partial u}\right)_x$ and $\left(\dfrac{\partial w}{\partial x}\right)_u$ using
a) the chain rule b) differentials .

2J-7 Let P be the point $(1, -1, 1)$, and assume $z = x^2 + y + 1$, and that $f(x, y, z)$ is a differentiable function for which $\nabla f(x, y, z) = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ at P.

Let $g(x, z) = f(x, y(x, z), z)$; find ∇g at the point $(1, 1)$, i.e., $x = 1$, $z = 1$.

2J-8 Interpreting r, θ as polar coordinates, let $w = \sqrt{r^2 - x^2}$.

a) Calculate $\left(\dfrac{\partial w}{\partial r}\right)_\theta$, by first writing w in terms of r and θ.



b)* Obtain the answer by intuitive geometrical reasoning (see picture).

2J-9* Prove the two-Jacobian rule in notes N section 10 ; use differentials.

2J-10* One of the laws of thermodynamics is expressed by the equation

$$\left(\frac{\partial U}{\partial p}\right)_T + T\left(\frac{\partial V}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial p}\right)_T = 0.$$

Show that the law takes the following form when the independent variables are changed to U and V:

$$\left(\frac{\partial T}{\partial V}\right)_U + T\left(\frac{\partial p}{\partial U}\right)_V - p\left(\frac{\partial T}{\partial U}\right)_V = 0.$$

2J-11* For the law in 2J-10, show the change of variables gives the law the following forms:

a) $$T - p\left(\frac{\partial T}{\partial p}\right)_V + \frac{\partial(T,U)}{\partial(V,p)} = 0, \qquad \text{(independent variables } V,p)$$

b) $$\left(\frac{\partial T}{\partial p}\right)_U - T\left(\frac{\partial V}{\partial U}\right)_p + p\,\frac{\partial(V,T)}{\partial(U,p)} = 0, \qquad \text{(independent variables } U,p)$$

2K. Partial Differential Equations

2K-1. Show that $w = \ln r$, where $r = \sqrt{x^2+y^2}$ is the usual polar coordinate, satisfies the two-dimensional Laplace equation (Notes P (1), without z), if $(x,y) \neq (0,0)$. What's wrong with $(0,0)$?

(The calculation will go faster if you remember that $\ln\sqrt{a} = \frac{1}{2}\ln a$.)

2K-2. For what value(s) of n will $w = (x^2+y^2+z^2)^n$ solve the 3-dimensional Laplace equation (Notes P, (1))? Where have you seen this function in physics?

2K-3. The solutions in exercises 2K-1 and 2K-2 have circular and spherical symmetry, respectively. But there are many other solutions. For example

a) find all solutions of the two-dimensional Laplace equation (see 2K-1) of the form

$$w = ax^2 + bxy + cy^2$$

and show they can all be written in the form $c_1 f_1(x,y) + c_2 f_2(x,y)$, where c_1, c_2 are arbitrary constants, and f_1, f_2 are two particular polynomials — that is, all such solutions are linear combinations of two particular polynomial solutions.

b)* Find and derive the analogue of part (a) for all of the cubic polynomial solutions $ax^3 + bx^2y + cxy^2 + dy^3$ to the two-dimensional Laplace equation.

2K-4. Show that the one-dimensional wave equation (Notes P, (4), first equation) is satisfied by any function of the form

$$w = f(x+ct) + g(x-ct)\ ,$$

where $f(u)$ and $g(u)$ are arbitrary twice-differentiable functions of one variable.

Take $g(u) = 0$, and interpret physically the solution $w = f(x+ct)$. What does $f(x)$ represent? What is the relation of $f(x+ct)$ to it?

> Note how this exercise shows that a solution to the wave equation can involve completely arbitrary functions; this is also clear from the remarks about the Laplace equation being solved by any gravitational or electrostatic potential function in a mass- or charge-free region of space.

2K-5. Find solutions to the one-dimensional heat equation (Notes P, (5), first equation) having the form

$$w = e^{rt}\sin kx \qquad k, r \text{ constants}$$

satisfying the additional conditions for all t:

$$w(0,t) = 0, \qquad w(1,t) = 0\ .$$

Interpret your solutions physically. What happens to the temperature as $t \to \infty$?

3. Double Integrals

3A. Double Integrals in Rectangular Coordinates

3A-1 Evaluate each of the following iterated integrals:

a) $\displaystyle\int_0^2 \int_{-1}^1 (6x^2 + 2y)\, dy\, dx$ b) $\displaystyle\int_0^{\pi/2} \int_0^{\pi} (u \sin t + t \cos u)\, dt\, du$

c) $\displaystyle\int_0^1 \int_{\sqrt{x}}^{x^2} 2x^2 y\, dy\, dx$ d) $\displaystyle\int_0^1 \int_0^u \sqrt{u^2+4}\, dv\, du$

3A-2 Express each double integral over the given region R as an iterated integral, using the given order of integration. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) R is the triangle with vertices at the origin, $(0,2)$, and $(-2,2)$.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

b) R is the finite region between the parabola $y = 2x - x^2$ and the x-axis.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

c) R is the sector of the circle with center at the origin and radius 2 lying between the x-axis and the line $y = x$.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

d)* R is the finite region lying between the parabola $y^2 = x$ and the line through $(2,0)$ having slope 1.
Express as an iterated integral: i) $\displaystyle\iint_R dy\, dx$ ii) $\displaystyle\iint_R dx\, dy$

3A-3 Evaluate each of the following double integrals over the indicated region R. Choose whichever order of integration seems easier — given the integrand, and the shape of R.

a) $\displaystyle\iint_R x\, dA$; R is the finite region bounded by the axes and $2y + x = 2$

b) $\displaystyle\iint_R (2x + y^2)\, dA$; R is the finite region in the first quadrant bounded by the axes and $y^2 = 1 - x$; ($dx\, dy$ is easier).

c) $\displaystyle\iint_R y\, dA$; R is the triangle with vertices at $(\pm 1, 0)$, $(0,1)$.

3A-4 Find by double integration the volume of the following solids.

a) the solid lying under the graph of $z = \sin^2 x$ and over the region R bounded below by the x-axis and above by the central arch of the graph of $\cos x$

b) the solid lying over the finite region R in the first quadrant between the graphs of x and x^2, and underneath the graph of $z = xy$..

c) the finite solid lying underneath the graph of $x^2 - y^2$, above the xy-plane, and between the planes $x = 0$ and $x = 1$

3A-5 Evaluate each of the following iterated integrals, by changing the order of integration (begin by figuring out what the region R is, and sketching it).

a) $\displaystyle\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$ b) $\displaystyle\int_0^{1/4} \int_{\sqrt{t}}^{1/2} \frac{e^u}{u}\, du\, dt$ c) $\displaystyle\int_0^1 \int_{x^{1/3}}^1 \frac{1}{1+u^4}\, du\, dx$

3A-6 Each integral below is over the disc consisting of the interior R of the unit circle, centered at the origin. For each integral, use the symmetries of R and the integrand

i) to identify its value as zero; or if its value is not zero,

ii) to find a double integral which is equivalent (i.e., has the same value), but which has a simpler integrand and/or is taken over the first quadrant (if possible), or over a half-disc. (Do not evaluate the integral.)

$$\iint_R x\, dA; \quad \iint_R e^x\, dA; \quad \iint_R x^2\, dA; \quad \iint_R x^2 y\, dA; \quad \iint_R (x^2+y) dA; \quad \iint_R xy\, dA$$

3A-7 By using the inequality $f \le g$ on $R \Rightarrow \iint_R f\, dA \le \iint_R g\, dA$, show the following estimates are valid:

a) $\displaystyle\iint_R \frac{dA}{1+x^4+y^4} \le$ area of R

b) $\displaystyle\iint_R \frac{x\, dA}{1+x^2+y^2} < .35,$ R is the square $0 \le x, y \le 1$.

3B. Double Integrals in Polar Coordinates

In evaluating the integrals, the following definite integrals will be useful:

$$\int_0^{\pi/2} sin^n x\, dx = \int_0^{\pi/2} cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (n-1)}{2\cdot 4\cdot \cdots \cdot n}\dfrac{\pi}{2}, & \text{if } n \text{ is an even integer} \ge 2 \\ \dfrac{2\cdot 4\cdot \cdots \cdot (n-1)}{1\cdot 3\cdot \cdots\cdots n}, & \text{if } n \text{ is an odd integer} \ge 3. \end{cases}$$

For example: $\displaystyle\int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}$, $\displaystyle\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}$, $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx = \frac{3\pi}{16}$,

and the same holds if $\cos x$ is substituted for $\sin x$.

3B-1 Express each double integral over the given region R as an iterated integral in polar coordinates. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) The region lying inside the circle with center at the origin and radius 2 and to the left of the vertical line through $(-1, 0)$.

b)* The circle of radius 1, and center at $(0, 1)$.

c) The region inside the cardioid $r = 1 - \cos\theta$ and outside the circle of radius $3/2$ and center at the origin.

d) The finite region bounded by the y-axis, the line $y = a$, and a quarter of the circle of radius a and center at $(a, 0)$.

3B-2 Evaluate by iteration the double integrals over the indicated regions. Use polar coordinates.

a) $\displaystyle\iint_R \frac{dA}{r}$; R is the region inside the first-quadrant loop of $r = \sin 2\theta$.

b) $\displaystyle\iint_R \frac{dx\,dy}{1+x^2+y^2}$; R is the first-quadrant portion of the interior of $x^2+y^2=a^2$

c) $\displaystyle\iint_R \tan^2\theta\,dA$; R is the triangle with vertices at $(0,0),\ (1,0),\ (1,1)$.

d) $\displaystyle\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$; R is the right half-disk of radius $\frac{1}{2}$ centered at $(0,\frac{1}{2})$.

3B-3 Find the volumes of the following domains by integrating in polar coordinates:

a) a solid hemisphere of radius a (place it so its base lies over the circle $x^2+y^2=a^2$)

b) the domain under the graph of xy and over the quarter-disc region R of 3B-2b

c) the domain lying under the cone $z=\sqrt{x^2+y^2}$ and over the circle of radius one and center at $(0,1)$

d) the domain lying under the paraboloid $z=x^2+y^2$ and over the interior of the right-hand loop of $r^2=\cos\theta$.

3B-4* Sometimes students wonder if you can do a double integral in polar coordinates iterating in the opposite order: $\iint_R d\theta\,dr$. Though this is uncommon, just to see if you can carry out in a new situation the basic procedure for putting in the limits, try supplying the limits for this integral over the region bounded above by the lines $x=1$ and $y=1$, and below by a quarter of the circle of radius 1 and center at the origin.

3C. Applications of Double Integration

If no coordinate system is specified for use, you can use either rectangular or polar coordinates, whichever is easier. In some of the problems, a good placement of the figure in the coordinate system simplifies the integration a lot.

3C-1 Let R be a right triangle, with legs both of length a, and density 1. Find the following ((b) and (c) can be deduced from (a) with no further calculation)
a) its moment of inertia about a leg;
b) its polar moment of inertia about the right-angle vertex;
c) its moment of inertia about the hypotenuse.

3C-2 Find the center of mass of the region inside one arch of $\sin x$, if: a) $\delta=1$ b) $\delta=y$

3C-3 D is a diameter of a disc of radius a, and C is a chord parallel to D with distance c from it. C divides the disc into two segments; let R be the smaller one. Assuming $\delta=1$, find the moment of R about D, giving the answer in simplest form, and using
(a) rectangular coordinates; (b) polar coordinates.

3C-4 Find the center of gravity of a sector of a circular disc of radius a, whose vertex angle is 2α. Take $\delta=1$.

3C-5 Find the polar moment of inertia of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the origin. Take $\delta = 1$.

3D. Changing Variables in Multiple Integrals

3D-1 Evaluate $\displaystyle\iint_R \frac{x-3y}{2x+y}\,dx\,dy$, where R is the parallelogram bounded on the sides by $y = -2x + 1$ and $y = -2x + 4$, and above and below by $y = x/3$ and $y = (x-7)/3$. Use a change of variables $u = x - 3y$, $v = 2x + y$.

3D-2 Evaluate $\displaystyle\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy$ by making the change of variables used in Example 2; take as the region R the triangle with vertices at the origin, (2,0) and (1,1).

3D-3 Find the volume underneath the surface $z = 16 - x^2 - 4y^2$ and over the xy-plane; simplify the integral by making the change of variable $u = x,\ v = 2y$.

3D-4 Evaluate $\displaystyle\iint_R (2x-3y)^2(x+y)^2\ dx\,dy$, where R is the triangle bounded by the positive x-axis, negative y-axis, and line $2x - 3y = 4$, by making a change of variable $u = x + y,\quad v = 2x - 3y$.

3D-5 Set up an iterated integral for the polar moment of inertia of the finite "triangular" region R bounded by the lines $y = x$ and $y = 2x$, and a portion of the hyperbola $xy = 3$. Use a change of coordinates which makes the boundary curves grid curves in the new coordinate system.

In the following problems, the numbered references to formulas, equations, and examples all refer to Notes CV.

3D-6* Verify that (25) gives the right volume element in spherical coordinates, using the equations (26).

3D-7* Using the coordinate change $u = xy,\ v = y/x$, set up an iterated integral for the polar moment of inertia of the region bounded by the hyperbola $xy = 1$, the x-axis, and the two lines $x = 1$ and $x = 2$. Choose the order of integration which makes the limits simplest.

3D-8 For the change of coordinates in 3D-7, give the uv-equations of the following curves: a) $y = x^2$ b) $x^2 + y^2 = 1$.

3D-9* Prove the relation (22) between Jacobians: $\displaystyle\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} = 1$; use the chain rule for partial differentiation, and the rule for multiplying determinants: $|AB| = |A||B|$, where A and B are square matrices of the same size.

3D-10* Set up the iterated integral in the order $\displaystyle\iint_R dv\,du$ for the region of Example 4 in Notes CV, and check your work by evaluating it. (You will have to break the region up into two pieces, using different limits of integration for the pieces.)

4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1 Describe geometrically how the vector fields determined by each of the following vector functions looks. Tell for each what the largest region in which $\mathbf{F}$ is continuously differentiable is.

a) $a\,\mathbf{i} + b\,\mathbf{j}$, $\quad a, b$ constants $\qquad$ b) $-x\,\mathbf{i} - y\,\mathbf{j}$ $\qquad$ c) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r}$ $\qquad$ d) $\dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{r}$

4A-2 Write down the gradient field ∇w for each of the following:

a) $w = ax + by$ $\qquad$ b) $w = \ln r$ $\qquad$ c) $w = f(r)$

4A-3 Write down an explicit expression for each of the following fields:

a) Each vector has the same direction and magnitude as $\mathbf{i} + 2\,\mathbf{j}$.

b) The vector at (x, y) is directed radially in towards the origin, with magnitude r^2.

c) The vector at (x, y) is tangent to the circle through (x, y) with center at the origin, clockwise direction, magnitude $1/r^2$.

d) Each vector is parallel to $\mathbf{i} + \mathbf{j}$, but the magnitude varies.

4A-4 The electromagnetic force field of a long straight wire along the z-axis, carrying a uniform current, is a two-dimensional field, tangent to horizontal circles centered along the z-axis, in the direction given by the right-hand rule (thumb pointed in positive z-direction), and with magnitude k/r. Write an expression for this field.

4B. Line Integrals in the Plane

4B-1 For each of the fields $\mathbf{F}$ and corresponding curve C or curves C_i, evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$.

Use any convenient parametrization of C, unless one is specified. Begin by writing the integral in the differential form $\displaystyle\int_C M\,dx + N\,dy$.

a) $\mathbf{F} = (x^2 - y)\,\mathbf{i} + 2x\,\mathbf{j}$; $\quad C_1$ and C_2 both run from $(-1, 0)$ to $(1, 0)$:
$\quad C_1$: the x-axis $\qquad C_2$: the parabola $y = 1 - x^2$

b) $\mathbf{F} = xy\,\mathbf{i} - x^2\,\mathbf{j}$; $\qquad C$: the quarter of the unit circle running from $(0, 1)$ to $(1, 0)$.

c) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$; $\quad C$: the triangle with vertices at $(0, 0)$, $(0, 1)$, $(1, 0)$, oriented clockwise.

d) $\mathbf{F} = y\,\mathbf{i}$; $\qquad C$ is the ellipse $x = 2\cos t$, $\;y = \sin t$, oriented counterclockwise.

e) $\mathbf{F} = 6y\,\mathbf{i} + x\,\mathbf{j}$; $\qquad C$ is the curve $x = t^2$, $\;y = t^3$, running from $(1, 1)$ to $(4, 8)$.

f) $\mathbf{F} = (x + y)\,\mathbf{i} + xy\,\mathbf{j}$; $\quad C$ is the broken line running from $(0, 0)$ to $(0, 2)$ to $(1, 2)$.

4B-2 For the following fields $\mathbf{F}$ and curves C, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ without any formal calculation, appealing instead to the geometry of $\mathbf{F}$ and C.

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; $\qquad C$ is the counterclockwise circle, center at $(0, 0)$, radius a.

b) $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$; $\qquad C$ is the counterclockwise circle, center at $(0, 0)$, radius a.

4B-3 Let $\mathbf{F} = \mathbf{i} + \mathbf{j}$. How would you place a directed line segment C of length one so that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ would be

a) a maximum; b) a minimum; c) zero;

d) what would the maximum and minimum values of the integral be?

4C. Gradient Fields and Exact Differentials

4C-1 Let $f(x,y) = x^3y + y^3$, and C be $y^2 = x$, between $(1,-1)$ and $(1,1)$, directed upwards.

a) Calculate $F = \nabla f$.

b) Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ three different ways:
(i) directly;
(ii) by using path-independence to replace C by a simpler path
(iii) by using the Fundamental Theorem for line integrals.

4C-2 Let $f(x,y) = x\,e^{xy}$, and C be the path $y = 1/x$ from $(1,1)$ to $(0,\infty)$.

a) Calculate $\mathbf{F} = \nabla f$.

b) Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$
(i) directly;
(ii) by using the Fundamental Theorem for line integrals.

4C-3 Let $f(x,y) = \sin x \cos y$.

a) Calculate $\mathbf{F} = \nabla f$.

b) What is the maximum value $\int_C \mathbf{F} \cdot d\mathbf{r}$ can have over all possible paths C in the plane? Give a path C for which this maximum value is attained.

4C-4* The Fundamental Theorem for line integrals should really be called the First Fundamental Theorem. There is an analogue for line integrals of the Second Fundamental Theorem also, where you first integrate, then differentiate; it provides the justification for Method 1 in this section. It runs:

$$\text{If } \int_C M\,dx + N\,dy \text{ is path-independent, and } f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}, \text{ then } \nabla f = M\,\mathbf{i} + N\,\mathbf{j}.$$

The conclusion says $f_x = M, f_y = N$; prove the second of these. (Hint: use the Second Fundamental Theorem of Calculus.)

4C-5 For each of the following, tell for what value of the constants the field will be a gradient field, and for this value, find the corresponding (mathematical) potential function.

a) $\mathbf{F} = (y^2 + 2x)\,\mathbf{i} + axy\,\mathbf{j}$ b) $\mathbf{F} = e^{x+y}\big((x+a)\,\mathbf{i} + x\,\mathbf{j}\big)$

4C-6 Decide which of the following differentials is exact. For each one that is exact, express it in the form df.

a) $y\,dx - x\,dy$ b) $y(2x+y)\,dx + x(2y+x)\,dy$

c)* $x \sin y\,dx + y \sin x\,dy$ d)* $\dfrac{y\,dx - x\,dy}{(x+y)^2}$

4D. Green's Theorem

4D-1 For each of the following fields $\mathbf{F}$ and closed positively oriented curves C, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ both directly, as a line integral, and also by applying Green's theorem and calculating a double integral.

a) $\mathbf{F}:\ 2y\,\mathbf{i} + x\,\mathbf{j}\,,\quad C:\ x^2+y^2=1$

b) $\mathbf{F}:\ x^2(\,\mathbf{i}\,+\,\mathbf{j}\,)\quad C$: rectangle joining (0,0), (2,0), (0,1), (2,1)

c) $\mathbf{F}:\ xy\,\mathbf{i} + y^2\,\mathbf{j}\,,\quad C: y=x^2$ and $y=x,\ 0\le x\le 1$

4D-2 Show that $\oint_C 4x^3y dx + x^4 dy = 0$ for all closed curves C.

4D-3 Find the area inside the hypocycloid $x^{2/3}+y^{2/3}=1$, by using Green's theorem. (This curve can be parametrized by $x=\cos^3\theta, y=\sin^3\theta$, between suitable limits on θ.)

4D-4 Show that the value of $\oint_C -y^3dx+x^3dy$ around any positively oriented simple closed curve C is always positive.

4D-5 Show that the value of $\oint_C xy^2dx+(x^2y+2x)dy$ around any positively oriented square C in the xy-plane depends only on the size of the square, and not upon its position.

4D-6* Show that $\oint_C -x^2y\,dx + xy^2\,dy\ >\ 0$ around any positively oriented simple closed curve C.

4D-7* Show that the value of $\oint_C y(y+3)\,dx + 2xy\,dy$ around any positively oriented equilateral triangle C depends only on the size of the triangle, and not upon its position in the xy-plane.

4E. Two-dimensional Flux

4E-1 Let $\mathbf{F} = -y\,\mathbf{i}\,+x\,\mathbf{j}$. Recalling the interpretation of this field (example 4, V1.5),. or just by remembering how it looks geometrically, evaluate with little or no calculation the flux integral $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, where

a) C is a circle of radius a centered at $(0,0)$, directed counterclockwise.
b) C is the line segment running from $(-1,0)$ to $(1,0)$
c) C is the line running from $(0,0)$ to $(1,0)$.

4E-2 Let $\mathbf{F}$ be the constant vector field $\mathbf{i}\,+\,\mathbf{j}$. Where would you place a directed line segment C of length one in the plane so that the flux across C would be

a) maximal b) minimal c) zero d) -1 e) what would the maximal and minimal values be?

4E-3 Let $\mathbf{F} = x^2\,\mathbf{i}\,+xy\,\mathbf{j}$. Evaluate $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ if C is given by $r(t) = (t+1)\,\mathbf{i}\,+t^2\,\mathbf{j}\,$, where $0\le t\le 1$; the positive direction on C is the direction of increasing t.

4E-4 Take C to be the square of side 1 with opposite vertices at $(0, 0)$ and $(1, 1)$, directed *clockwise*. Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$; find the flux across C.

4E-5 Let $\mathbf{F}$ be defined everywhere except at the origin by the description:

dir $\mathbf{F}$ = radially outward, $|\mathbf{F}| = r^m$, m an integer.

a) Evaluate the flux of $\mathbf{F}$ across a circle of radius a and center at the origin, directed counterclockwise.

b) For which value(s) of m will the flux be independent of a?

4E-6* Let $\mathbf{F}$ be a constant vector field, and let C be a closed polygon, directed counterclockwise. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Hint evaluate the integral along one of the directed sides; then add up the integrals over the successive sides, using properties of vectors.)

4E-7* Let $\mathbf{F}$ be a constant vector field, and C a closed polygon, as in the preceding exercise. Show that $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = 0$.

4F. Green's Theorem in Normal Form

4F-1 Calculate the functions div $\mathbf{F}$ and curl $\mathbf{F}$ for each of the following fields.

a) $a\,\mathbf{i} + b\,\mathbf{j}$ $(a, b$ constants$)$ b) $x^2\,\mathbf{i} + y^2\,\mathbf{j}$ c) $xy(\mathbf{i} + \mathbf{j})$

4F-2 Let $\mathbf{F} = \omega(-y\,\mathbf{i} + x\,\mathbf{j})$ be the vector field of Section V1, Example 4.

a) Calculate div $\mathbf{F}$ and curl $\mathbf{F}$.

b) Using the physical interpretation of this vector field, explain why it is reasonable that div $\mathbf{F} = 0$.

c) Using the physical interpretation of curl $\mathbf{F}$, explain why it is reasonable that curl $\mathbf{F} = 2\omega$ at the origin.

4F-3 Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$, and C is formed by the upper half of the unit circle and the x-axis interval $[-1, 1]$.

4F-4 Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, and C is the square with opposite vertices at $(0, 0)$ and $(1, 1)$.

4F-5 Calculate div $\mathbf{F}$ and curl $\mathbf{F}$ for $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j})$. (Simplify the differentiation by using $r_x = x/r$, $r_y = y/r$.)

For which value(s) of n is div $\mathbf{F} = 0$? For which value(s) of n is curl $\mathbf{F} = 0$?

4F-6* a) Suppose that all the vectors of a field $\mathbf{F}$ point radially outward and their magnitude is a differentiable function $f(r)$ of r alone. Show that curl $\mathbf{F} = 0$.

b) Suppose all the vectors of a field $\mathbf{F}$ are parallel. Reasoning from the physical interpretation of curl $\mathbf{F}$, would you expect it to be zero everywhere? Illustrate your answer by an example.

4G. Simply-connected Regions.

4G-1 Using the criterion of this section, tell which of the following fields and differentials definitely are respectively conservative or exact, which of them are definitely not, and for which of them the criterion fails.

a) $(y^2+2)\,\mathbf{i}+2xy\,\mathbf{j}$ b) $x(\cos y)\,dx+y(\cos x)\,dy$

c) $\dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{\sqrt{r^2-1}}$ d) $\dfrac{x\,dx+y\,dy}{\sqrt{1-r^2}}$ e) $\sqrt{x}\,\mathbf{i}+\sqrt{y}\,\mathbf{j}$

4G-2 For each of the following fields $\mathbf{F}$, find $f(x,y)$ such that $\mathbf{F}=\nabla f$.

a) the field of 1a b) the field of 1e

c) the field of 1d; use polar coordinates.

4G-3 Evaluate $\displaystyle\int_{(1,1)}^{(3,4)}\mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{r^3}$. Use the results of Example 3 in Notes V5.

4G-4 Even though the field of Example 1, $\mathbf{F}=xy\,\mathbf{i}+x^2\,\mathbf{j}$, is not a gradient field, show that $\oint\mathbf{F}\cdot d\mathbf{r}=0$ around every simple closed curve which is symmetric about the y-axis.

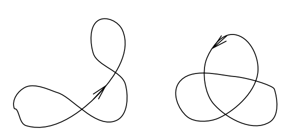
4G-5 Which of the following regions are simply-connected?
a) the half-plane lying above the x-axis
b) the plane minus the line segment joining (0,0) and (0,1)
c) the plane minus the positive x-axis
d) the plane minus the entire x-axis
e) in polar coordinates, the region where $r>0,\ 0<\theta<\theta_0$
f) the region between two concentric circles
g) the region in the plane between the two branches of the hyperbola $xy=1$

4G-6 For which of the following vector fields is the domain where it is defined and continuously differentiable a simply-connected region?

a) $\sqrt{x}\,\mathbf{i}+\sqrt{y}\,\mathbf{j}$ b) $\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{1-x^2-y^2}}$ c) $\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{x^2+y^2-1}}$

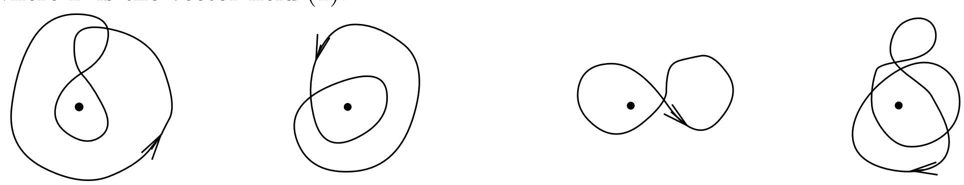
d) $\dfrac{-y\,\mathbf{i}+x\,\mathbf{j}}{r}$ e) $(\mathbf{i}+\mathbf{j})\ln(x^2+y^2)$

4G-7* By following the method outlined in the proof of (3), show that if curl $\mathbf{F}=0$ in the whole xy-plane, then $\displaystyle\oint_C\mathbf{F}\cdot d\mathbf{r}=0$ over each of the following closed paths (break them into as few pieces as possible):

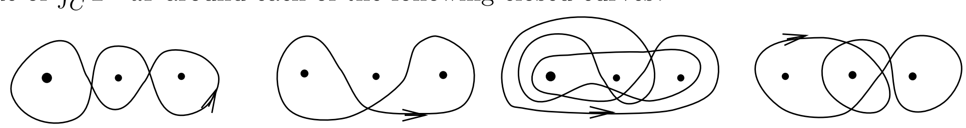


4H. Multiply-connected Regions

4H-1 For each of the closed curves shown determine the winding number about the indicated point. Then assume the point is $(0, 0)$, and tell what the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around C is, where $\mathbf{F}$ is the vector field (1).



4H-2 Suppose $\mathbf{F}$ is continuous differentiable everywhere in the xy-plane except at the three points $-1, 0, 1$ on the x-axis. Suppose the line integrals of $\mathbf{F}$ around small circles surrounding each of these points have respectively the values $2, \sqrt{2}$, and $\sqrt{3}$. What is the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around each of the following closed curves?



4I. Laplace's Equation and Harmonic Functions

4I-1* Suppose $f(x, y)$ is a polynomial in x and y all of whose terms have the same total degree n in x and y (such a polynomial is called "homogeneous of degree n"). Show that, a and b being arbitrary constants,

a) if $n = 2$, $f(x, y)$ is harmonic $\Leftrightarrow$ $f = a(x^2 - y^2) + bxy$

b) if $n = 3$, $f(x, y)$ is harmonic $\Leftrightarrow$ $f = a(x^3 - 3xy^2) + b(3x^2y - y^3)$.

In each case, prove $\Leftarrow$ by using (5).

4I-2* a) Show that the functions $c \ln r$ are harmonic (c constant; use $r_x = x/r$, etc.)

b) Show that if $w = f(r)$ and w is harmonic, then $f(r) = c_1 \ln r + c_2$.

(Hint: introduce a new variable $y = dw/dr$ and solve the resulting differential equation in y and r.)

4I-3* Verify that $e^{kx} \sin(ky + c)$ is harmonic for any constants k and c.

4I-4* Can you find a function ϕ harmonic in a region containing the circle of radius 1 and center at the origin, if the boundary values are prescribed by the following (s is the arclength along the circle, measured counterclockwise from $(1, 0)$):

a) $\dfrac{\partial \phi}{\partial \eta} = \cos s$ b) $\dfrac{\partial \phi}{\partial \eta} = s^2 - 2\pi s$ c) $\phi(s) = s$.

4I-5* "Derive" the equation beneath (12) symbolically, assuming that the product rule for differentiation applies to the operator ∇.

4I-6* The function $\ln r$ is zero on the unit circle and harmonic; why doesn't this contradict Theorem 3, (13)?

4I-7* Prove the **reciprocity law**: *if ϕ and ψ are harmonic in a region containing a simple closed curve C and its interior R, then*

$$\oint_C \psi \frac{\partial \phi}{\partial \eta}\, ds \;=\; \oint_C \phi \frac{\partial \psi}{\partial \eta}\, ds.$$

(Hint: use Theorem 2.)

4I-8* An important property of a harmonic function is that its value at the center of any circle is equal to its average value over the circle. (This is the basis of one important numerical method for calculating it by successive approximations.)

What would be the value of $\phi(0,0)$, if on the unit circle $\phi(s) = \sin^2 s$? (cf. Exercise 4 for the notations.)

5. Triple Integrals

5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 Evaluate: a) $\displaystyle\int_0^2\int_{-1}^1\int_0^1 (x+y+z)dx\,dy\,dz$ b) $\displaystyle\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy$

5A-2. Follow the three steps in the notes to supply limits for the triple integrals over the following regions of 3-space.

a) The rectangular prism having as its two bases the triangle in the yz-plane cut out by the two axes and the line $y+z=1$, and the corresponding triangle in the plane $x=1$ obtained by adding 1 to the x-coordinate of each point in the first triangle. Supply limits for three different orders of integration:

(i) $\displaystyle\iiint dz\,dy\,dx$ (ii) $\displaystyle\iiint dx\,dz\,dy$ (iii) $\displaystyle\iiint dy\,dx\,dz$

b)* The tetrahedron having with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$. Use the order $\displaystyle\iiint dz\,dy\,dx$.

c) The quarter of a solid circular cylinder of radius 1 and height 2 lying in the first octant, with its central axis the interval $0 \le y \le 2$ on the y-axis, and base the quarter circle in the xz-plane with center at the origin, radius 1, and lying in the first quadrant. Integrate with respect to y first; use suitable cylindrical coordinates.

d) The region bounded below by the cone $z^2 = x^2 + y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin. Use cylindrical coordinates.

5A-3 Find the center of mass of the tetrahedron D in the first octant formed by the coordinate planes and the plane $x+y+z=1$. Assume $\delta = 1$.

5A-4 A solid right circular cone of height h with 90^0 vertex angle has density at point P numerically equal to the distance from P to the central axis. Choosing the placement of the cone which will give the easiest integral, find

a) its mass b) its center of mass

5A-5 An engine part is a solid S in the shape of an Egyptian-type pyramid having height 2 and a square base with diagonal D of length 2. Inside the engine it rotates about D. Set up (but do not evaluate) an iterated integral giving its moment of inertia about D. Assume $\delta = 1$. (Place S so the positive z axis is its central axis.)

5A-6 Using cylindrical coordinates, find the moment of inertia of a solid hemisphere D of radius a about the central axis perpendicular to the base of D. Assume $\delta = 1$..

5A-7 The paraboloid $z = x^2 + y^2$ is shaped like a wine-glass, and the plane $z = 2x$ slices off a finite piece D of the region above the paraboloid (i.e., inside the wine-glass). Find the moment of inertia of D about the z-axis, assuming $\delta = 1$.

5B. Triple Integrals in Spherical Coordinates

5B-1 Supply limits for iterated integrals in spherical coordinates $\iiint d\rho\, d\phi\, d\theta$ for each of the following regions. (No integrand is specified; $d\rho\, d\phi\, d\theta$ is given so as to determine the order of integration.)

a) The region of 5A-2d: bounded below by the cone $z^2 = x^2 + y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin.

b) The first octant.

c) That part of the sphere of radius 1 and center at $z = 1$ on the z-axis which lies above the plane $z = 1$.

5B-2 Find the center of mass of a hemisphere of radius a, using spherical coordinates. Assume the density $\delta = 1$.

5B-3 A solid D is bounded below by a right circular cone whose generators have length a and make an angle $\pi/6$ with the central axis. It is bounded above by a portion of the sphere of radius a centered at the vertex of the cone. Find its moment of inertia about its central axis, assuming the density δ at a point is numerically equal to the distance of the point from a plane through the vertex perpendicular to the central axis.

5B-4 Find the average distance of a point in a solid sphere of radius a from
a) the center b) a fixed diameter c) a fixed plane through the center

5C. Gravitational Attraction

5C-1.* Consider the solid V bounded by a right circular cone of vertex angle 60^o and slant height a, surmounted by the cap of a sphere of radius a centered at the vertex of the cone. Find the gravitational attraction of V on a unit test mass placed at the vertex of V. Take the density to be

(a) 1 (b) the distance from the vertex. Ans.: a) $\pi G a/4$ b) $\pi G a^2/8$

5C-2. Find the gravitational attraction of the region bounded above by the plane $z = 2$ and below by the cone $z^2 = 4(x^2 + y^2)$, on a unit mass at the origin; take $\delta = 1$.

5C-3. Find the gravitational attraction of a solid sphere of radius 1 on a unit point mass Q on its surface, if the density of the sphere at $P(x, y, z)$ is $|PQ|^{-1/2}$.

5C-4. Find the gravitational attraction of the region which is bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the sphere $x^2 + y^2 + z^2 = 2z$, on a unit mass at the origin. (Take $\delta = 1$.)

5C-5.* Find the gravitational attraction of a solid hemisphere of radius a and density 1 on a unit point mass placed at its pole. Ans: $2\pi G a(1 - \sqrt{2}/3)$

5C-6.* Let V be a uniform solid sphere of mass M and radius a. Place a unit point mass a distance b from the center of V. Show that the gravitational attraction of V on the point mass is

a) GM/b^2, if $b \geq a$; b) GM'/b^2, if $b \leq a$, where $M' = \dfrac{b^3}{a^3} M$.

Part (a) is Newton's theorem, described in the Remark. Part (b) says that the outer portion of the sphere—the spherical shell of inner radius b and outer radius a —exerts no force on the test mass: all of it comes from the inner sphere of radius b, which has total mass $\dfrac{b^3}{a^3} M$.

5C-7.* Use Problem 6b to show that if we dig a straight hole through the earth, it takes a point mass m a total of $\pi\sqrt{R/g} \approx 42$ minutes to fall from one end to the other, no matter what the length of the hole is.

(Write $\mathbf{F} = m\mathbf{a}$, letting x be the distance from the middle of the hole, and obtain an equation of simple harmonic motion for $x(t)$. Here

R = earth's radius, M = earth's mass, $g = GM/R^2$.)

6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 Describe geometrically the following vector fields: a) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho}$ b) $-x\,\mathbf{i} - z\,\mathbf{k}$

6A-2 Write down the vector field where each vector runs from (x, y, z) to a point half-way towards the origin.

6A-3 Write down the velocity field $\mathbf{F}$ representing a rotation about the x-axis in the direction given by the right-hand rule (thumb pointing in positive x-direction), and having constant angular velocity ω.

6A-4 Write down the most general vector field all of whose vectors are parallel to the plane $3x - 4y + z = 2$.

6B. Surface Integrals and Flux

6B-1 Without calculating, find the flux of $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the sphere of radius a and center at the origin. Take $\mathbf{n}$ pointing outward.

6B-2 Without calculation, find the flux of $\mathbf{k}$ through the infinite cylinder $x^2 + y^2 = 1$. (Take $\mathbf{n}$ pointing outward.)

6B-3 Without calculation, find the flux of $\mathbf{i}$ through that portion of the plane $x+y+z = 1$ lying in the first octant (take $\mathbf{n}$ pointed away from the origin).

6B-4 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$, and $S =$ the half of the sphere $x^2 + y^2 + z^2 = a^2$ for which $y \geq 0$, oriented so that $\mathbf{n}$ points away from the origin.

6B-5 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where where $\mathbf{F} = z\,\mathbf{k}$, and S is the surface of Exercise 6B-3 above.

6B-6 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ lying underneath the plane $z = 1$, with $\mathbf{n}$ pointing generally upwards. Explain geometrically why your answer is negative.

6B-7* Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{x^2 + y^2 + z^2}$, and S is the surface of Exercise 6B-2.

6B-8 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{j}$ and S is that portion of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$, and to the right of the xz-plane; $\mathbf{n}$ points outwards.

6B-9* Find the center of gravity of a hemispherical shell of radius a. (Assume the density is 1, and place it so its base is on the xy-plane.

6B-10* Let S be that portion of the plane $-12x + 4y + 3z = 12$ projecting vertically onto the plane region $(x - 1)^2 + y^2 \leq 4$. Evaluate

a) the area of S b) $\displaystyle\iint_S z\,dS$ c) $\displaystyle\iint_S (x^2 + y^2 + 3z)\,dS$

6B-11* Let S be that portion of the cylinder $x^2 + y^2 = a^2$ bounded below by the xy-plane and above by the cone $z = \sqrt{(x-a)^2 + y^2}$.

a) Find the area of S. Recall that $\sqrt{1 - \cos\theta} = \sqrt{2}\sin(\theta/2)$. (Hint: remember that the upper limit of integration for the z-integral will be a function of θ determined by the intersection of the two surfaces.)

b) Find the moment of inertia of S about the z-axis. There should be nothing to calculate once you've done part (a).

c) Evaluate $\displaystyle\iint_S z^2\, dS$.

6B-12 Find the average height above the xy-plane of a point chosen at random on the surface of the hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geq 0$.

6C. Divergence Theorem

6C-1 Calculate div $\mathbf{F}$ for each of the following fields

a) $x^2 y\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$ b)* $3x^2yz\,\mathbf{i} + x^3z\,\mathbf{j} + x^3y\,\mathbf{k}$ c)* $\sin^3 x\,\mathbf{i} + 3y\cos^3 x\,\mathbf{j} + 2x\,\mathbf{k}$

6C-2 Calculate div $\mathbf{F}$ if $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, and tell for what value(s) of n we have div $\mathbf{F} = 0$. (Use $\rho_x = x/\rho$, etc.)

6C-3 Verify the divergence theorem when $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the surface composed of the upper half of the sphere of radius a and center at the origin, together with the circular disc in the xy-plane centered at the origin and of radius a.

6C-4* Verify the divergence theorem if $\mathbf{F}$ is as in Exercise 3 and S is the surface of the unit cube having diagonally opposite vertices at (0,0,0) and (1,1,1), with three sides in the coordinate planes. (All the surface integrals are easy and do not require any formulas.)

6C-5 By using the divergence theorem, evaluate the surface integral giving the flux of $\mathbf{F} = x\,\mathbf{i} + z^2\,\mathbf{j} + y^2\,\mathbf{k}$ over the tetrahedron with vertices at the origin and the three points on the positive coordinate axes at distance 1 from the origin.

6C-6 Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ over the closed surface S formed below by a piece of the cone $z^2 = x^2 + y^2$ and above by a circular disc in the plane $z = 1$; take $\mathbf{F}$ to be the field of Exercise 6B-5; use the divergence theorem.

6C-7 Verify the divergence theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 1$ and for its top and bottom circular portions of the planes $z = 1$ and $z = 0$; take $\mathbf{F}$ to be

a) $x^2\,\mathbf{i} + xy\,\mathbf{j}$ b)* $zy\,\mathbf{k}$ c)* $x^2\,\mathbf{i} + xy\,\mathbf{j} + zy\,\mathbf{k}$ (use (a) and (b))

6C-8 Suppose div $\mathbf{F} = 0$ and S_1 and S_2 are the upper and lower hemispheres of the unit sphere centered at the origin. Direct both hemispheres so that the unit normal is "up", i.e., has positive $\mathbf{k}$-component.

a) Show that $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$, and interpret this physically in terms of flux.

b) State a generalization to an arbitrary closed surface S and a field $\mathbf{F}$ such that div $\mathbf{F} = 0$.

6C-9* Let $\mathbf{F}$ be the vector field for which all vectors are aimed radially away from the origin, with magnitude $1/\rho^2$.

a) What is the domain of $\mathbf{F}$?

b) Show that div $\mathbf{F} = 0$.

c) Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere of radius a centered at the origin. Does the fact that the answer is not zero contradict the divergence theorem? Explain.

d) Prove using the divergence theorem that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over a positively oriented closed surface S has the value zero if the surface does not enclose the origin, and the value 4π if it does.

($\mathbf{F}$ is the vector field for the flow arising from a *source of strength* 4π at the origin.)

6C-10 A flow field $\mathbf{F}$ is said to be *incompressible* if $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for all closed surfaces S. Assume that $\mathbf{F}$ is continuously differentiable. Show that

$$\mathbf{F} \textit{ is the field of an incompressible flow} \quad \Longleftrightarrow \quad \text{div } \mathbf{F} = 0 \ .$$

6C-11 Show that the flux of the position vector $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

6D. Line Integrals in Space

6D-1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following fields $\mathbf{F}$ and curves C:

a) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$; C is the twisted cubic curve $x = t,\ y = t^2,\ z = t^3$ running from $(0, 0, 0)$ to $(1, 1, 1)$.

b) $\mathbf{F}$ is the field of (a); C is the line running from $(0, 0, 0)$ to $(1, 1, 1)$

c) $\mathbf{F}$ is the field of (a); C is the path made up of the succession of line segments running from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$.

d) $\mathbf{F} = zx\,\mathbf{i} + zy\,\mathbf{j} + x\,\mathbf{k}$; C is the helix $x = \cos t,\ y = \sin t,\ z = t$, running from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

6D-2 Let $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C lying on a sphere of radius a centered at the origin.

6D-3* a) Let C be the directed line segment running from P to Q, and let $\mathbf{F}$ be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot PQ$.

b) Let C be a closed space polygon $P_1P_2 \ldots P_nP_1$, and let $\mathbf{F}$ be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (a).)

c) Let C be a closed space curve, $\mathbf{F}$ a constant vector field. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (b).)

6D-4 a) Let $f(x,y,z) = x^2 + y^2 + z^2$; calculate $\mathbf{F} = \nabla f$.

b) Let C be the helix of 6D-1d above, but running from $t = 0$ to $t = 2n\pi$. Calculate the work done by $\mathbf{F}$ moving a unit point mass along C; use three methods:

(i) directly

(ii) by using the path-independence of the integral to replace C by a simpler path

(iii) by using the first fundamental theorem for line integrals.

6D-5 Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \sin(xyz)$. What is the maximum value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ over all possible paths C? Give a path C for which this maximum value is attained.

6D-6* Let $\mathbf{F} = \nabla f$, where $f(x,y,z) = \dfrac{1}{x+y+z+1}$. Find the work done by $\mathbf{F}$ carrying a unit point mass from the origin out to ∞ along a ray in the first octant.

(Take the ray to be $x = at, y = bt, z = ct$, with a, b, c positive and $t \geq 0$.)

6E. Gradient Fields in Space

6E-1 Which of the following differentials are exact? For each one which is, express it in the form *df* for a suitable function $f(x,y,z)$, using one of the systematic methods.

a) $x^2\,dx + y^2\,dy + z^2\,dz$ b) $y^2 z\,dx + 2xyz\,dy + xy^2\,dz$

c) $y(6x^2 + z)\,dx + x(2x^2 + z)\,dy + xy\,dz$

6E-2 Find curl $\mathbf{F}$, if $\mathbf{F} = x^2 y\,\mathbf{i} + yz\,\mathbf{j} + xyz^2\,\mathbf{k}$.

6E-3 The fields $\mathbf{F}$ below are defined for all x, y, z. For each,

a) show that curl $\mathbf{F} = \mathbf{0}$;

b) find a potential function $f(x,y,z)$, using either method, or inspection.

(i) $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ (ii) $(2xy + z)\,\mathbf{i} + x^2\,\mathbf{j} + x\,\mathbf{k}$ (iii) $yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

6E-4 Show that if $f(x,y,z)$ and $g(x,y,z)$ are two functions having the same gradient, then $f = g + c$ for some constant c. (Use the Fundamental Theorem for Line Integrals.)

6E-5 For what values of a and b will $\mathbf{F} = yz^2\,\mathbf{i} + (xz^2 + ayz)\,\mathbf{j} + (bxyz + y^2)\,\mathbf{k}$ be a conservative field? Using these values, find the corresponding potential function $f(x,y,z)$ by one of the systematic methods.

6E-6 a) Define what it means for $M dx + N dy + P dz$ to be an exact differential.

b) Find all values of a, b, c for which

$$(axyz + y^3 z^2)dx + (a/2)x^2 z + 3xy^2 z^2 + byz^3)dy + (3x^2 y + cxy^3 z + 6y^2 z^2)dz$$

will be exact.

c) For those values of a, b, c, express the differential as *df* for a suitable $f(x,y,z)$.

6F. Stokes' Theorem

6F-1 Verify Stokes' theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector fields

a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$; b) $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

6F-2 Verify Stokes' theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

6F-3 Verify Stokes' theorem when S is the rectangle with vertices at (0,0,0), (1,1,0), (0,0,1), and (1,1,1), and $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

6F-4* Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where M, N, P have continuous second partial derivatives.

a) Show by direct calculation that div(curl F) = 0.

b) Using (a), show that $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = 0$ for any closed surface S.

6F-5 Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Find the flux of $\nabla \times \mathbf{F}$ through S

(a) directly, by calculating two surface integrals;
(b) by using Stokes' theorem.

6G. Topological Questions

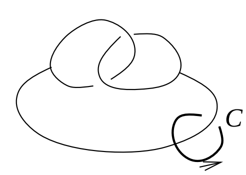
6G-1 Which regions are simply-connected?

a) first octant b) exterior of a torus c) region between two concentric spheres
d) three-space with one of the following removed:
i) a line ii) a point iii) a circle iv) the letter H v) the letter R vi) a ray

6G-2 Show that the fields $\mathbf{F} = \rho^n(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$, where $\rho = \sqrt{x^2 + y^2 + z^2}$, are gradient fields for any value of the integer n. (Use $\rho_x = x/\rho$, etc.)

Then, find the potential function $f(x, y, z)$. (It is easiest to phrase the question in terms of differentials: one wants $df = \rho^n(x\,dx + y\,dy + z\,dz)$; for $n = 0$, you can find f by inspection; from this you can guess the answer for $n \neq 0$ as well. The case $n = -2$ is an exception, and must be handled separately. The printed solutions use this method, somewhat more formally phrased using the fundamental theorem of line integrals.)

6G-3* If D is taken to be the exterior of the wire link shown, then the little closed curve C cannot be shrunk to a point without leaving D, i.e., without crossing the link. Nonetheless, show that C is the boundary of a two-sided surface lying entirely inside D. (So if $\mathbf{F}$ is a field in D such that curl $\mathbf{F} = \mathbf{0}$, the above considerations show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.)



6G-4* In cylindrical coordinates r, θ, z, let $\mathbf{F} = \nabla\varphi$, where $\varphi = \tan^{-1}\dfrac{z}{r-1}$.

a) Interpret φ geometrically. What is the domain of $\mathbf{F}$?

b) From the geometric interpretation what will be the value of $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around a closed path C that links with the unit circle in the xy-plane (for example, take C to be the circle in the yz-plane with radius 1 and center at $(0,1,0)$?

6H. Applications to Physics

6H-1 Prove that $\nabla\cdot\nabla\times\mathbf{F} = 0$. What are the appropriate hypotheses about the field $\mathbf{F}$?

6H-2 Show that for any closed surface S, and continuously differentiable vector field $\mathbf{F}$,

$$\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} \;=\; 0\ .$$

Do it two ways: a) using the divergence theorem; b) using Stokes' theorem.

6H-3* Prove each of the following (ϕ is a (scalar) function):

a) $\nabla\cdot(\phi\,\mathbf{F}) \;=\; \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi$

b) $\nabla\times(\phi\mathbf{F}) \;=\; \phi\nabla\times\mathbf{F} + (\nabla\phi)\times\mathbf{F}$

c) $\nabla\cdot(\mathbf{F}\times\mathbf{G}) \;= \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}$

6H-4* The *normal derivative.* If S is an oriented surface with unit normal vector $\mathbf{n}$, and ϕ is a function defined and differentiable on some domain containing S, then the **normal derivative** of ϕ on S is defined to be the directional derivative of ϕ in the direction $\mathbf{n}$. In symbols (on the left is the notation for the normal derivative):

$$\frac{\partial\phi}{\partial n} \;=\; \nabla\phi\cdot\mathbf{n}.$$

Prove that if S is closed and D its interior, and if ϕ has continuous second derivatives inside D, then

$$\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \nabla^2\phi\,dV\ .$$

(This shows for example that if you are trying to find a *harmonic* function ϕ defined in D and having a prescribed normal derivative on S, you must be sure that $\dfrac{\partial\phi}{\partial n}$ has been prescribed so that $\displaystyle\iint_S \frac{\partial\phi}{\partial n}\,dS \;=\; 0.$

6H-5* Formulate and prove the analogue of the preceding exercise for the plane.

6H-6* Prove that, if S is a closed surface with interior D, and ϕ has continuous second derivatives in D, then

$$\iint_S \phi\frac{\partial\phi}{\partial n}\,dS \;=\; \iiint_D \phi(\nabla^2\phi) + (\nabla\phi)^2\,dV\ .$$

6H-7* Formulate and prove the analogue of the preceding exercise for a plane.

6H-8 A boundary value problem.* Suppose you want to find a function ϕ defined in a domain containing a closed surface S and its interior D, such that (i) ϕ is harmonic in D and (ii) $\phi = 0$ on S.

a) Show that the two conditions imply that $\phi = 0$ on all of D. (Use Exercise 6.)

b) Instead of assuming (ii), assume instead that the values of ϕ on S are prescribed as some continuous function on S. Prove that if a function ϕ exists which is harmonic in D and has these prescribed boundary values, then it is unique — there is only one such function. (In other words, the values of a harmonic function function on the boundary surface S determine its values everywhere inside S.) (Hint: Assume there are two such functions and consider their difference.)

6H-9 Vector potential* In the same way that $\mathbf{F} = \nabla\phi \;\Rightarrow\; \nabla \times \mathbf{F} = \mathbf{0}$ has the partial converse

$$\nabla \times \mathbf{F} = 0 \quad \text{in a simply-connected region} \quad \Rightarrow \quad \mathbf{F} = \nabla f,$$

so the theorem $\quad \mathbf{F} = \nabla \times \mathbf{G} \quad \Rightarrow \quad \nabla \cdot \mathbf{F} = 0 \quad$ has the partial converse

$$(*) \qquad \nabla \cdot \mathbf{F} = 0 \quad \text{in a suitable region} \quad \Rightarrow \quad \mathbf{F} = \nabla \times \mathbf{G}, \quad \text{for some } \mathbf{G}.$$

$\mathbf{G}$ is called a **vector potential** for $\mathbf{F}$. A suitable region is one with this property: whenever P lies in the region, the whole line segment joining P to the origin lies in the region. (Instead of the origin, one could use some other fixed point.) For instance, a sphere, a cube, or all of 3-space would be suitable regions.

Suppose for instance that $\nabla \cdot \mathbf{F} = 0$ in all of 3-space. Then $\mathbf{G}$ exists in all of 3-space, and is given by the formula

$$(**) \qquad \mathbf{G} \;=\; \int_0^1 t\,\mathbf{F}(tx, ty, tz) \times \mathbf{R}\,dt, \qquad\qquad \mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The integral means: integrate separately each component of the vector function occurring in the integrand, and you'll get the corresponding component of $\mathbf{G}$.

We shall not prove this formula here; the proof depends on Leibniz' rule for differentiating under an integral sign. We can however try out the formula.

a) Let $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$. Check that div $\mathbf{F}$=0, find $\mathbf{G}$ from the formula (**), and check your answer by verifying that $\mathbf{F}$ = curl $\mathbf{G}$.

b) Show that $\mathbf{G}$ is unique up to the addition of an arbitrary gradient field; i.e., if $\mathbf{G}$ is one such field, then all others are of the form

$$(***) \qquad \mathbf{G}' = \mathbf{G} + \nabla f,$$

for an arbitrary function $f(x, y, z)$. (Show that if $\mathbf{G}'$ has the form (***), then $\mathbf{F}$ = curl $\mathbf{G}'$; then show conversely that if $\mathbf{G}'$ is a field such that curl $\mathbf{G}' = \mathbf{F}$, then $\mathbf{G}'$ has the form (***).)

6H-10 Let **B** be a magnetic field produced by a moving electric field **E**. Assume there are no charges in the region. Then one of Maxwell's equations in differential form reads

$$\nabla \times \mathbf{B} \;=\; \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t}\ .$$

What is the integrated form of this law? Prove your answer, as in the notes; you can assume that the partial differentiation can be moved outside of the integral sign.

6H-11* In the preceding problem if we also allow for a field $\mathbf{j}$ which gives the current density at each point of space, we get Ampere's law in differential form (as modified by Maxwell):

$$\nabla \times \mathbf{B} \;=\; \frac{1}{c}\left(4\pi\,\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\right).$$

Give the integrated form of this law, and deduce it from the differential form, as done in the notes.

SP. Supplementary Problems

SP1. Vectors and Vector Functions

SP1-A-1. Using the two expressions for the dot product of two vectors, prove the law of cosines, $|C|^2 = |A|^2 + |B|^2 - 2|A||B|\cos\theta$, using vector methods.

(Since the equality of the two expressions for the dot product was proved using the law of cosines, this shows that the cosine law and the dot product equality are equivalent theorems: either implies the other.)

SP1-A-2. Prove using vectors that the diagonals of a parallelogram are equal if and only if it is a rectangle.

SP1-A-3. Prove using vector methods that the four midpoints of a quadrilateral in 3-space lie in a plane and are the vertices of a parallelogram.

SP1-A-4. a) Let $\mathbf{i}'$ and $\mathbf{j}'$ be perpendicular unit vectors in the plane. Then any vector A can be expressed in terms of $\mathbf{i}'$ and $\mathbf{j}'$:

$$A = a\,\mathbf{i}' + b\,\mathbf{j}'\ .$$

Show how to determine the coefficients a and b in terms of A, $\mathbf{i}'$, and $\mathbf{j}'$, by using scalar products.

b) Suppose $\mathbf{i}' = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Find $\mathbf{j}'$, and then express the vector $A = 2\,\mathbf{i} + 3\,\mathbf{j}$ in the $\mathbf{i}'$-$\mathbf{j}'$ system by using the method of part (a).

c) Do part (b) another way by first expressing $\mathbf{i}, \mathbf{j}$ in terms of $\mathbf{i}', \mathbf{j}'$, and then substituting. (This method is substantially harder to carry out by hand in higher dimensions; one needs good computer software.)

SP1-A-5. Let A and B be vectors in space. Prove that $|A+B|^2 = |A|^2 + |B|^2$ if and only if A and B are orthogonal.

SP1-B-1. Let $A = \mathbf{i} - 3\,\mathbf{j} + 2\,\mathbf{k}$. Find a vector C such that $A \cdot C = 1$ and $A \times C = -18\,\mathbf{i} + 9\,\mathbf{k}$.

SP1-B-2. Three vertices of a parallelogram $ABCD$ (labeled in clockwise order) are $A(0,1,-2)$, $B(1,3,-1)$, and $D(-1,4,0)$. Find the fourth vertex and the area of the parallelogram.

SP1-C-1. Invert the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix}$ and use your answer to solve $AX = \begin{pmatrix} 6 \\ 0 \\ -12 \end{pmatrix}$

SP1-D-1. a) Find the intersection of the planes $x - y + 2z = 10$ and $z - y = 2$.

b) Find the angle between the two planes.

SP1-D-2. a) Find in parametric form the equation of the line through the points $(1, 2, -1)$ and $(2, 1, 3)$.

b) Where does this line intersect the plane $2x - 3y + z + 14 = 0$?

SP1-D-3. At noon, a snail starts at the center of an open clock face on a public building. It creeps at a steady rate along the minute hand, reaching the end of the hand at 1:00 PM. The minute hand is 1 meter long.

Write parametric equations for the position of the snail at time t (in hours), taking the center of the clock face as the origin in the xy-plane.

SP1-D-4. a) What part of a train is moving backwards when the train moves forwards?

b) A circular disc has inner radius a and outer radius b. Its inner circle rolls along the positive x-axis without slipping.

Find parametric equations for the motion of a point P on its outer edge, assuming P starts at the point $(0, a-b)$. Use as parameter the angle θ through which the disc has rolled.

c) Sketch the curve that P traces out.

d) Show that the parametric equations imply that P is moving backwards in an interval containing its lowest point.

SP1-E-1. Suppose a particle moves along a circle of radius a and center at the origin.

a) Express its position vector R in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

b) Without using your book, derive expressions for the velocity and acceleration vectors for the above motion, in terms of $\mathbf{u}_r, \mathbf{u}_\theta$.

c) Compare your answers in part (b) with the general expression for $\mathbf{v}$ and $\mathbf{a}$ in your book, and explain why certain terms are missing in your answer.

SP1-E-2. a) Suppose a particle moves along a fixed ray going out from the origin. Show its acceleration vector is $\mathbf{a} = r''\mathbf{u}_r$.

b) Conversely, show that if a particle moves so that $\mathbf{a} = r''\mathbf{u}_r + f(t)\mathbf{u}_\theta$, where $f(t)$ is any differentiable function of t, then it moves along a ray going out from the origin, and $f(t) = 0$. (Use the book's formulas called for in problem E1 above.)

SP1-E-3. Using the same formula in your book as in E1 and E2 above, show:

a) if the motion takes place on a circle of radius a centered at the origin, then the angular acceleration is $a\omega'$, and the radial acceleration is $-a\omega^2$, where ω is the angular velocity;

b) if the angular velocity is a non-zero constant and the force is central, then we have uniform motion on a circle centered at the origin;

c) if there is no radial component to the acceleration and r is a linear function of time, then we have uniform motion along a fixed ray.

SP2- Partial Differentiation

SP2-A-1. A frustum of a solid right circular cone is the piece between two horizontal planes perpendicular to the axis of the cone. (It looks like a solid lampshade.) If a and b are the radii of the two circular faces of the frustum, its volume is

$$V = \frac{\pi}{3}\,(a^2 + ab + b^2)h\ .$$

a) Give an approximate expression for ΔV in terms of $h, a, b, \Delta a, \Delta b, \Delta h$.

b) If $a = 1, b = 2, h = 2$, to which variable is V most sensitive?

SP2-B-1. Use the method of least squares to find the best line through the three points $(0,0), (1,2)$, and $(2,2)$. Sketch the points and the line. (Do the work from scratch, as a minimum problem; don't use a formula.)

SP2-B-2. a) Guess what line the method of least squares will give as the best fit to the three points $(-1,0), (0,1)$, and $(1,0)$. (Plot the points and eyeball them.)

b) Then calculate the least squares line. (Do it as a minimum problem; don't use a formula.)

SP2-C-1. Let $f(u)$ be differentiable, and let $g(x,y) = f\left(\dfrac{x+y}{xy}\right)$. Prove that $x^2 g_x = y^2 g_y$.

SP2-C-2. a) Find the directional derivative of $f(x,y) = \cos \pi xy + xy^2$ in the direction of $A = \mathbf{i} + \mathbf{j}$, at $P(-1,1)$.

b) Use the definition of directional derivative and your answer to part (a) to find approximately how much $f(x,y)$ changes as we move from P a distance .1 along A.

SP2-C-3. a) Define what is meant by a contour curve of $f(x,y)$.

b) Prove that at any point (a,b), the vector ∇f is perpendicular to the contour curve of f passing through (a,b).

SP2-C-4. a) Find the equation of the tangent plane to the surface $3x^2 - y^2 + 3z^2 = 0$ at (x_0, y_0, z_0).

b) Show that the tangent plane always makes an angle of 60^o with the xy-plane.

SP2-C-5. a) Find the equation of the tangent plane to the surface $e^{2xy} + 2x^3z^2 - \sin \pi yz = 1$ at (0,1,1).

b) Use your answer to approximate z by a linear function of x and y for $(x,y) = (0,1)$.

SP3. Double Integrals

SP3-A-1. Evaluate $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x}\, dx\, dy$ by changing the order of integration.

SP3-A-2. Find the average distance from the origin to the points in the triangle with vertices at the origin, $(a,0)$ and $(a, a/\sqrt{2})$.

SP3-B-1. Find the center of mass of the plane region lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

SP3-B-2. Find the average distance from the origin to the points inside the circles

(a) $x^2 + y^2 = a^2$; (b) $(x-1)^2 + y^2 = 1$.

SP4. Line integrals, Conservative Fields, Green's Theorem

SP4-A-1. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = 2xy\,\mathbf{i} + (x^21 + z)\,\mathbf{j} + yz\,\mathbf{k}$, and C is the curve:

a) the line from $(1,1,1)$ to $(2,1,-2)$

b) the semicircle running counterclockwise from $(1,1,0)$ to $(-1,-1,0)$.

SP4-B-1. Let $\mathbf{F} = (y^2 + 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$. Find a function f such that $F = \nabla f$. Do this by calculating $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ along a curve C consisting of line segments running from the origin to (x_1, y_1), first in the $\mathbf{i}$ direction, then in the $\mathbf{j}$ direction.

SP4-B-2. Carry out the work of the previous problem for $\mathbf{F} = (2xy + z)\,\mathbf{i} + x^2\,\mathbf{j} + x\,\mathbf{k}$ (the curve C runs from the origin to (x_1, y_1, z_1) and is made up of three line segments).

SP4-B-3. Define "conservative vector field", and prove that ∇f is a conservative vector field.

SP4-B-4. a) Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ be a continuously differentiable vector field. Prove that if $\mathbf{F}$ is a gradient field, then $M_y = N_x$.

b) Show the converse is false, by calculating $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$ and C is $x^2 + y^2 = 1$. (Why does the answer show the converse is false?)

SP4-B-5. If the $\mathbf{j}$-component of ∇f is proportional to y^3, find $f(1,2,-1) - f(1,-2,-1)$.

SP4-B-6. If $f_y = 3xz^2 + x^2y$ and $f_x = 3yz^2 + xy^2 + \cos(\pi x/2)$, find $f(1,2,-1) - f(5,-1,-1)$.

SP4-C-6. By using Green's theorem, show that if the derivatives are continuous and $M_y = N_x$ in a simply-connected region of the plane (one with no holes), then $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a conservative field in this region.

SP4-D-1. Evaluate $\int \mathbf{F}\cdot\mathbf{n}ds$ where $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$ and C is $(t+1)\,\mathbf{i} + t^2\,\mathbf{j}\,,\ \ 0 \le t \le 1.$

SP4-D-2. Find the flux of $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$ outward across the triangle running from (0,0) to (1,0) to (0,1) and back to (0,0).

SP4-D-3. Let $\mathbf{F} = r^n(x\,\mathbf{i} + y\,\mathbf{j}\,)$. For what values of the integer n will the flux of $\mathbf{F}$ across a circle of radius a and center at the origin be independent of a?

SP4-D-4. Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}\,$. Show that if div $\mathbf{F} = 0$, then the flux of $\mathbf{F}$ across all closed curves C is zero.

SP5. Triple Integrals

SP5-A-1. Find the volume of the region bounded by the elliptic paraboloids $z = x^2+4y^2$ and $z = 8 - x^2 - 4y^2$.

(Hint: make a change of variable $x = 2u$, calculate the volume in the uyz-system; how will this be related to the volume in the xyz-system?)

SP5-A-2. Set up an iterated integral in cylindrical coordinates giving the moment of inertia about the z-axis of the region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the paraboloid $2z = x^2 + y^2$.

SP5-A-3. Find the center of mass of the region bounded above by the plane $2x+4y-z = 0$ and below by the paraboloid $z = x^2 + y^2$.

SP5-B-1. Find the average distance of the points in a sphere of radius a from

(a) its center (b) a fixed plane passing through its center.

SP5-B-2. Find the average distance from the origin to the points in the solid cone given in spherical coordinates by $0 \le \phi \le \pi/4, \quad 0 \le \rho\cos\phi \le 1$.

SP5-B-3. Find the gravitational attraction of a solid right circular cone (base radius is R, height h) on a unit mass at its vertex. Take the density $\delta = 1$. (Place the cone so as to make the calculation easiest.)

SP5-B-4. Evaluate $\iiint_V z^2 dV$, where V is the region bounded above by the unit sphere with center at the origin, and below by the unit sphere with center at $x = y = 0, z = 1$.

(Begin by writing the equations of both spheres in spherical coordinates.)

SP5-B-5. Let the density of a region in space be given by $\delta(x,y,z) = xyz$. Let V be the volume bounded above by the sphere of radius a with center at the origin, and below by the cone $3z = 2\sqrt{3}\sqrt{x^2+y^2}$. Find the average density of the region V.

SP6. Surface Integrals, Divergence Theorem, Stokes' Theorem

SP6-A-1. Let S denote the closed, positively oriented surface whose top is the cap of the sphere $x^2 + y^2 + z^2 = 2a^2$, $z \geq a$, and whose bottom is the flat circular disc at the height $z = a$. Find the flux of $\mathbf{F} = xz\,\mathbf{i} - yz\,\mathbf{k} + y^2\,\mathbf{k}$ outward through S

a) directly, by calculating the two surface integrals;

b) by applying the divergence theorem.

SP6-A-2 Evaluate $\int\int_S F \cdot \mathbf{n}\, dS$, where S is the sphere of radius a centered at the origin, and $F = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

a) directly, by evaluating the surface integral;

b) by applying the divergence theorem.

SP6-A-3 Suppose div $\mathbf{F} = 0$. Show that $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ has the same value for any two similarly oriented and non-intersecting surfaces spanning the same simple closed curve C. Interpret the result physically in terms of fluid flow.

(Take two such surfaces, S and T; apply the divergence theorem to the region they surround.)

SP6-A-4. a) Show that the gravitational field exerted by a mass at the origin is of the form

$$\mathbf{F}(x, y, z) = c(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})(x^2 + y^2 + z^2)^{3/2} \ .$$

b) Find the flux of $\mathbf{F}$ through a sphere of radius a centered at the origin.

c) Calculate div $\mathbf{F}$. What is its value at (0,0,0)?

d) Show the flux of $\mathbf{F}$ through a closed surface not containing the origin is zero.

e) Show that the flux of $\mathbf{F}$ through a closed surface S containing the origin is $4\pi c$. (Take a sufficiently small sphere S' centered at the origin, and apply the divergence theorem to the region between S and S'.)

SP6-A-5. a) Show that the flux of the position vector $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

b) Verify part (a), by calculating the surface integral, when S is a sphere of radius a, centered at the origin.

c) Verify part (a), by calculating the surface integral, when S is the cube centered at the origin and having edges of length 2 parallel to the three coordinate axes. (It is not necessary to calculate six surface integrals: use symmetry.)

d) Let $\mathbf{n}$ be the unit normal (pointing outwards) for a closed surface S. Show that it is impossible for the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ to be orthogonal to $\mathbf{n}$ at every point on the surface.

SP6-B-1. a) Calculate curl $y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

b) Prove that under suitable hypotheses about $f(x, y, z)$, we have curl $\nabla f = 0$.

Selected Solutions and Hints

SP1.

SP1-A1. $|C|^2 = C \cdot C = (A - B) \cdot (A - B)$. SP1-A2. Two diagonals are $A - B$ and $A + B$.

SP1-A4. a) $a = A \cdot \mathbf{i}'; b = A \cdot \mathbf{j}'$. b) $\mathbf{j}' = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$; $A = (5\mathbf{i}' + \mathbf{j}')/\sqrt{2}$

SP1-A5. Essentially same as SP1-A2.

SP1-B1. $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ SP1-B2. $(0, 6, 1), \sqrt{35}$

SP1-C1. $A^{-1} = [5/6, -2/3, -1/2; 1/6, -1/3, -1/2; -1/2, 1, 1/2]$; $X = [11; 7; -9]$ (column vector)

SP1-D1. a) $x = t, y = 6 - t, z = 8 - t$ b) $\pi/6$ SP1-D2. a) $x = 1 + t, y = 2 - t, z = -1 + 4t$ b) (0,3,-5)

SP2.

SP2-A1. b)b SP2-B1. $y = x + 1/3$ SP2-B2. $y = 1/3$ SP2-C2. a) $-1/\sqrt{2}$ b) $-1/10\sqrt{2}$

SP2-C3. b) Parametrize the curve by $x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and differentiate the function $f(x(t), y(t))$.

SP2-C5. a) $2x + \pi y + \pi z = 2\pi$ b) $z = 2 - (2/\pi)x - y$

SP3.

SP3-A1. $1 - \sin 1$ SP3-A2. $(2a/\sqrt{3})(1/3 + (\ln 3)/4)$ SP3-B1.. $\overline{x} = a(15\pi + 32)/6(8 + \pi)$, $\overline{y} = 0$

SP3-B2. a)$2a/3$ b) $32/9\pi$

SP4.

SP4-A1. a) $-\pi$ b) 4 c) $-1/12$

SP4-B1 a) x^2+xy^2 b) x^2y+xz SP4-B4. b) $\oint F \cdot d\mathbf{r} = 2\pi$, so field is not conservative.

SP4-B5. 0 B6. -42

SP4-C1. Hint: the double integral is like a center of mass integral – use this to simplify calculations (use symmetry, etc.)

SP4-D1. 9/4 SP4-D2. 1 SP4-D3. $n = -2$

SP5.

SP5-A1. 8π SP5-A2 $\int_0^{2\pi} \int_0^2 \int_{a^2/2}^{\sqrt{8-r^2}} r^3 dz dr d\theta$ SP5-A3. $(1, 2, 25/3)$

SP5-B1. a) $3a/4$ b) $3a/8$ SP5-B2. $\sqrt{2} - 1/2$ SP5-B3. $2\pi G h(1 - h(h^2 + R^2)^{-1/2})\,\mathbf{k}$

SP5-B4. $59\pi/480$

SP6.

SP6-B1. a) $-(\mathbf{i} + \mathbf{j} + \mathbf{k})$ SP6-B2. a) $a = 6, b = 4, c = 2$ b) $f = 3x^2yz + xy^3z^2 + 2y^2z^3$

SP6-C1. a) 0 b) $-\pi$ C3. 0

1. Vectors and Matrices

1A. Vectors

1A-1 a) $|\mathbf{A}| = \sqrt{3}$, dir $\mathbf{A} = \mathbf{A}/\sqrt{3}$ b) $|\mathbf{A}| = 3$, dir $\mathbf{A} = \mathbf{A}/3$
c) $|\mathbf{A}| = 7$, dir $\mathbf{A} = \mathbf{A}/7$

1A-2 $1/25 + 1/25 + c^2 = 1 \;\Rightarrow\; c = \pm\sqrt{23}/5$

1A-3 a) $\mathbf{A} = -\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$, $|\mathbf{A}| = 3$, dir $\mathbf{A} = \mathbf{A}/3$.

b) $\mathbf{A} = |\mathbf{A}|$ dir $\mathbf{A} = 2\,\mathbf{i} + 4\,\mathbf{j} - 4\,\mathbf{k}$. Let P be its tail and Q its head. Then $OQ = OP + \mathbf{A} = 4\,\mathbf{j} - 3\,\mathbf{k}$; therefore $Q = (0, 4, -3)$.

1A-4 a) $OX = OP + PX = OP + \frac{1}{2}(PQ) = OP + \frac{1}{2}(OQ - OP) = \frac{1}{2}(OP + OQ)$
b) $OX = s\,OP + r\,OQ$; replace $\frac{1}{2}$ by r in above; use $1 - r = s$.

1A-5 $\mathbf{A} = \frac{3}{2}\sqrt{3}\,\mathbf{i} + \frac{3}{2}\,\mathbf{j}$. The condition is not redundant since there are two vectors of length 3 making an angle of 30^o with $\mathbf{i}$.

1A-6 wind $\mathbf{w} = 50(-\,\mathbf{i} - \mathbf{j})/\sqrt{2})$, $\mathbf{v} + \mathbf{w} = 200\,\mathbf{j} \;\Rightarrow\; \mathbf{v} = 50/\sqrt{2}\,\mathbf{i} + (200 + 50/\sqrt{2})\,\mathbf{j}$.

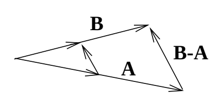
1A-7 a) $b\,\mathbf{i} - a\,\mathbf{j}$ b) $-b\,\mathbf{i} + a\,\mathbf{j}$ c) $(3/5)^2 + (4/5)^2 = 1$; $\mathbf{j}' = -(4/5)\,\mathbf{i} + (3/5)\,\mathbf{j}$

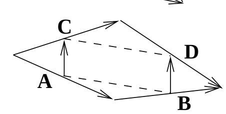
1A-8 a) is elementary trigonometry;

b) $\cos\alpha = a/\sqrt{a^2 + b^2 + c^2}$, etc.; dir $\mathbf{A} = (-1/3, 2/3, 2/3)$

c) if t, u, v are direction cosines of some $\mathbf{A}$, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k}$ = dir $\mathbf{A}$, a unit vector, so $t^2 + u^2 + v^2 = 1$; conversely, if this relation holds, then $t\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k} = \mathbf{u}$ is a unit vector, so dir $\mathbf{u} = \mathbf{u}$ and t, u, v are the direction cosines of $\mathbf{u}$.

1A-9 Letting $\mathbf{A}$ and $\mathbf{B}$ be the two sides, the third side is $\mathbf{B} - \mathbf{A}$; the line joining the two midpoints is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}$, which $= \frac{1}{2}(\mathbf{B} - \mathbf{A})$, a vector parallel to the third side and half its length.





1A-10 Letting $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be the four sides; then if the vectors are suitably oriented, we have $\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$.

The vector from the midpoint of $\mathbf{A}$ to the midpoint of $\mathbf{C}$ is $\frac{1}{2}\mathbf{C} - \frac{1}{2}\mathbf{A}$; similarly, the vector joining the midpoints of the other two sides is $\frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{D}$, and

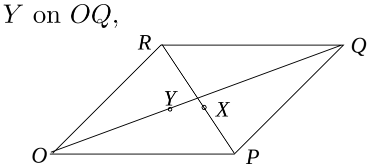
$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D} \quad\Rightarrow\quad \mathbf{C} - \mathbf{A} = \mathbf{B} - \mathbf{D} \quad\Rightarrow\quad \tfrac{1}{2}(\mathbf{C} - \mathbf{A}) = \tfrac{1}{2}(\mathbf{B} - \mathbf{D})\;;$$

thus two opposite sides are equal and parallel, which shows the figure is a parallelogram.

1A-11 Letting the four vertices be O, P, Q, R, with X on PR and Y on OQ,

$$\begin{aligned} OX &= OP + PX = OP + \tfrac{1}{2}PR \\ &= OP + \tfrac{1}{2}(OR - OP) \\ &= \tfrac{1}{2}(OR + OP) = \tfrac{1}{2}OQ = OY; \end{aligned}$$

therefore $X = Y$.



1B. Dot Product

1B-1 a) $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \dfrac{4+2}{\sqrt{2}\cdot 6} = \dfrac{1}{\sqrt{2}}, \quad \theta = \dfrac{\pi}{4}$ b) $\cos\theta = \dfrac{3}{\sqrt{6}\cdot\sqrt{6}} = \dfrac{1}{2}, \quad \theta = \dfrac{\pi}{3}$.

1B-2 $\mathbf{A}\cdot\mathbf{B} = c-4$; therefore (a) orthogonal if $c = 4$,

b) $\cos\theta = \dfrac{c-4}{\sqrt{c^2+5}\sqrt{6}}$; the angle θ is acute if $\cos\theta > 0$, i.e., if $c > 4$.

1B-3 Place the cube in the first octant so the origin is at one corner P, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three edges. The longest diagonal $PQ = \mathbf{i}+\mathbf{j}+\mathbf{k}$; a face diagonal $PR = \mathbf{i}+\mathbf{j}$.

a) $\cos\theta = \dfrac{PQ\cdot PR}{|PQ|\cdot|PR|} = \dfrac{2}{\sqrt{3}\sqrt{2}}; \quad \theta = \cos^{-1}\sqrt{2/3}$

b) $\cos\theta = \dfrac{PQ\cdot\mathbf{i}}{|PQ||\mathbf{i}|} = \dfrac{1}{\sqrt{3}}, \quad \theta = \cos^{-1} 1/\sqrt{3}$.

1B-4 $QP = (a, 0, -2), \quad QR = (a, -2, 2)$, therefore

a) $QP\cdot QR = a^2 - 4$; therefore PQR is a right angle if $a^2 - 4 = 0$, i.e., if $a = \pm 2$.

b) $\cos\theta = \dfrac{a^2-4}{\sqrt{a^2+4}\sqrt{a^2+8}}$; the angle is acute if $\cos\theta > 0$, i.e., if $a^2 - 4 > 0$, or $|a| > 2$, i.e., $a > 2$ or $a < -2$.

1B-5 a) $\mathbf{F}\cdot\mathbf{u} = -1/\sqrt{3}$ b) $\mathbf{u} = \text{dir}\,\mathbf{A} = \mathbf{A}/7$, so $\mathbf{F}\cdot\mathbf{u} = -4/7$

1B-6 After dividing by $|OP|$, the equation says $\cos\theta = c$, where θ is the angle between OP and $\mathbf{u}$; call its solution $\theta_0 = \cos^{-1} c$. Then the locus is the nappe of a right circular cone with axis in the direction $\mathbf{u}$ and vertex angle $2\theta_0$. In particular this cone is

a) a plane if $\theta_0 = \pi/2$, i.e., if $c = 0$ b) a ray if $\theta_0 = 0,\ \pi$, i.e., if $c = \pm 1$.

c) Locus is the origin, if $c > 1$ or $c < -1$ (division by $|OP|$ is illegal, notice).

1B-7 a) $|\mathbf{i}'| = |\mathbf{j}'| = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$; a picture shows the system is right-handed.

b) $\mathbf{A}\cdot\mathbf{i}' = -1/\sqrt{2}; \qquad \mathbf{A}\cdot\mathbf{j}' = -5/\sqrt{2}$;

since they are perpendicular unit vectors, $\mathbf{A} = \dfrac{-\mathbf{i}' - 5\mathbf{j}'}{\sqrt{2}}$.

c) Solving, $\mathbf{i} = \dfrac{\mathbf{i}'-\mathbf{j}'}{\sqrt{2}}, \quad \mathbf{j} = \dfrac{\mathbf{i}'+\mathbf{j}'}{\sqrt{2}}$;

thus $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} = \dfrac{2(\mathbf{i}'-\mathbf{j}')}{\sqrt{2}} - \dfrac{3(\mathbf{i}'+\mathbf{j}')}{\sqrt{2}} = \dfrac{-\mathbf{i}'-5\mathbf{j}'}{\sqrt{2}}$, as before.

1B-8 a) Check that each has length 1, and the three dot products $\mathbf{i}'\cdot\mathbf{j}',\ \mathbf{i}'\cdot\mathbf{k}',\ \mathbf{j}'\cdot\mathbf{k}'$ are 0; make a sketch to check right-handedness.

b) $\mathbf{A}\cdot\mathbf{i}' = \sqrt{3}, \quad \mathbf{A}\cdot\mathbf{j}' = 0, \quad \mathbf{A}\cdot\mathbf{k}' = \sqrt{6}$, therefore, $\mathbf{A} = \sqrt{3}\,\mathbf{i}' + \sqrt{6}\,\mathbf{k}'$.

1B-9 Let $\mathbf{u} = \text{dir}\,\mathbf{A}$, then the vector $\mathbf{u}$-component of $\mathbf{B}$ is $(\mathbf{B}\cdot\mathbf{u})\mathbf{u}$. Subtracting it off gives a vector perpendicular to $\mathbf{u}$ (and therefore also to $\mathbf{A}$); thus

$$\mathbf{B} = (\mathbf{B}\cdot\mathbf{u})\mathbf{u} + \big(\mathbf{B} - (\mathbf{B}\cdot\mathbf{u})\mathbf{u}\big)$$

or in terms of A, remembering that $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$,

$$\mathbf{B} \;=\; \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A} + \left(\mathbf{B} \;-\; \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A}\right)$$

.

1B-10 Let two adjacent edges of the parallelogram be the vectors $\mathbf{A}$ and $\mathbf{B}$; then the two diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$. Remembering that for any vector $\mathbf{C}$ we have $\mathbf{C}\cdot\mathbf{C} = |\mathbf{C}|^2$, the two diagonals have equal lengths

$$\begin{aligned}
\Leftrightarrow \quad (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) &= (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) \\
\Leftrightarrow \quad (\mathbf{A}\cdot\mathbf{A}) + 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\mathbf{A}) - 2(\mathbf{A}\cdot\mathbf{B}) + (\mathbf{B}\cdot\mathbf{B}) \\
\Leftrightarrow \quad \mathbf{A}\cdot\mathbf{B} &= 0,
\end{aligned}$$

which says the two sides are perpendicular, i.e., the parallelogram is a rectangle.

1B-11 Using the notation of the previous exercise, we have successively,

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B}; \quad \text{therefore,} \\
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = 0 \quad &\Leftrightarrow \quad \mathbf{A}\cdot\mathbf{A} \;=\; \mathbf{B}\cdot\mathbf{B},
\end{aligned}$$

i.e., the diagonals are perpendicular if and only if two adjacent edges have equal length, in other words, if the parallelogram is a rhombus.

1B-12 Let O be the center of the semicircle, Q and R the two ends of the diameter, and P the vertex of the inscribed angle; set $\mathbf{A} = QO = OR$ and $\mathbf{B} = OP$; then $|\mathbf{A}| = |\mathbf{B}|$.

The angle sides are $QP = \mathbf{A}+\mathbf{B}$ and $PR = \mathbf{A}-\mathbf{B}$; they are perpendicular since

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B} \\
&= 0, \qquad \text{since} \;\; |\mathbf{A}| = |\mathbf{B}|.
\end{aligned}$$

1B-13 The unit vectors are $\mathbf{u}_i = \cos\theta_i\,\mathbf{i} + \sin\theta_i\,\mathbf{j}$, for $i = 1, 2$; the angle between them is $\theta_2 - \theta_1$. We then have by the geometric definition of the dot product

$$\begin{aligned}
\cos(\theta_2 - \theta_1) &= \frac{\mathbf{u}_1\cdot\mathbf{u}_2}{|\mathbf{u}_1||\mathbf{u}_2|}, \\
&= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2,
\end{aligned}$$

according to the formula for evaluating the dot product in terms of components.

1B-14 Let the coterminal vectors $\mathbf{A}$ and $\mathbf{B}$ represent two sides of the triangle, and let θ be the included angle. Suitably directed, the third side is then $\mathbf{C} = \mathbf{A} - \mathbf{B}$, and

$$\begin{aligned}
|\mathbf{C}|^2 \;=\; (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) &= \mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} - 2\mathbf{A}\cdot\mathbf{B} \\
&= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta,
\end{aligned}$$

by the geometric interpretation of the dot product.

1C. Determinants

1C-1 a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = -1-8=-9$ b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = -10.$

1C-2 $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} = 2+0-8-(24+4+0)=-34.$

a) By the cofactors of row one: $= -1\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 4\cdot\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -34$

b) By the cofactors of column one: $= -1\cdot\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 0 & 4 \\ -2 & -1 \end{vmatrix} + 3\cdot\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} = -34.$

1C-3 a) $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3;$ so area of the parallelogram is 3, area of the triangle is 3/2

b) sides are $PQ=(0,-3)$, $PR=(1,1)$, $\begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} = 3$, so area of the parallelogram is 3, area of the triangle is 3/2

1C-4 $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = x_2x_3^2 + x_1x_2^2 + x_1^2x_3 - x_1^2x_2 - x_2^2x_3 - x_1x_3^2$

$(x_1-x_2)(x_1-x_3)(x_2-x_3) = x_1^2x_2 - x_1^2x_3 - x_1x_3x_2 + x_1x_3^2 - x_2^2x_1 + x_2x_1x_3 + x_2^2x_3 - x_2x_3^2.$

Two terms cancel, and the other six are the same as those above, except they have the opposite sign.

1C-5 a) $\begin{vmatrix} x_1 & y_1 \\ x_2+ax_1 & y_2+ay_1 \end{vmatrix} = x_1y_2 + ax_1y_1 - x_2y_1 - ay_1x_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$

b) is similar.

1C-6 Use the Laplace expansion by the cofactors of the first row.

1C-7 The heads of two vectors are on the unit circle. The area of the parallelogram they span is biggest when the vectors are perpendicular, since area $= ab\sin\theta = 1\cdot 1\cdot\sin\theta$, and $\sin\theta$ has its maximum when $\theta=\pi/2$.

Therefore the maximum value of $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} =$ area of unit square $= 1$.

1C-9 $PQ=(0,-1,2)$, $PR=(0,1,-1)$, $PS=(1,2,1)$;

$$\text{volume parallelepiped} = \pm\begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \pm(-1) = 1\ .$$

vol. tetrahedron $= \frac{1}{3}$(base)(ht.) $= \frac{1}{3}\cdot\frac{1}{2}$ (p'piped base)(ht.) $= \frac{1}{6}$(vol. p'piped) $= 1/6$.

1C-10 Thinking of them all as origin vectors, **A** lies in the plane of **B** and **C**, therefore the volume of the parallelepiped spanned by the three vectors is zero.

1D. Cross Product

1D-1 a) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 3\,\mathbf{i} - (-3)\,\mathbf{j} + 3\,\mathbf{k}$ b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$

1D-2 $PQ = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $PR = -3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -1\,\mathbf{i} + 5\,\mathbf{j} + 4\,\mathbf{k}$;

area of the triangle $= \frac{1}{2}|PQ \times PR| = \frac{1}{2}\sqrt{42}$.

1D-3 We get a third vector (properly oriented) perpendicular to $\mathbf{A}$ and $\mathbf{B}$ by using $\mathbf{A}\times\mathbf{B}$:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -\,\mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k}$$

Make these unit vectors: $\mathbf{i}' = \mathbf{A}/\sqrt{5}$, $\mathbf{j}' = \mathbf{B}/\sqrt{6}$, $\mathbf{k}' = (-\,\mathbf{i} - 2\,\mathbf{j} + 5\,\mathbf{k})/\sqrt{30}$.

1D-4 $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$; $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

1D-5 For both, use $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

a) $\sin\theta = 1 \Rightarrow \theta = \pi/2$; the two vectors are orthogonal.

b) $|\mathbf{A}||\mathbf{B}|\sin\theta = |\mathbf{A}||\mathbf{B}|\cos\theta$, therefore $\tan\theta = 1$, so $\theta = \pi/4$

1D-6 Taking the cube so P is at the origin and three coterminous edges are $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the three diagonals of the faces are $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{k}$, so

$$\text{volume of parallelepiped spanned by diagonals} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2\ .$$

1D-7 We have $PQ = (-2, 1, 1)$, $PR = (-1, 0, 1)$, $PS = (2, 1, -2)$;

$$\text{volume parallelepiped} = \pm \begin{vmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \pm 1 = 1; \quad \text{volume tetrahedron} = \frac{1}{6}.$$

1D-8 One determinant has rows in the order $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the other represents $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and therefore has its rows in the order $\mathbf{C}, \mathbf{A}, \mathbf{B}$.

To change the first determinant into the second, interchange the second and third rows, then the first and second row; each interchange multiplies the determinant by -1, according to **D-1** (see Notes D), therefore the net effect of two successive interchanges is to leave its value unchanged; thus $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

1D-9 a) Lift the triangle up into the plane $z = 1$, so its vertices are at the three points $P_i = (x_i, y_i, 1)$, $i = 1, 2, 3$.

volume tetrahedron $OP_1P_2P_3 = \frac{1}{3}$(height)(base) $= \frac{1}{3}1\cdot$(area of triangle);

volume tetrahedron $OP_1P_2P_3 = \frac{1}{6}$ (volume parallelepiped spanned by the OP_i)

$= \frac{1}{6}$ (determinant);

Therefore: area of triangle $= \frac{1}{2}$ (determinant)

b) Subtracting the first row from the second, and the first row from the third does not alter the value of the determinant, by **D-4**, and gives

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{vmatrix}$$
$$= \frac{1}{2}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix},$$

using the Laplace expansion by the cofactors of the last column; but this 2×2 determinant gives the area of the parallelogram spanned by the vectors representing two sides of the plane triangle, and the triangle has half this area.

1E. Lines and Planes

1E-1 a) $(x-2)+2y-2(z+1)=0 \quad\Rightarrow\quad x+2y-2z=4.$

b) $\mathbf{N} = (1,1,0)\times(2,-1,3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 3\,\mathbf{i} - 3\,\mathbf{j} - 3\,\mathbf{k}$

Removing the common factor 3, the equation is $x-y-z=0$.

c) Calling the points respectrively P,Q,R, we have $PQ=(1,-1,1)$, $PR=(-2,3,1)$;
$\mathbf{N} = PQ\times PR = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = -4\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$

Equation (through $P:(1,0,1)$): $-4(x-1)-3y+(z-1)=0$, or $-4x-3y+z=-3$.

d) $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1.$

e) $\mathbf{N}$ must be perpendicular to both $\mathbf{i}-\mathbf{j}+2\,\mathbf{k}$ and $PQ=(-1,1,0)$. Therefore
$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\,\mathbf{i} - 2\,\mathbf{j}$; a plane through $(1,0,1)$ with this $\mathbf{N}$ is then $x+y=1$.

1E-2 The dihedral angle between two planes is the same as the angle θ between their normal vectors. The normal vectors to the planes are respectively $(2,-1,1)$ and $(1,1,2)$; therefore $\cos\theta = \dfrac{3}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2}$, so that $\theta = 60^o$ or $\pi/3$.

1E-3 a) $x=1+2t,\ y=-t,\ z=-1+3t.$

b) $x=2+t,\ y=-1-t,\ z=-1+2t,$ since the line has the direction of the normal to the plane.

c) The direction vector of the line should be parallel to the plane, i.e., perpendicular to its normal vector $\mathbf{i}+2\,\mathbf{j}-\mathbf{k}$; so the answer is

$x=1+at,\ y=1+bt,\ z=1+ct,$ where $a+2b-c=0$, a,b,c not all 0, or better,

$x=1+at,\ y=1+bt,\ z=1+(a+2b)t$ for any constants a,b.

1E-4 The line has direction vector $PQ = (2, -1, 1)$, so its parametric equations are:

$$x = 2t,\ y = 1 - t,\ z = 2 + t.$$

Substitute these into the equation of the plane to find a point that lies in both the line and the plane:

$$2t + 4(1-t) + (2+t) = 4, \quad \text{or} \quad -t + 6 = 4;$$

therefore $t = 2$, and the point is (substituting into the parametric equations): $(4, -1, 4)$.

1E-5 The line has the direction of the normal to the plane, so its parametric equations are

$$x = 1 + t,\ y = 1 + 2t,\ z = -1 - t;$$

substituting, it intersects the plane when

$$2(1+t) - (1+2t) + (-1-t) = 1, \quad \text{or} \quad -t = 1;$$

therefore, at $(0, 1, 0)$.

1E-6 Let $P_0 : (x_0, y_0, z_0)$ be a point on the plane, and $\mathbf{N} = (a, b, c)$ be a normal vector to the plane. The distance we want is the length of that origin vector which is perpendicular to the plane; but this is exactly the component of OP_0 in the direction of $\mathbf{N}$. So we get (choose the sign which makes it positive):

$$\text{distance from point to plane} = \pm\, OP_0 \cdot \frac{\mathbf{N}}{|\mathbf{N}|} = \pm\,(x_0, y_0, z_0) \cdot \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2+b^2+c^2}} = \frac{|d|}{\sqrt{a^2+b^2+c^2}};$$

the last equality holds since the point satisfies the equation of the plane.

1F. Matrix Algebra

1F-3 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{pmatrix}$

We want all four entries of the product to be zero; this gives the equations:

$$a^2 = -bc, \qquad b(a+d) = 0, \qquad c(a+d) = 0, \qquad d^2 = -bc.$$

case 1: $a + d \neq 0$; then $b = 0$ and $c = 0$; thus $a = 0$ and $d = 0$.

case 2: $a + d = 0$; then $d = -a$, $bc = -a^2$

Answer: $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $bc = -a^2$, i.e., $\begin{vmatrix} a & b \\ c & -a \end{vmatrix} = 0.$

1F-5 a) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$; $\quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Guess: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; $\quad$ Proof by induction:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}.$$

1F-8 a) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\d\\g\end{pmatrix}$; $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}b\\e\\h\end{pmatrix}$; etc.

Answer: $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 1 & 4 & -1 \end{pmatrix}$ (See p. 13 for a more .

1F-9 For the entries of the product matrix $A \cdot A^T = C$, we have

$$c_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j, \end{cases} \qquad \text{since } A \cdot A^T = I.$$

On the other hand, by the definition of matrix multiplication,

$$c_{ij} = (i\text{-th row of } A)\cdot(j\text{-th column of } A^T) = (i\text{-th row of } A)\cdot(j\text{-th row of } A).$$

Since the right-hand sides of the two expressions for c_{ij} must be equal, when $j = i$ it shows that the i-th row has length 1; while for $j \neq i$, it shows that different rows are orthogonal to each other.

1G. Solving Square Systems; Inverse Matrices

1G-3 $M = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & 2 \\ -2 & -1 & 1 \end{pmatrix}$; $M^T = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$; $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$,

where M is the matrix of cofactors (watch the signs), M^T is its transpose (the adjoint matrix), and we calculated that $\det A = 5$, to get A^{-1} . Thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix}2\\0\\3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}0\\-5\\5\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

1G-4 The system is $A\mathbf{x} = \mathbf{y}$; the solution is $\mathbf{x} = A^{-1}\mathbf{y}$, or $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 1 & -2 \\ -1 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$;

written out, this is the system of equations

$$x_1 = \tfrac{3}{5}y_1 + \tfrac{1}{5}y_2 - \tfrac{2}{5}y_3, \qquad x_2 = -\tfrac{1}{5}y_1 + \tfrac{3}{5}y_2 - \tfrac{1}{5}y_3, \qquad x_3 = \tfrac{1}{5}y_1 + \tfrac{2}{5}y_2 + \tfrac{1}{5}y_3 .$$

1G-5 Using in turn the associative law, definition of the inverse, and identity law,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Similarly, $(AB)(B^{-1}A^{-1}) = I$. Therefore, $B^{-1}A^{-1}$ is the inverse to AB.

1H. Cramer's Rule; Theorems about Square Systems

1H-1 b) $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2;$ $\quad x = \dfrac{1}{|A|}\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \dfrac{1}{2}\cdot 4 = 2.$

1H-2 Using Cramer's rule, the determinants in the numerators for x, y, and z all have a column of zeros, therefore have the value zero, by the determinant law **D-2**.

1H-3 a) The condition for it to have a non-zero solution is $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{vmatrix} = 0$; expanding,

$2 + 2c + 1 - (-1 + c - 4) = 0$, or $c = -8$.

b) $\begin{cases} (2-c)x + y = 0 \\ (-1-c)y = 0 \end{cases}$ has a nontrivial solution if $\begin{vmatrix} 2-c & 1 \\ 0 & -1-c \end{vmatrix} = 0$, i.e.,
if $(2-c)(-1-c) = 0$, or $c = 2$, $c = -1$.

c) Take $c = -8$. The equations say we want a vector (x_1, x_2, x_3) which is orthogonal to the three vectors

$$(1, -1, 1), \quad (2, 1, 1), \quad (-1, -8, 2)\ .$$

A vector orthogonal to the first two is $(1, -1, 1) \times (2, 1, 1) = (-2, 1, 3)$ (by calculation). And this is orthogonal to $(-1, -8, 2)$ also: $(-2, 1, 3) \cdot (-1, -8, 2) = 0$.

1H-5 If (x_0, y_0) is a solution, then $\begin{cases} a_1x_0 + b_1y_0 = c_1 \\ a_2x_0 + b_2y_0 = c_2 \end{cases}$.

Eliminating x_0 gives $(a_2b_1 - a_1b_2)y_0 = a_2c_1 - a_1c_2$.

The left side is zero by hypothesis, so the right side is also zero: $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$.

Conversely, if this holds, then a solution is $x_0 = \dfrac{c_1}{a_1}$, $y_0 = 0$ (or $x_0 = \dfrac{c_2}{a_2}$, if $a_1 = 0$).

1H-7 a) $\begin{cases} a\cos x_1 + b\sin x_1 &= y_1 \\ a\cos x_2 + b\sin x_2 &= y_2; \end{cases}$ has a unique solution if $\begin{vmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{vmatrix} \neq 0$,
i.e.,

if $\cos x_1 \sin x_2 - \cos x_2 \sin x_1 \neq 0$, or equivalently, $\sin(x_2 - x_1) \neq 0$, and this last holds if and only if $x_2 - x_1 \neq n\pi$, for any integer n.

b) Since $\cos(x + n\pi) = (-1)^n \cos x$ and $\sin(x + n\pi) = (-1)^n \sin x$, the equations are solvable if and only if $y_2 = (-1)^n y_1$.

1I. Vector Functions and Parametric Equations

1I-1 Let $\mathbf{u} = \text{dir}\,(a\,\mathbf{i} + b\,\mathbf{j}) = \dfrac{a\,\mathbf{i} + b\,\mathbf{j}}{\sqrt{a^2+b^2}}$, and $\mathbf{x}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j}$. Then

$$\mathbf{r}(t) = \mathbf{x}_0 + vt\,\mathbf{u}$$

1I-2 a) Since the motion is the reflection in the x-axis of the usual counterclockwise motion, $\mathbf{r} = a\,\cos(\omega t)\,\mathbf{i} - a\,\sin(\omega t)\,\mathbf{j}$. (This is a little special; part (b) illustrates an approach more generally applicable.)

b) The position vector is $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$. At time $t = 0$, the angle $\theta = \pi/2$; then it decreases linearly at the rate ω. Therefore $\theta = \pi/2 - \omega t$; substituting and then using the trigonometric identities for $\cos(A + B)$ and $\sin(A + B)$, we get

$$\mathbf{r} = a\cos(\pi/2 - \omega t)\,\mathbf{i} + a\sin(\pi/2 - \omega t)\,\mathbf{j} = a\sin\omega t\,\mathbf{i} + a\cos\omega t\,\mathbf{j}$$

(In retrospect, we could have given another "special" derivation by observing that this motion is the reflection in the diagonal line $y = x$ of the usual counterclockwise motion starting at $(a, 0)$, so we get its position vector $\mathbf{r}(t)$ by interchanging the x and y in the usual position vector function $\mathbf{r} = \cos(\omega t)\,\mathbf{i} + \sin(\omega t)\,\mathbf{j}$.)

1I-3 a) $x = 2\cos^2 t,\ y = \sin^2 t,$ so $x + 2y = 2$; only the part of this line between $(0, 1)$ and $(2, 0)$ is traced out, back and forth.

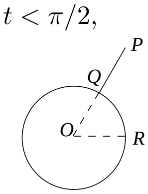
b) $x = \cos 2t,\ y = \cos t$; eliminating t to get the xy-equation, we have

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 \quad \Rightarrow \quad x = 2y^2 - 1\ ;$$

only the part of this parabola between $(1, 1)$ and $(1, -1)$ is traced out, back and forth.

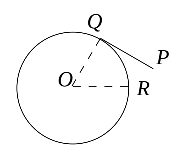
c) $x = t^2 + 1,\ y = t^3$; eliminating t, we get $y^2 = (x - 1)^3$; the entire curve is traced out as t increases, with y going from $-\infty$ to ∞.

d) $x = \tan t,\ y = \sec t$; eliminate t via the trigonometric identity $\tan^2 t + 1 = \sec^2 t$, getting $y^2 - x^2 = 1$. This is a hyperbola; the upper branch is traced out for $-\pi/2 < t < \pi/2$, the lower branch for $\pi/2 < t < 3\pi/2$. Then it repeats.



1I-4 $OP = |OP| \cdot \operatorname{dir} OP$; $\operatorname{dir} OP = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$;
$|OP| = |OQ| + |QP| = a + a\theta$, since $|QP| = \text{arc } QR = a\theta$.

So $OP = a(1 + \theta)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ or $x = a(1 + \theta)\cos\theta,\ y = a(1 + \theta)\sin\theta$.



1I-5 $OP = OQ + QP$; $OQ = a(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$
$QP = |QP| \operatorname{dir} QP = a\theta(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$, since $|QP| = \text{arc } QR = a\theta$
(cf. Exer. 1A-7a for dir QP)

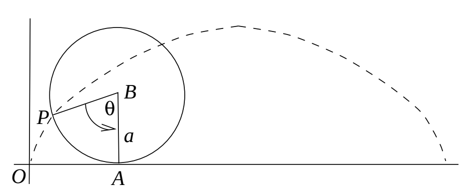
Therefore, $OP = \mathbf{r} = a(\cos\theta + \theta\sin\theta)\,\mathbf{i} + a(\sin\theta - \theta\cos\theta)\,\mathbf{j}$

1I-6 a) $\mathbf{r}_1(t) = (10 - t)\,\mathbf{i}$ (hunter); $\mathbf{r}_2(t) = t\,\mathbf{i} + 2t\,\mathbf{j}$ (rabbit; note that $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, so indeed $|\mathbf{v}| = \sqrt{5}$)

$$\text{Arrow} = HA = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \text{ since the arrow has unit length}$$
$$= \frac{(t-5)\,\mathbf{i} + t\,\mathbf{j}}{\sqrt{2t^2 - 10t + 25}}, \text{ after some algebra.}$$

b) It is easier mathematically to minimize the square of the distance between hunter and rabbit, rather than the distance itself; you get the same t-value in either case.

Let $f(t) = |\mathbf{r}_2 - \mathbf{r}_1|^2 = 2t^2 - 10t + 25$; then $f'(t) = 4t - 10 = 0$ when $t = 2.5$.



1I-7 $OP = OA + AB + BP$;

$OA = \text{arc}AP = a\theta$; $AB = a\,\mathbf{j}$;

$BP = a(-\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$

Therefore, $OP = a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j}$.

1J. Vector Derivatives

1J-1 a) $\mathbf{r} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$; $\mathbf{v} = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $|\mathbf{v}| = \sqrt{e^{2t}+e^{-2t}}$, $\mathbf{T} = \dfrac{e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}}{\sqrt{e^{2t}+e^{-2t}}}$,

$\mathbf{a} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

b) $\mathbf{r} = t^2\,\mathbf{i} + t^3\,\mathbf{j}$; $\mathbf{v} = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$; $|\mathbf{v}| = t\sqrt{4+9t^2}$; $\mathbf{T} = \dfrac{2\,\mathbf{i} + 3t\,\mathbf{j}}{\sqrt{4+9t^2}}$;

$\mathbf{a} = 2\,\mathbf{i} + 6t\,\mathbf{j}$

c) $\mathbf{r} = (1-2t^2)\,\mathbf{i} + t^2\,\mathbf{j} + (-2+2t^2)\,\mathbf{k}$; $\mathbf{v} = 2t(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $|\mathbf{v}| = 6t$;

$\mathbf{T} = \frac{1}{3}(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$; $\mathbf{a} = 2(-2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k})$

1J-2 a) $\mathbf{r} = \dfrac{1}{1+t^2}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j}$; $\mathbf{v} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{(1+t^2)^2}$; $|\mathbf{v}| = \dfrac{1}{1+t^2}$; $\mathbf{T} = \dfrac{-2t\,\mathbf{i} + (1-t^2)\,\mathbf{j}}{1+t^2}$

b) $|\mathbf{v}|$ is largest when $t=0$, therefore at the point $(1,0)$. There is no point at which $|\mathbf{v}|$ is smallest; as $t \to \infty$ or $t \to -\infty$, the point $P \to (0,0)$, and $|\mathbf{v}| \to 0$.

c) The position vector shows $y = tx$, so $t = y/x$; substituting into $x = 1/(1+t^2)$ yields after some algebra the equation $x^2+y^2-x=0$; completing the square gives the equation $(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$, which is a circle with center at $(\frac{1}{2},0)$ and radius $\frac{1}{2}$.

1J-3 $$\frac{d}{dt}(x_1y_1 + x_2y_2) = \begin{cases} x_1'y_1 + x_1y_1' + \\ x_2'y_2 + x_2y_2' \end{cases}$$

Adding the columns, we get: $(x_1,x_2)'\cdot(y_1,y_2) + (x_1,x_2)\cdot(y_1,y_2)' = \dfrac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\dfrac{d\mathbf{s}}{dt}$.

1J-4 a) Since P moves on a sphere, say of radius a,

$$x(t)^2 + y(t)^2 + z(t)^2 = a^2;$$

Differentiating,

$$2xx' + 2yy' + 2zz' = 0,$$

which says that $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \cdot x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k} = 0$ for all t, i.e., $\mathbf{r}\cdot\mathbf{r}' = 0$.

b) Since by hypothesis, $\mathbf{r}(t)$ has length a, for all t, we get the chain of implications

$$|\mathbf{r}| = a \quad\Rightarrow\quad \mathbf{r}\cdot\mathbf{r} = a^2 \quad\Rightarrow\quad 2r\cdot\frac{d\mathbf{r}}{dt} = 0 \quad\Rightarrow\quad \mathbf{r}\cdot\mathbf{v} = 0.$$

c) Using first the result in Exercise **1J-3**, then $\mathbf{r}\cdot\mathbf{r} = |\mathbf{r}|^2$, we have

$$\mathbf{r}\cdot\mathbf{v} = 0 \quad\Rightarrow\quad \frac{d}{dt}\mathbf{r}\cdot\mathbf{r} = 0 \quad\Rightarrow\quad \mathbf{r}\cdot\mathbf{r} = c, \text{ a constant}, \quad\Rightarrow\quad |\mathbf{r}| = \sqrt{c},$$

which shows that the head of $\mathbf{r}$ moves on a sphere of radius $\sqrt{c}$.

1J-5 a) $|\mathbf{v}| = c \quad\Rightarrow\quad \mathbf{v}\cdot\mathbf{v} = c^2 \quad\Rightarrow\quad \dfrac{d}{dt}\mathbf{v}\cdot\mathbf{v} = 2\mathbf{v}\cdot\mathbf{a} = 0$, by **1J-3**. Therefore the velocity and acceleration vectors are perpendicular.

b) $\mathbf{v}\cdot\mathbf{a} = 0 \quad\Rightarrow\quad \frac{d}{dt}\mathbf{v}\cdot\mathbf{v} = 0 \quad\Rightarrow\quad \mathbf{v}\cdot\mathbf{v} = a \quad\Rightarrow\quad |\mathbf{v}| = \sqrt{a}$, which shows the speed is constant.

1J-6 a) $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$; $\mathbf{v} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}$; $|\mathbf{v}| = \sqrt{a^2+b^2}$;

$\mathbf{T} = \dfrac{\mathbf{v}}{\sqrt{a^2+b^2}}$; $\mathbf{a} = -a(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$

b) By direct calculation using the components, we see that $\mathbf{v} \cdot \mathbf{a} = 0$; this also follows theoretically from Exercise **1J-5b**, since the speed is constant.

1J-7 a) The criterion is $|\mathbf{v}| = 1$; namely, if we measure arclength s so $s = 0$ when $t = 0$, then since s increases with t,

$$|\mathbf{v}| = 1 \quad \Rightarrow \quad ds/dt = 1 \quad \Rightarrow \quad s = t + c \quad \Rightarrow \quad s = t\ .$$

b) $\mathbf{r} = (x_0 + at)\,\mathbf{i} + (y_0 + at)\,\mathbf{j}$; $\mathbf{v} = a(\,\mathbf{i} + \mathbf{j}\,)$; $|\mathbf{v}| = a\sqrt{2}$; therefore choose $a = 1/\sqrt{2}$.

c) Choose a and b to be non-negative numbers such that $a^2 + b^2 = 1$; then $\mathbf{v}| = 1$.

1J-8 a) Let $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$; then $u(t)\mathbf{r}(t) = ux\,\mathbf{i} + uy\,\mathbf{j}$, and differentiation gives

$(u\mathbf{r})' = (ux)'\,\mathbf{i} + (uy)'\,\mathbf{j} = (u'x + ux')\,\mathbf{i} + (u'y + uy')\,\mathbf{j} = u'(x\,\mathbf{i} + y\,\mathbf{j}) + u(x'\,\mathbf{i} + y'j) = u'\mathbf{r} + u\mathbf{r}'$.

b)
$$\begin{aligned}\frac{d}{dt}\, e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) &= e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) + e^t(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) \\ &= e^t\big((\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big).\end{aligned}$$

Therefore $|\mathbf{v}| \;=\; e^t\big|(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\big| = 2e^t$, after calculation.

1J-9 a) $\mathbf{r} = 3\cos t\,\mathbf{i} + 5\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{r}| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5.$

b) $\mathbf{v} = -3\sin t\,\mathbf{i} + 5\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$; therefore $|\mathbf{v}| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5.$

c) $\mathbf{a} = d\mathbf{v}/dt = -3\cos t\,\mathbf{i} - 5\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k} = -\mathbf{r}$

d) By inspection, the x, y, z coordinates of p satisfy $4x - 3z = 0$, which is a plane through the origin.

e) Since by part (a) the point P moves on the surface of a sphere of radius 5 centered at the origin, and by part (d) also in a plane through the origin, its path must be the intersection of these two surfaces, which is a great circle of radius 5 on the sphere.

1J-10 a) Use the results of Exercise **1J-6**:

$$\mathbf{T} \;=\; \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2 + b^2}}; \qquad |\mathbf{v}| \;=\; \sqrt{a^2 + b^2}\ .$$

By the chain rule,

$$\left|\frac{d\mathbf{T}}{dt}\right| \;=\; \left|\frac{d\mathbf{T}}{ds}\right|\left|\frac{ds}{dt}\right|;$$

therefore

$$\frac{|-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}\,|}{\sqrt{a^2 + b^2}} \;=\; \kappa\sqrt{a^2 + b^2}\ ;$$

since the numerator on the left has the value $|a|$, we get

$$\kappa \;=\; \frac{|a|}{a^2 + b^2}\ .$$

b) If $b = 0$, the helix is a circle of radius $|a|$ in the xy-plane, and $\kappa = \dfrac{1}{|a|}$.

1K. Kepler's Second Law

1K-1 $\displaystyle \frac{d}{dt}(x_1y_1 + x_2y_2) \;=\; \begin{cases} x_1'y_1 + x_1y_1' + \\ x_2'y_2 + x_2y_2' \end{cases}$

Adding the columns, we get: $\langle x_1, x_2\rangle' \cdot \langle y_1, y_2\rangle + \langle x_1, x_2\rangle \cdot \langle y_1, y_2\rangle' \;=\; \dfrac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \dfrac{d\mathbf{s}}{dt}.$

1K-2 In two dimensions, $\mathbf{s}(t) = \langle x(t), y(t)\rangle$, $\mathbf{s}'(t) = \langle x'(t), y'(t)\rangle$.

Therefore $\mathbf{s}'(t) = \mathbf{0} \;\Rightarrow\; x'(t) = 0,\; y'(t) = 0 \;\Rightarrow\; x(t) = k_1,\; y(t) = k_2 \;\Rightarrow\; \mathbf{s}(t) = (k_1, k_2)$ where k_1, k_2 are constants.

1K-3 Since $\mathbf{F}$ is central, we have $\mathbf{F} = c\mathbf{r}$; using Newton's law, $\mathbf{a} = \mathbf{F}/m \;=\; (c/m)\mathbf{r}$; so

$$
\begin{aligned}
\mathbf{F} = c\mathbf{r} \quad &\Rightarrow \quad \mathbf{r} \times \mathbf{a} \;=\; \mathbf{r} \times \frac{c}{m}\mathbf{r} \;=\; \mathbf{0}, \\
&\Rightarrow \quad \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) \;=\; \mathbf{0}, \qquad \text{by (7)} \\
(*) \qquad &\Rightarrow \quad \mathbf{r} \times \mathbf{v} \;=\; \mathbf{K}, \quad \text{a constant vector, by Exercise K-2.}
\end{aligned}
$$

This last line (*) shows that $\mathbf{r}$ is perpendicular to $\mathbf{K}$, and therefore its head (the point P) lies in the plane through the origin which has $\mathbf{K}$ as normal vector. Also, since

$$
\begin{aligned}
|\mathbf{r} \times \mathbf{v}| \;&=\; 2\,\frac{dA}{dt}, \quad \text{by (2)}, \\
|\mathbf{r} \times \mathbf{v}| \;&=\; |K|, \quad \text{by (*)},
\end{aligned}
$$

we conclude that

$$\frac{dA}{dt} \;=\; \frac{1}{2}\,|K|\;,$$

which shows the area is swept out at a constant rate.

2. Partial Differentiation

2A. Functions and Partial Derivatives

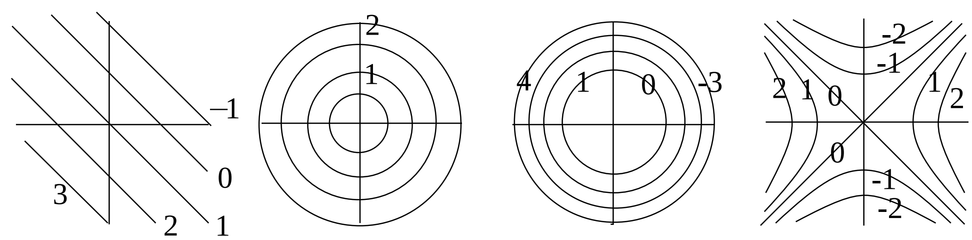
2A-1 In the pictures below, not all of the level curves are labeled. In (c) and (d), the picture is the same, but the labelings are different. In more detail:

b) the origin is the level curve 0; the other two unlabeled level curves are .5 and 1.5;

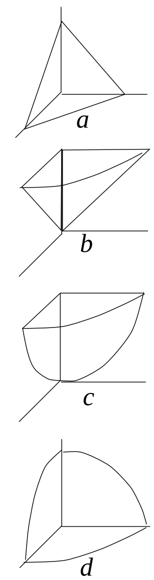
c) on the left, two level curves are labeled; the unlabeled ones are 2 and 3; the origin is the level curve 0;

d) on the right, two level curves are labeled; the unlabeled ones are -1 and -2; the origin is the level curve 1;

The crude sketches of the graph in the first octant are at the right.



a *b* *c, d* *e*



2A-2 a) $f_x = 3x^2y - 3y^2$, $f_y = x^3 - 6xy + 4y$ b) $z_x = \dfrac{1}{y}$, $z_y = -\dfrac{x}{y^2}$

c) $f_x = 3\cos(3x+2y)$, $f_y = 2\cos(3x+2y)$

d) $f_x = 2xye^{x^2y}$, $f_y = x^2e^{x^2y}$ e) $z_x = \ln(2x+y) + \dfrac{2x}{2x+y}$, $z_y = \dfrac{x}{2x+y}$

f) $f_x = 2xz$, $f_y = -2z^3$, $f_z = x^2 - 6yz^2$

2A-3 a) both sides are $mnx^{m-1}y^{n-1}$

b) $f_x = \dfrac{y}{(x+y)^2}$, $f_{xy} = (f_x)_y = \dfrac{x-y}{(x+y)^3}$; $f_y = \dfrac{-x}{(x+y)^2}$, $f_{yx} = \dfrac{-(y-x)}{(x+y)^3}$.

c) $f_x = -2x\sin(x^2+y)$, $f_{xy} = (f_x)_y = -2x\cos(x^2+y)$;
$f_y = -\sin(x^2+y)$, $f_{yx} = -\cos(x^2+y)\cdot 2x$.

d) both sides are $f'(x)g'(y)$.

2A-4 $(f_x)_y = ax + 6y$, $(f_y)_x = 2x + 6y$; therefore $f_{xy} = f_{yx} \Leftrightarrow a = 2$. By inspection, one sees that if $a = 2$, $f(x,y) = x^2y + 3xy^2$ is a function with the given f_x and f_y.

2A-5

a) $w_x = ae^{ax}\sin ay$, $w_{xx} = a^2e^{ax}\sin ay$;
$w_y = e^{ax}a\cos ay$, $w_{yy} = e^{ax}a^2(-\sin ay)$; therefore $w_{yy} = -w_{xx}$.

b) We have $w_x = \dfrac{2x}{x^2+y^2}$, $w_{xx} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$. If we interchange x and y, the function $w = \ln(x^2+y^2)$ remains the same, while w_{xx} gets turned into w_{yy}; since the interchange just changes the sign of the right hand side, it follows that $w_{yy} = -w_{xx}$.

2B. Tangent Plane; Linear Approximation

2B-1 a) $z_x = y^2$, $z_y = 2xy$; therefore at (1,1,1), we get $z_x = 1$, $z_y = 2$, so that the tangent plane is $z = 1 + (x-1) + 2(y-1)$, or $z = x + 2y - 2$.

b) $w_x = -y^2/x^2$, $w_y = 2y/x$; therefore at (1,2,4), we get $w_x = -4$, $w_y = 4$, so that the tangent plane is $w = 4 - 4(x-1) + 4(y-2)$, or $w = -4x + 4y$.

2B-2 a) $z_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{z}$; by symmetry (interchanging x and y), $z_y = \dfrac{y}{z}$; then the tangent plane is $z = z_0 + \dfrac{x_0}{z_0}(x - x_0) + \dfrac{y_0}{z_0}(y - y_0)$, or $z = \dfrac{x_0}{z_0}x + \dfrac{y_0}{z_0}y$, since $x_0^2 + y_0^2 = z_0^2$.

b) The line is $x = x_0 t$, $y = y_0 t$, $z = z_0 t$; substituting into the equations of the cone and the tangent plane, both are satisfied for all values of t; this shows the line lies on both the cone and tangent plane (this can also be seen geometrically).

2B-3 Letting x, y, z be respectively the lengths of the two legs and the hypotenuse, we have $z = \sqrt{x^2+y^2}$; thus the calculation of partial derivatives is the same as in **2B-2**, and we get $\Delta z \approx \dfrac{3}{5}\Delta x + \dfrac{4}{5}\Delta y$. Taking $\Delta x = \Delta y = .01$, we get $\Delta z \approx \dfrac{7}{5}(.01) = .014$.

2B-4 From the formula, we get $R = \dfrac{R_1 R_2}{R_1 + R_2}$. From this we calculate

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1+R_2}\right)^2, \text{ and by symmetry, } \frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1+R_2}\right)^2.$$

Substituting $R_1 = 1$, $R_2 = 2$ the approximation formula then gives $\Delta R = \dfrac{4}{9}\Delta R_1 + \dfrac{1}{9}\Delta R_2$.

By hypothesis, $|\Delta R_i| \le .1$, for $i = 1, 2$, so that $|\Delta R| \le \dfrac{4}{9}(.1) + \dfrac{1}{9}(.1) = \dfrac{5}{9}(.1) \approx .06$; thus

$$R = \frac{2}{3} = .67 \ \pm .06.$$

2B-5 a) We have $f(x,y) = (x+y+2)^2$, $f_x = 2(x+y+2)$, $f_y = 2(x+y+2)$. Therefore

at $(0,0)$, $f_x(0,0) = f_y(0,0) = 4$, $f(0,0) = 4$; linearization is $4 + 4x + 4y$;

at $(1,2)$, $f_x(1,2) = f_y(1,2) = 10$, $f(1,2) = 25$;
linearization is $10(x-1) + 10(y-2) + 25$, or $10x + 10y - 5$.

b) $f = e^x \cos y$; $f_x = e^x \cos y$; $f_y = -e^x \sin y$.

linearization at $(0,0)$: $1 + x$; linearization at $(0, \pi/2)$: $-(y - \pi/2)$

2B-6 We have $V = \pi r^2 h$, $\dfrac{\partial V}{\partial r} = 2\pi r h$, $\dfrac{\partial V}{\partial h} = \pi r^2$; $\Delta V \approx \left(\dfrac{\partial V}{\partial r}\right)_0 \Delta r + \left(\dfrac{\partial V}{\partial h}\right)_0 \Delta h$.
Evaluating the partials at $r = 2$, $h = 3$, we get

$$\Delta V \approx 12\pi \Delta r + 4\pi \Delta h.$$

Assuming the same accuracy $|\Delta r| \le \epsilon$, $|\Delta h| \le \epsilon$ for both measurements, we get

$$|\Delta V| \le 12\pi\,\epsilon + 4\pi\,\epsilon = 16\pi\,\epsilon, \quad \text{which is } < .1 \text{ if } \epsilon < \frac{1}{160\pi} < .002 .$$

2B-7 We have $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Therefore at $(3,4)$, $r = 5$, and $\Delta r \approx \frac{3}{5}\Delta x + \frac{4}{5}\Delta y$. If $|\Delta x|$ and $|\Delta y|$ are both $\le .01$, then

$$|\Delta r| \le \tfrac{3}{5}|\Delta x| + \tfrac{4}{5}|\Delta y| = \tfrac{7}{5}(.01) = .014 \text{ (or .02)}.$$

Similarly, $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2}$; $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2}$, so at the point $(3,4)$,

$$|\Delta\theta| \;\le\; |\tfrac{-4}{25}\Delta x| + |\tfrac{3}{25}\Delta y| \;\le\; \tfrac{7}{25}(.01) \;=\; .0028 \text{ (or } .003).$$

Since at $(3,4)$ we have $|r_y| > |r_x|$, r is more sensitive there to changes in y; by analogous reasoning, θ is more sensitive there to x.

2B-9 a) $w = x^2(y+1)$; $w_x = 2x(y+1) = 2$ at $(1,0)$, and $w_y = x^2 = 1$ at $(1,0)$; therefore w is more sensitive to changes in x around this point.

b) To first order approximation, $\Delta w \approx 2\Delta x + \Delta y$, using the above values of the partial derivatives.

If we want $\Delta w = 0$, then by the above, $2\Delta x + \Delta y = 0$, or $\Delta y/\Delta x = -2$.

2C. Differentials; Approximations

2C-1 a) $dw = \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$ b) $dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2dz$

c) $dz = \dfrac{2y\,dx - 2x\,dy}{(x+y)^2}$ d) $dw = \dfrac{t\,du - u\,dt}{t\sqrt{t^2-u^2}}$

2C-2 The volume is $V = xyz$; so $dV = yz\,dx + xz\,dy + xy\,dz$. For $x = 5$, $y = 10$, $z = 20$,

$$\Delta V \approx dV = 200\,dx + 100\,dy + 50\,dz,$$

from which we see that $|\Delta V| \le 350(.1)$; therefore $V = 1000 \pm 35$.

2C-3 a) $A = \frac{1}{2}ab\sin\theta$. Therefore, $dA = \frac{1}{2}(b\sin\theta\,da + a\sin\theta\,db + ab\cos\theta\,d\theta)$.

b) $dA = \frac{1}{2}(2\cdot\frac{1}{2}\,da + 1\cdot\frac{1}{2}\,db + 1\cdot 2\cdot\frac{1}{2}\sqrt{3}\,d\theta) = \frac{1}{2}(da + \frac{1}{2}\,db + \sqrt{3}\,d\theta)$;
therefore most sensitive to θ, least senstitive to b, since $d\theta$ and db have respectively the largest and smallest coefficients.

c) $dA = \frac{1}{2}(.02 + .01 + 1.73(.02) \approx \frac{1}{2}(.065) \approx .03$

2C-4 a) $P = \dfrac{kT}{V}$; therefore $dP = \dfrac{k}{V}\,dT - \dfrac{kT}{V^2}\,dV$

b) $V\,dP + P\,dV = k\,dT$; therefore $dP = \dfrac{k\,dT - P\,dV}{V}$.

c) Substituting $P = kT/V$ into (b) turns it into (a).

2C-5 a) $-\dfrac{dw}{w^2} = -\dfrac{dt}{t^2} - \dfrac{du}{u^2} - \dfrac{dv}{v^2}$; therefore $dw = w^2\left(\dfrac{dt}{t^2} + \dfrac{du}{u^2} + \dfrac{dv}{v^2}\right)$.

b) $2u\,du + 4v\,dv + 6w\,dw = 0$; therefore $dw = -\dfrac{u\,du + 2v\,dv}{3w}$.

2D. Gradient; Directional Derivative

2D-1 a) $\nabla f = 3x^2\,\mathbf{i} + 6y^2\,\mathbf{j}$; $(\nabla f)_P = 3\,\mathbf{i} + 6\,\mathbf{j}$; $\left.\dfrac{df}{ds}\right|_{\mathbf{u}} = (3\,\mathbf{i} + 6\,\mathbf{j})\cdot\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = -\dfrac{3\sqrt{2}}{2}$

b) $\nabla w = \dfrac{y}{z}\,\mathbf{i} + \dfrac{x}{z}\,\mathbf{j} - \dfrac{xy}{z^2}\,\mathbf{k}$; $(\nabla w)_P = -\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$; $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (\nabla w)_P\cdot\dfrac{\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}}{3} = -\dfrac{1}{3}$

c) $\nabla z = (\sin y - y\sin x)\,\mathbf{i} + (x\cos y + \cos x)\,\mathbf{j}$; $(\nabla z)_P = \mathbf{i} + \mathbf{j}$;

$$\left.\frac{dz}{ds}\right|_{\mathbf{u}} = (\,\mathbf{i} + \mathbf{j}\,)\cdot\frac{-3\,\mathbf{i} + 4\,\mathbf{j}}{5} = \frac{1}{5}$$

d) $\nabla w = \dfrac{2\,\mathbf{i} + 3\,\mathbf{j}}{2t+3u}$; $\qquad (\nabla w)_P = 2\,\mathbf{i} + 3\,\mathbf{j}$; $\qquad \left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \dfrac{4i - 3\,\mathbf{j}}{5} = -\dfrac{1}{5}$

e) $\nabla f = 2(u+2v+3w)(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$; $\qquad (\nabla f)_P = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,)$

$$\left.\frac{df}{ds}\right|_{\mathbf{u}} = 4(\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}\,) \cdot \frac{-2\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{3} = -\frac{4}{3}$$

2D-2 a) $\nabla w = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{4x-3y}$; $\qquad (\nabla w)_P = 4\,\mathbf{i} - 3\,\mathbf{j}$

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = (4\,\mathbf{i} - 3\,\mathbf{j}\,) \cdot \mathbf{u}$ has maximum 5, in the direction $\mathbf{u} = \dfrac{4\,\mathbf{i} - 3\,\mathbf{j}}{5}$,
and minimum -5 in the opposite direction.

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{3\,\mathbf{i} + 4\,\mathbf{j}}{5}$.

b) $\nabla w = \langle y+z, x+z, x+y\rangle$; $\qquad (\nabla w)_P = \langle 1,3,0\rangle$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{10}$, direction $\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$; $\qquad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{10}$, direction $-\dfrac{\mathbf{i} + 3\,\mathbf{j}}{\sqrt{10}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\mathbf{u} = \pm\dfrac{-3\,\mathbf{i} + \mathbf{j} + c\,\mathbf{k}}{\sqrt{10+c^2}}$ (for all c)

c) $\nabla w = 2\sin(t-u)\cos(t-u)(\,\mathbf{i} - \mathbf{j}\,) = \sin 2(t-u)(\,\mathbf{i} - \mathbf{j}\,)$; $\qquad (\nabla w)_P = \mathbf{i} - \mathbf{j}$;

max $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = \sqrt{2}$, direction $\dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$; $\qquad$ min $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = -\sqrt{2}$, direction $-\dfrac{-\,\mathbf{i} + \mathbf{j}}{\sqrt{2}}$;

$\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} = 0$ in the directions $\pm\dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$

2D-3 a) $\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$; $\qquad (\nabla f)_P = \langle 4,12,36\rangle$; $\quad$ normal at P: $\langle 1,3,9\rangle$;
tangent plane at P: $x+3y+9z=18$

b) $\nabla f = \langle 2x, 8y, 18z\rangle$; $\quad$ normal at P: $\langle 1,4,9\rangle$, $\quad$ tangent plane: $x+4y+9z=14$.

c) $(\nabla w)_P = \langle 2x_0, 2y_0, -2z_0\rangle$; $\quad$ tangent plane: $x_0(x-x_0)+y_0(y-y_0)-z_0(z-z_0)=0$,
or $x_0x+y_0y-z_0z=0$, since $x_0^2+y_0^2-z_0^2=0$.

2D-4 a) $\nabla T = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{x^2+y^2}$; $\qquad (\nabla T)_P = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5}$;

T is increasing at P most rapidly in the direction of $(\nabla T)_P$, which is $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$.

b) $|\nabla T| = \dfrac{2}{\sqrt{5}}$ = rate of increase in direction $\dfrac{\mathbf{i} + 2\,\mathbf{j}}{\sqrt{5}}$. Call the distance to go Δs, then

$$\frac{2}{\sqrt{5}}\,\Delta s = .20 \quad\Rightarrow\quad \Delta s = \frac{.2\sqrt{5}}{2} = \frac{\sqrt{5}}{10} \approx .22.$$

c) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \dfrac{2\,\mathbf{i} + 4\,\mathbf{j}}{5} \cdot \dfrac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \dfrac{6}{5\sqrt{2}}$;

$$\frac{6}{5\sqrt{2}}\,\Delta s = .12 \quad\Rightarrow\quad \Delta s = \frac{5\sqrt{2}}{6}\,(.12) \;\approx\; (.10)(\sqrt{2}) \;\approx\; .14$$

d) In the directions orthogonal to the gradient: $\pm\dfrac{2\,\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

2D-5 a) isotherms = the level surfaces $x^2 + 2y^2 + 2z^2 = c$, which are ellipsoids.

b) $\nabla T = \langle 2x, 4y, 4z\rangle$; $(\nabla T)_P = \langle 2, 4, 4\rangle$; $|(\nabla T)_P| = 6$;
for most rapid decrease, use direction of $-(\nabla T)_P$: $-\frac{1}{3}\langle 1, 2, 2\rangle$

c) let Δs be distance to go; then $-6(\Delta s) = -1.2$; $\Delta s \approx .2$

d) $\left.\dfrac{dT}{ds}\right|_{\mathbf{u}} = (\nabla T)_P \cdot \mathbf{u} = \langle 2, 4, 4\rangle \cdot \dfrac{\langle 1, -2, 2\rangle}{3} = \dfrac{2}{3}$; $\dfrac{2}{3}\Delta s \approx .10 \Rightarrow \Delta s \approx .15.$

2D-6 $\nabla uv = \langle (uv)_x, (uv)_y\rangle = \langle uv_x + vu_x, uv_y + vu_y\rangle = \langle uv_x, uv_y\rangle + \langle vu_x + vu_y\rangle = u\nabla v + v\nabla u$

$$\nabla(uv) = u\nabla v + v\nabla u \quad \Rightarrow \quad \nabla(uv)\cdot\mathbf{u} = u\nabla v\cdot\mathbf{u} + v\nabla u\cdot\mathbf{u} \quad \Rightarrow \quad \left.\frac{d(uv)}{ds}\right|_{\mathbf{u}} = u\left.\frac{dv}{ds}\right|_{\mathbf{u}} + v\left.\frac{du}{ds}\right|_{\mathbf{u}}.$$

2D-7 At P, let $\nabla w = a\,\mathbf{i} + b\,\mathbf{j}$. Then

$$a\,\mathbf{i} + b\,\mathbf{j} \cdot \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = 2 \quad \Rightarrow \quad a + b = 2\sqrt{2}$$

$$a\,\mathbf{i} + b\,\mathbf{j} \cdot \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = 1 \quad \Rightarrow \quad a - b = \sqrt{2}$$

Adding and subtracting the equations on the right, we get $a = \frac{3}{2}\sqrt{2}$, $b = \frac{1}{2}\sqrt{2}$.

2D-8 We have $P(0,0,0) = 32$; we wish to decrease it to 31.1 by traveling the shortest distance from the origin $\mathbf{0}$; for this we should travel in the direction of $-(\nabla P)_{\mathbf{0}}$.

$$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z\rangle; \quad (\nabla P)_{\mathbf{0}} = \langle 2, 1, 2\rangle. \qquad |(\nabla P)_{\mathbf{0}}| = 3.$$

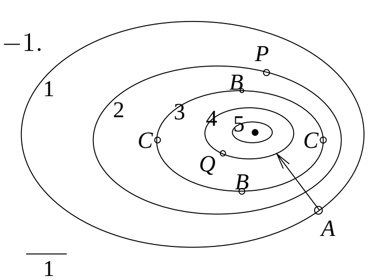
Since $(-3)\cdot(\Delta s) = -.9 \quad \Rightarrow \quad \Delta s = .3$, we should travel a distance .3 in the direction of $-(\nabla P)_{\mathbf{0}}$. Since $|-\langle 2, 1, 2\rangle| = 3$, the distance .3 will be $\frac{1}{10}$ of the distance from $(0,0,0)$ to $(-2,-1,-2)$, which will bring us to $(-.2,-.1,-.2)$.

2D-9 In these, we use $\left.\dfrac{dw}{ds}\right|_{\mathbf{u}} \approx \dfrac{\Delta w}{\Delta s}$: we travel in the direction $\mathbf{u}$ from a given point P to the nearest level curve C; then Δs is the distance traveled (estimate it by using the unit distance), and Δw is the corresponding change in w (estimate it by using the labels on the level curves).

a) The *direction* of ∇f is perpendicular to the level curve at A, in the increasing sense (the "uphill" direction). The *magnitude* of ∇f is the directional derivative in that direction: from the picture, $\dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2.$

b), c) $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}}$, $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}}$, so B will be where $\mathbf{i}$ is tangent to the level curve and C where $\mathbf{j}$ is tangent to the level curve.

d) At P, $\dfrac{\partial w}{\partial x} = \left.\dfrac{dw}{ds}\right|_{\mathbf{i}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/3} = -.6$; $\dfrac{\partial w}{\partial y} = \left.\dfrac{dw}{ds}\right|_{\mathbf{j}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{1} = -1.$



e) If $\mathbf{u}$ is the direction of $\mathbf{i} + \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_u \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{1}{.5} = 2$

f) If $\mathbf{u}$ is the direction of $\mathbf{i} - \mathbf{j}$, we have $\left.\dfrac{dw}{ds}\right|_u \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{-1}{5/4} = -.8$

g) The gradient is 0 at a local extremum point: here at the point marked giving the location of the hilltop.

2E. Chain Rule

2E-1

a) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = yz \cdot 1 + xz \cdot 2t + xy \cdot 3t^2 = t^5 + 2t^5 + 3t^5 = 6t^5$

(ii) $w = xyz = t^6$; $\quad \dfrac{dw}{dt} = 6t^5$

b) (i) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2x(-\sin t) - 2y(\cos t) = -4\sin t\cos t$

(ii) $w = x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t$; $\quad \dfrac{dw}{dt} = -2\sin 2t$

c) (i) $\dfrac{dw}{dt} = \dfrac{2u}{u^2+v^2}(-2\sin t) + \dfrac{2v}{u^2+v^2}(2\cos t) = -\cos t\sin t + \sin t\cos t = 0$

(ii) $w = \ln(u^2+v^2) = \ln(4\cos^2 t + 4\sin^2 t) = \ln 4$; $\quad \dfrac{dw}{dt} = 0.$

2E-2 a) The value $t = 0$ corresponds to the point $(x(0), y(0)) = (1, 0) = P$.

$$\left.\frac{dw}{dt}\right|_0 = \left.\frac{\partial w}{\partial x}\right|_P \left.\frac{dx}{dt}\right|_0 + \left.\frac{\partial w}{\partial y}\right|_P \left.\frac{dy}{dt}\right|_0 = -2\sin t + 3\cos t\Big]_0 = 3.$$

b) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t.$

$\dfrac{dw}{dt} = 0$ when $2t = \dfrac{\pi}{2} + n\pi$, therefore when $t = \dfrac{\pi}{4} + \dfrac{n\pi}{2}$.

c) $t = 1$ corresponds to the point $(x(1), y(1), z(1)) = (1, 1, 1)$.

$$\left.\frac{df}{dt}\right|_1 = 1 \cdot \left.\frac{dx}{dt}\right|_1 - 1 \cdot \left.\frac{dy}{dt}\right|_1 + 2 \cdot \left.\frac{dz}{dt}\right|_1 = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 3 = 5.$$

d) $\dfrac{df}{dt} = 3x^2y\dfrac{dx}{dt} + (x^3+z)\dfrac{dy}{dt} + y\dfrac{dz}{dt} = 3t^4 \cdot 1 + 2x^3 \cdot 2t + t^2 \cdot 3t^2 = 10t^4.$

2E-3 a) Let $w = uv$, where $u = u(t)$, $v = v(t)$; $\quad \dfrac{dw}{dt} = \dfrac{\partial w}{\partial u}\dfrac{du}{dt} + \dfrac{\partial w}{\partial v}\dfrac{dv}{dt} = v\,\dfrac{du}{dt} + u\,\dfrac{dv}{dt}.$

b) $\dfrac{d(uvw)}{dt} = vw\dfrac{du}{dt} + uw\dfrac{dv}{dt} + uv\dfrac{dw}{dt}$; $\quad e^{2t}\sin t + 2te^{2t}\sin t + te^{2t}\cos t$

2E-4 The values $u = 1$, $v = 1$ correspond to the point $x = 0$, $y = 1$. At this point,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = 2 \cdot 2u + 3 \cdot v = 2 \cdot 2 + 3 = 7.$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = 2 \cdot (-2v) + 3 \cdot u = 2 \cdot (-2) + 3 \cdot 1 = -1.$$

2E-5 a) $w_r = w_x x_r + w_y y_r = w_x \cos\theta + w_y \sin\theta$

$w_\theta = w_x x_\theta + w_y y_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$

Therefore,

$(w_r)^2 + (w_\theta/r)^2$

$= (w_x)^2(\cos^2\theta + \sin^2\theta) + (w_y)^2(\sin^2\theta + \cos^2\theta) + 2w_xw_y\cos\theta\sin\theta - 2w_xw_y\sin\theta\cos\theta$

$= (w_x)^2 + (w_y)^2.$

b) The point $r = \sqrt{2}$, $\theta = \pi/4$ in polar coordinates corresponds in rectangular coordinates to the point $x = 1$, $y = 1$. Using the chain rule equations in part (a),

$$w_r = w_x \cos\theta + w_y \sin\theta; \qquad w_\theta = w_x(-r\sin\theta) + w_y(r\cos\theta)$$

but evaluating all the partial derivatives at the point, we get

$$w_r = 2\cdot\tfrac{1}{2}\sqrt{2} - 1\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}; \qquad \frac{w_\theta}{r} = 2(-\tfrac{1}{2})\sqrt{2} - \tfrac{1}{2}\sqrt{2} = -\tfrac{3}{2}\sqrt{2};$$

$$(w_r)^2 + \frac{1}{r}(w_\theta)^2 = \tfrac{1}{2} + \tfrac{9}{2} = 5; \qquad (w_x)^2 + (w_y)^2 = 2^2 + (-1)^2 = 5.$$

2E-6 $w_u = w_x \cdot 2u + w_y \cdot 2v$; $w_v = w_x \cdot (-2v) + w_y \cdot 2u$, by the chain rule. Therefore

$$\begin{aligned}(w_u)^2 + (w_v)^2 &= [4u^2(w_x) + 4v^2(w_y)^2 + 4uvw_xw_y] + [4v^2(w_x) + 4u^2(w_y)^2 - 4uvw_xw_y] \\ &= 4(u^2+v^2)[(w_x)^2 + (w_y)^2].\end{aligned}$$

2E-7 By the chain rule, $f_u = f_x x_u + f_y y_u$, $f_v = f_x x_v + f_y y_v$; therefore

$$\langle f_u\ f_v\rangle = \langle f_x\ f_y\rangle \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

2E-8 a) By the chain rule for functions of one variable,

$$\frac{\partial w}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = f'(u)\cdot -\frac{y}{x^2}; \qquad \frac{\partial w}{\partial y} = f'(u)\cdot\frac{\partial u}{\partial y} = f'(u)\cdot\frac{1}{x};$$

Therefore,

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = f'(u)\cdot -\frac{y}{x} + f'(u)\cdot\frac{y}{x} = 0.$$

2F. Maximum-minimum Problems

2F-1 In these, denote by $D = x^2+y^2+z^2$ the square of the distance from the point (x,y,z) to the origin; then the point which minimizes D will also minimize the actual distance.

a) Since $z^2 = \dfrac{1}{xy}$, we get on substituting, $D = x^2 + y^2 + \dfrac{1}{xy}$. with x and y independent; setting the partial derivatives equal to zero, we get

$$D_x = 2x - \frac{1}{x^2y} = 0; \quad D_y = 2y - \frac{1}{y^2x} = 0; \quad \text{or} \quad 2x^2 = \frac{1}{xy}, \quad 2y^2 = \frac{1}{xy}.$$

Solving, we see first that $x^2 = \dfrac{1}{2xy} = y^2$, from which $y = \pm x$.

If $y = x$, then $x^4 = \frac{1}{2}$ and $x = y = 2^{-1/4}$, and so $z = 2^{1/4}$; if $y = -x$, then $x^4 = -\frac{1}{2}$ and there are no solutions. Thus the unique point is $(1/2^{1/4}, 1/2^{1/4}, 2^{1/4})$.

b) Using the relation $x^2 = 1 + yz$ to eliminate x, we have $D = 1 + yz + y^2 + z^2$, with y and z independent; setting the partial derivatives equal to zero, we get

$$D_y = 2y + z = 0, \quad D_z = 2z + y = 0;$$

solving, these equations only have the solution $y = z = 0$; therefore $x = \pm 1$, and there are two points: $(\pm 1, 0, 0)$, both at distance 1 from the origin.

2F-2 Letting x be the length of the ends, y the length of the sides, and z the height, we have

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \qquad \text{volume } V = xyz = 1.$$

Eliminating z to make the remaining variables independent, and equating the partials to zero, we get

$$A = 3xy + \frac{4}{y} + \frac{2}{x}; \qquad A_x = 3y - \frac{2}{x^2} = 0, \quad A_y = 3x - \frac{4}{y^2} = 0.$$

From these last two equations, we get

$$3xy = \frac{2}{x}, \quad 3xy = \frac{4}{y} \quad \Rightarrow \quad \frac{2}{x} = \frac{4}{y} \quad \Rightarrow \quad y = 2x$$

$$\Rightarrow \quad 3x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{3^{1/3}}, \;\; y = \frac{2}{3^{1/3}}, \;\; z = \frac{1}{xy} = \frac{3^{2/3}}{2} = \frac{3}{2 \cdot 3^{1/3}};$$

therefore the proportions of the most economical box are $x : y : z = 1 : 2 : \frac{3}{2}$.

2F-5 The cost is $C = xy + xz + 4yz + 4xz$, where the successive terms represent in turn the bottom, back, two sides, and front; i.e., the problem is:

minimize: $C = xy + 5xz + 4yz$, with the constraint: $xyz = V = 2.5$

Substituting $z = V/xy$ into C, we get

$$C = xy + \frac{5V}{y} + \frac{4V}{x}; \qquad \frac{\partial C}{\partial x} = y - \frac{4V}{x^2}, \quad \frac{\partial C}{\partial y} = x - \frac{5V}{y^2}.$$

We set the two partial derivatives equal to zero and solving the resulting equations simultaneously, by eliminating y; we get $x^3 = \dfrac{16V}{5} = 8$, (using $V = 5/2$), so $x = 2,\; y = \frac{5}{2},\; z = \frac{1}{2}$.

2G. Least-squares Interpolation

2G-1 Find $y = mx + b$ that best fits $(1,1),\ (2,3),\ (3,2)$.

$$\begin{aligned} D &= (m+b-1)^2 + (2m+b-3)^2 + (3m+b-2)^2 \\ \frac{\partial D}{\partial m} &= 2(m+b-1) + 4(2m+b-3) + 6(3m+b-2) \;=\; 2(14m+6b-13) \\ \frac{\partial D}{\partial b} &= 2(m+b-1) + 2(2m+b-3) + 2(3m+b-2) \;=\; 2(6m+3b-6). \end{aligned}$$

Thus the equations $\dfrac{\partial D}{\partial m} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ are $\begin{cases} 14m + 6b = 13 \\ 6m + 3b = 6 \end{cases}$, whose solution is $m = \frac{1}{2},\; b = 1,$ and the line is $y = \frac{1}{2}x + 1$.

2G-4 $D = \sum_i (a + bx_i + cy_i - z_i)^2$. The equations are

$$\begin{aligned} \partial D/\partial a &= \textstyle\sum 2(a + bx_i + cy_i - z_i) \;=\; 0 \\ \partial D/\partial b &= \textstyle\sum 2x_i(a + bx_i + cy_i - z_i) \;=\; 0 \\ \partial D/\partial c &= \textstyle\sum 2y_i(a + bx_i + cy_i - z_i) \;=\; 0 \end{aligned}$$

Cancel the 2's; the equations become (on the right, $\mathbf{x} = [x_1, \ldots, x_n]$, $\mathbf{1} = [1, \ldots, 1]$, etc.)

$$\begin{aligned} na + (\textstyle\sum x_i)b + (\sum y_i)c &= \textstyle\sum z_i \\ (\textstyle\sum x_i)a + (\sum x_i^2)b + (\sum x_i y_i)c &= \textstyle\sum x_i z_i \\ (\textstyle\sum y_i)a + (\sum x_i y_i)b + (\sum y_i^2)c &= \textstyle\sum y_i z_i \end{aligned} \qquad \text{or} \qquad \begin{aligned} n\,a + (\mathbf{x}\cdot\mathbf{1})\,b + (\mathbf{y}\cdot\mathbf{1})\,c &= \mathbf{z}\cdot\mathbf{1} \\ (\mathbf{x}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{x})\,b + (\mathbf{x}\cdot\mathbf{y})\,c &= \mathbf{x}\cdot\mathbf{z} \\ (\mathbf{y}\cdot\mathbf{1})\,a + (\mathbf{x}\cdot\mathbf{y})\,b + (\mathbf{y}\cdot\mathbf{y})\,c &= \mathbf{y}\cdot\mathbf{z} \end{aligned}$$

2H. Max-min: 2nd Derivative Criterion; Boundary Curves

2H-1

a) $f_x = 0: \ 2x - y = 3; \quad f_y = 0: \ -x - 4y = 3 \qquad$ critical point: $(1, -1)$
$A = f_{xx} = 2; \ B = f_{xy} = -1; \ C = f_{yy} = -4; \qquad AC - B^2 = -9 < 0$; saddle point

b) $f_x = 0: \ 6x + y = 1; \quad f_y = 0: \ x + 2y = 2 \qquad$ critical point: $(0, 1)$
$A = f_{xx} = 6; \ B = f_{xy} = 1; \ C = f_{yy} = 2; \qquad AC - B^2 = 11 > 0$; local minimum

c) $f_x = 0: \ 8x^3 - y = 0; \quad f_y = 0: \ 2y - x = 0; \qquad$ eliminating y, we get

$16x^3 - x = 0$, or $x(16x^2 - 1) = 0 \ \Rightarrow \ x = 0, \ x = \frac{1}{4}, \ x = -\frac{1}{4}$, giving the critical points

$$(0, 0), \quad (\tfrac{1}{4}, \tfrac{1}{8}), \quad (-\tfrac{1}{4}, -\tfrac{1}{8}).$$

Since $f_{xx} = 24x^2, \quad f_{xy} = -1, \quad f_{yy} = 2$, we get for the three points respectively:

$(0, 0): \ \Delta = -1$ (saddle); $\quad (\frac{1}{4}, \frac{1}{8}): \ \Delta = 2$ (minimum); $\qquad (-\frac{1}{4}, -\frac{1}{8}): \ \Delta = 2$ (minimum)

d) $f_x = 0: \quad 3x^2 - 3y = 0; \qquad f_y = 0: \quad -3x + 3y^2 = 0$. Eliminating y gives

$-x + x^4 = 0$, or $x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0, \ y = 0$ or $x = 1, \ y = 1$.

Since $f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 6y$, we get for the two critical points respectively:

$(0, 0): \quad AC - B^2 = -9$ (saddle); $\qquad (1, 1): \quad AC - B^2 = 27$ (minimum)

e) $f_x = 0: \quad 3x^2(y^3 + 1) = 0; \qquad f_y = 0: \quad 3y^2(x^3 + 1) = 0$; solving simultaneously, we get from the first equation that either $x = 0$ or $y = -1$; finding in each case the other coordinate then leads to the two critical points $(0, 0)$ and $(-1, -1)$.

Since $f_{xx} = 6x(y^3 + 1), \quad f_{xy} = 3x^2 \cdot 3y^2, \quad f_{yy} = 6y(x^3 + 1)$, we have

$(-1, -1): \quad AC - B^2 = -9$ (saddle); $\qquad (0, 0): \quad AC - B^2 = 0$, test fails.

(By studying the behavior of $f(x, y)$ on the lines $y = mx$, for different values of m, it is possible to see that also $(0, 0)$ is a saddle point.)

2H-3 The region R has no critical points; namely, the equations $f_x = 0$ and $f_y = 0$ are

$$2x + 2 = 0, \quad 2y + 4 = 0 \quad \Rightarrow \quad x = -1, \ y = -2,$$

but this point is not in R. We therefore investigate the diagonal boundary of R, using the parametrization $x = t, \ y = -t$. Restricted to this line, $f(x, y)$ becomes a function of t alone, which we denote by $g(t)$, and we look for its maxima and minima.

$$g(t) = f(t, -t) = 2t^2 - 2t - 1; \qquad g'(t) = 4t - 2, \text{ which is 0 at } t = 1/2.$$

This point is evidently a minimum for $g(t)$; there is no maximum: $g(t)$ tends to ∞. Therefore for $f(x, y)$ on R, the minimum occurs at the point $(1/2, -1/2)$, and there is no maximum; $f(x, y)$ tends to infinity in different directions in R.

2H-4 We have $f_x = y - 1, \quad f_y = x - 1,$ so the only critical point is at $(1, 1)$.

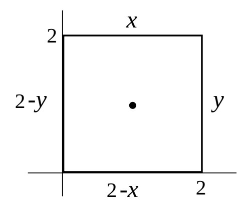
a) On the two sides of the boundary, the function $f(x, y)$ becomes respectively

$$y = 0: \quad f(x, y) \;=\; -x + 2; \qquad x = 0: \quad f(x, y) \;=\; -y + 2.$$

Since the function is linear and decreasing on both sides, it has no minimum points (informally, the minimum is $-\infty$). Since $f(1, 1) = 1$ and $f(x, x) = x^2 - 2x + 2 \to \infty$ as $x \to \infty$, the maximum of f on the first quadrant is ∞.

b) Continuing the reasoning of (a) to find the maximum and minimum points of $f(x, y)$ on the boundary, on the other two sides of the boundary square, the function $f(x, y)$ becomes

$$y = 2: \quad f(x, y) = x \qquad\qquad x = 2: \quad f(x, y) = y$$



Since $f(x, y)$ is thus increasing or decreasing on each of the four sides, the maximum and minimum points on the boundary square R can only occur at the four corner points; evaluating $f(x, y)$ at these four points, we find

$$f(0, 0) = 2; \qquad f(2, 2) = 2; \qquad\qquad f(2, 0) = 0; \qquad f(0, 2) = 0.$$

As in (a), since $f(1, 1) = 1$,

maximum points of f on R: $(0, 0)$ and $(2, 2)$; minimum points: $(2, 0)$ and $(0, 2)$.

c) The data indicates that $(1, 1)$ is probably a saddle point. Confirming this, we have $f_{xx} = 0,\ f_{xy} = 1,\ f_{yy} = 0$ for all x and y; therefore $AC - B^2 = -1 < 0$, so $(1, 1)$ is a saddle point, by the 2nd-derivative criterion.

2H-5 Since $f(x, y)$ is linear, it will not have critical points: namely, for all x and y we have $f_x = 1,\ f_y = \sqrt{3}$. So any maxima or minima must occur on the boundary circle.

We parametrize the circle by $x = \cos\theta,\ y = \sin\theta$; restricted to this boundary circle, $f(x, y)$ becomes a function of θ alone which we call $g(\theta)$:

$$g(\theta) = f(\cos\theta, \sin\theta) = \cos\theta + \sqrt{3}\sin\theta + 2.$$

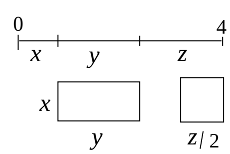
Proceeding in the usual way to find the maxima and minima of $g(\theta)$, we get

$$g'(\theta) = -\sin\theta + \sqrt{3}\cos\theta = 0, \qquad \text{or} \qquad \tan\theta = \sqrt{3}.$$

It follows that the two critical points of $g(\theta)$ are $\theta = \dfrac{\pi}{3}$ and $\dfrac{4\pi}{3}$; evaluating g at these two points, we get $g(\pi/3) = 4$ (the maximum), and $g(4\pi/3) = 0$ (the minimum).

Thus the maximum of $f(x, y)$ in the circular disc R is at $(1/2, \sqrt{3}/2)$, while the minimum is at $(-1/2, -\sqrt{3}/2)$.

2H-6 a) Since $z = 4 - x - y$, the problem is to find on R the maximum and minimum of the total area



$$f(x, y) = xy + \tfrac{1}{4}(4 - x - y)^2$$

where R is the triangle given by$R: \ 0 \le x, \quad 0 \le y, \quad x + y \le 4.$

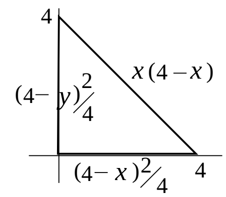
To find the critical points of $f(x, y)$, the equations $f_x = 0$ and $f_y = 0$ are respectively

$$y - \tfrac{1}{2}(4 - x - y) = 0; \quad x - \tfrac{1}{2}(4 - x - y) = 0,$$

which imply first that $x = y$, and from this, $x - \frac{1}{2}(4 - 2x)$; the unique solution is $x = 1,\ y = 1$.

The region R is a triangle, on whose sides $f(x,y)$ takes respectively the values

$$\text{bottom: } y=0;\ f=\tfrac{1}{4}(4-x)^2; \qquad \text{left side: } x=0;\ f=\tfrac{1}{4}(4-y)^2;$$
$$\text{diagonal } y=4-x;\ f=x(4-x).$$

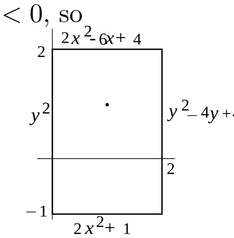


On the bottom and side, f is decreasing; on the diagonal, f has a maximum at $x=2,\ y=2$. Therefore we need to examine the three corner points and $(2,2)$ as candidates for maximum and minimum points, as well as the critical point $(1,1)$. We find

$$f(0,0)=4; \quad f(4,0)=0; \quad f(0,4)=0; \quad f(2,2)=4 \quad f(1,1)=2.$$

It follows that the critical point is just a saddle point; to get the maximum total area 4, make $x=y=0,\ z=4$, or $x=y=2,\ z=0$, either of which gives a point "rectangle" and a square of side 2; for the minimum total area 0, take for example $x=0, y=4, z=0$, which gives a "rectangle" of length 4 with zero area, and a point square.

b) We have $f_{xx}=\frac{1}{2},\ f_{xy}=\frac{3}{2},\ f_{yy}=\frac{1}{2}$ for all x and y; therefore $AC-B^2=-2<0$, so $(1,1)$ is a saddle point, by the 2nd-derivative criterion.



2H-7 a) $f_x = 4x-2y-2, \quad f_y=-2x+2y$; setting these $=0$ and solving simultaneously, we get $x=1,\ y=1$, which is therefore the only critical point.

On the four sides of the boundary rectangle R, the function $f(x,y)$ becomes:

$$\text{on } y=-1: \ f(x,y)=2x^2+1; \qquad \text{on } y=2: \ f(x,y)=2x^2-6x+4$$
$$\text{on } x=0: \ f(x,y)=y^2; \qquad \text{on } x=2: \ f(x,y)=y^2-4y+4$$

By one-variable calculus, $f(x,y)$ is increasing on the bottom and decresing on the right side; on the left side it has a minimum at $(0,0)$, and on the top a minimum at $(\frac{3}{2},2)$. Thus the maximum and minimum points on the boundary rectangle R can only occur at the four corner points, or at $(0,0)$ or $(\frac{3}{2},2)$. At these we find:

$$f(0,-1)=1; \quad f(0,2)=4; \quad f(2,-1)=9; \quad f(2,2)=0; \quad f(\tfrac{3}{2},2)=-\tfrac{1}{2}, \quad f(0,0)=0.$$

At the critical point $f(1,1)=-1$; comparing with the above, it is a minimum; therefore,
maximum point of $f(x,y)$ on R: $(2,-1)$ minimum point of $f(x,y)$ on R: $(1,1)$

b) We have $f_{xx}=4,\ f_{xy}=-2,\ f_{yy}=2$ for all x and y; therefore $AC-B^2=4>0$ and $A=4>0$, so $(1,1)$ is a minimum point, by the 2nd-derivative criterion.

2I. Lagrange Multipliers

2I-1 Letting $P:(x,y,z)$ be the point, in both problems we want to maximize $V=xyz$, subject to a constraint $f(x,y,z)=c$. The Lagrange equations for this, in vector form, are

$$\nabla(xyz)=\lambda\cdot\nabla f(x,y,z), \qquad f(x,y,z)=c.$$

a) Here $f=c$ is $x+2y+3z=18$; equating components, the Lagrange equations become

$$yz=\lambda, \quad xz=2\lambda, \quad xy=3\lambda; \qquad x+2y+3z=18.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda x = 2\lambda y = 3\lambda z, \quad \text{or} \quad x=2y=3z=6, \ \text{since the sum is 18.}$$

We get therefore as the answer $x = 6, \quad y = 3, \quad z = 2$. This is a maximum point, since if P lies on the triangular boundary of the region in the first octant over which it varies, the volume of the box is zero.

b) Here $f = c$ is $x^2 + 2y^2 + 4z^2 = 12$; equating components, the Lagrange equations become

$$yz = \lambda \cdot 2x, \quad xz = \lambda \cdot 4y, \quad xy = \lambda \cdot 8z; \qquad x^2 + 2y^2 + 4z^2 = 12.$$

To solve these symmetrically, multiply the left sides respectively by x, y, and z to make them equal; this gives

$$\lambda \cdot 2x^2 = \lambda \cdot 4y^2 = \lambda \cdot 8z^2, \quad \text{or} \quad x^2 = 2y^2 = 4z^2 = 4, \quad \text{since the sum is 12.}$$

We get therefore as the answer $x = 2, \quad y = \sqrt{2}, \quad z = 1$. This is a maximum point, since if P lies on the boundary of the region in the first octant over which it varies (1/8 of the ellipsoid), the volume of the box is zero.

2I-2 Since we want to minimize $x^2 + y^2 + z^2$, subject to the constraint $x^3y^2z = 6\sqrt{3}$, the Lagrange multiplier equations are

$$2x = \lambda \cdot 3x^2y^2z, \quad 2y = \lambda \cdot 2x^3yz, \quad 2z = \lambda \cdot x^3y^2; \quad x^3y^2z = 6\sqrt{3}.$$

To solve them symmetrically, multiply the first three equations respectively by x, y, and z, then divide them through respectively by 3, 2, and 1; this makes the right sides equal, so that, after canceling 2 from every numerator, we get

$$\frac{x^2}{3} = \frac{y^2}{2} = z^2; \quad \text{therefore } x = z\sqrt{3},\ y = z\sqrt{2}.$$

Substituting into $x^3y^2z = 6\sqrt{3}$, we get $3\sqrt{3}z^3 \cdot 2z^2 \cdot z = 6\sqrt{3}$, which gives as the answer, $x = \sqrt{3},\ y = \sqrt{2},\ z = 1$.

This is clearly a minimum, since if P is near one of the coordinate planes, one of the variables is close to zero and therefore one of the others must be large, since $x^3y^2z = 6\sqrt{3}$; thus P will be far from the origin.

2I-3 Referring to the solution of 2F-2, we let x be the length of the ends, y the length of the sides, and z the height, and get

$$\text{total area of cardboard } A = 3xy + 4xz + 2yz, \qquad \text{volume } V = xyz = 1.$$

The Lagrange multiplier equations $\nabla A = \lambda \cdot \nabla(xyz); \quad xyz = 1$, then become

$$3y + 4z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad 4x + 2y = \lambda xy, \quad xyz = 1.$$

To solve these equations for x, y, z, λ, treat them symmetrically. Divide the first equation through by yz, and treat the next two equations analogously, to get

$$3/z + 4/y = \lambda, \quad 3/z + 2/x = \lambda, \quad 4/y + 2/x = \lambda,$$

which by subtracting the equations in pairs leads to $3/z = 4/y = 2/x$; setting these all equal to k, we get $x = 2/k, y = 4/k, z = 3/k$, which shows the proportions using least cardboard are $x : y : z = 2 : 4 : 3$.

To find the actual values of x, y, and z, we set $1/k = m$; then substituting into $xyz = 1$ gives $(2m)(4m)(3m) = 1$, from which $m^3 = 1/24$, $m = 1/2 \cdot 3^{1/3}$, giving finally

$$x = \frac{1}{3^{1/3}}, \quad y = \frac{2}{3^{1/3}}, \quad z = \frac{3}{2 \cdot 3^{1/3}}.$$

2I-4 The equations for the cost C and the volume V are $xy+4yz+6xz = C$ and $xyz = V$. The Lagrange multiplier equations for the two problems are

a) $\quad yz = \lambda(y+6z), \quad xz = \lambda(x+4z), \quad xy = \lambda(4y+6x); \qquad xy+4yz+6xz = 72$

b) $\quad y+6z = \mu\cdot yz, \quad x+4z = \mu\cdot xz, \quad 4y+6x = \mu\cdot xy; \qquad xyz = 24$

The first three equations are the same in both cases, since we can set $\mu = 1/\lambda$. Solving the first three equations in (a) symmetrically, we multiply the equations through by x, y, and z respectively, which makes the left sides equal; since the right sides are therefore equal, we get after canceling the λ,

$$xy+6xz = xy+4yz = 4yz+6xz, \quad \text{which implies} \quad xy = 4yz = 6xz.$$

a) Since the sum of the three equal products is 72, by hypothesis, we get

$$xy = 24, \quad yz = 6, \quad xz = 4;$$

from the first two we get $x = 4z$, and from the first and third we get $y = 6z$, which lead to the solution $x = 4$, $y = 6$, $z = 1$.

b) Dividing $xy = 4yz = 6xz$ by xyz leads after cross-multiplication to $x = 4z$, $y = 6z$; since by hypothesis, $xyz = 24$, again this leads to the solution $x = 4$, $y = 6$, $z = 1$.

2J. Non-independent Variables

2J-1 a) $\left(\dfrac{\partial w}{\partial y}\right)_z$ means that x is the dependent variable; get rid of it by writing $w = (z-y)^2 + y^2 + z^2 = z + z^2$. This shows that $\left(\dfrac{\partial w}{\partial y}\right)_z = 0$.

b) To calculate $\left(\dfrac{\partial w}{\partial z}\right)_y$, once again x is the dependent variable; as in part (a), we have $w = z + z^2$ and so $\left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-2 a) Differentiating $z = x^2+y^2$ w.r.t. y: $\quad 0 = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y$; so $\left(\dfrac{\partial x}{\partial y}\right)_z = -\dfrac{y}{x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial y}\right)_z = 2x\left(\dfrac{\partial x}{\partial y}\right)_z + 2y = 2x\left(\dfrac{-y}{x}\right) + 2y = 0$.

Differentiating $z = x^2+y^2$ with respect to z: $\quad 1 = 2x\left(\dfrac{\partial x}{\partial z}\right)_y$; so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{1}{2x}$;

By the chain rule, $\left(\dfrac{\partial w}{\partial z}\right)_y = 2x\left(\dfrac{\partial x}{\partial z}\right)_y + 2z = 1 + 2z$.

b) Using differentials, $\quad dw = 2x\,dx + 2y\,dy + 2z\,dz, \qquad dz = 2x\,dx + 2y\,dy$;
since the independent variables are y and z, we eliminate dx by substracting the second equation from the first, which gives $\quad dw = 0\,dy + (1+2z)\,dz$;
therefore we get $\quad \left(\dfrac{\partial w}{\partial y}\right)_z = 0, \quad \left(\dfrac{\partial w}{\partial z}\right)_y = 1 + 2z$.

2J-3 a) To calculate $\left(\frac{\partial w}{\partial t}\right)_{x,z}$, we see that y is the dependent variable; solving for it, we get $y = \frac{zt}{x}$; using the chain rule, $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2 = x^3\frac{z}{x} - z^2 = x^2z - z^2$.

b) Similarly, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$ means that t is the dependent variable; since $t = \frac{xy}{z}$, we have by the chain rule, $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2zt - z^2\left(\frac{\partial t}{\partial z}\right)_{x,y} = -2zt - z^2 \cdot \frac{-xy}{z^2} = -zt$.

2J-4 The differentials are calculated in equation (4).

a) Since x, z, t are independent, we eliminate dy by solving the second equation for $x\,dy$, substituting this into the first equation, and grouping terms:
$dw = 2x^2y\,dx + (x^2z - z^2)dt + (x^2t - 2zt)dz$, which shows that $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2z - z^2$.

b) Since x, y, z are independent, we eliminate dt by solving the second equation for $z\,dt$, substituting this into the first equation, and grouping terms:
$dw = (3x^2y - zy)dx + (x^3 - zx)dy - zt\,dz$, which shows that $\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt$.

2J-5 a) If $pv = nRT$, then $\left(\frac{\partial S}{\partial p}\right)_v = S_p + S_T \cdot \left(\frac{\partial T}{\partial p}\right)_v = S_p + S_T \cdot \frac{v}{nR}$.

b) Similarly, we have $\left(\frac{\partial S}{\partial T}\right)_v = S_T + S_p \cdot \left(\frac{\partial p}{\partial T}\right)_v = S_T + S_p \cdot \frac{nR}{v}$.

2J-6 a) $\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - u \cdot 2v\left(\frac{\partial v}{\partial u}\right)_x = 3u^2 - v^2 - 2uv$.

$\left(\frac{\partial w}{\partial x}\right)_u = -u \cdot 2v\left(\frac{\partial v}{\partial x}\right)_u = -2uv$.

b) $dw = (3u^2 - v^2)du - 2uvdv$; $du = x\,dy + y\,dx$; $dv = du + dx$;
for both derivatives, u and x are the independent variables, so we eliminate dv, getting
$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv\,dx,$$
whose coefficients are $\left(\frac{\partial w}{\partial u}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_u$.

2J-7 Since we need both derivatives for the gradient, we use differentials.
$df = 2dx + dy - 3dz$ at P; $dz = 2x\,dx + dy = 2\,dx + dy$ at P;
the independent variables are to be x and z, so we eliminate dy, getting
$df = 0\,dx - 2\,dz$ at the point $(x, z) = (1, 1)$. So $\nabla g = \langle 0, -2\rangle$ at $(1, 1)$.

2J-8 To calculate $\left(\frac{\partial w}{\partial r}\right)_\theta$, note that $w = r|\sin\theta|$. Therefore, $\left(\frac{\partial w}{\partial r}\right)_\theta = |\sin\theta|$.

2K. Partial Differential Equations

2K-1 $w = \frac{1}{2}\ln(x^2+y^2)$. If $(x,y) \neq (0,0)$, then

$$w_{xx} = \frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$w_{yy} = \frac{\partial}{\partial y}(w_y) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2},$$

Therefore w satisfies the two-dimensional Laplace equation, $w_{xx}+w_{yy}=0$; we exclude the point $(0,0)$ since $\ln 0$ is not defined.

2K-2 If $w = (x^2+y^2+z^2)^n$, then $\frac{\partial}{\partial x}(w_x) = \frac{\partial}{\partial x}\big(2x\cdot n(x^2+y^2+z^2)^{n-1}\big)$

$$= 2n(x^2+y^2+z^2)^{n-1} + 4x^2n(n-1)(x^2+y^2+z^2)^{n-2}$$

We get w_{yy} and w_{zz} by symmetry; adding and combining, we get

$$w_{xx}+w_{yy}+w_{zz} = 6n(x^2+y^2+z^2)^{n-1} + 4(x^2+y^2+z^2)n(n-1)(x^2+y^2+z^2)^{n-2}$$

$= 2n(2n+1)(x^2+y^2+z^2)^{n-1}$, which is identically zero if $n=0$, or if $n=-1/2$.

2K-3 a) $w = ax^2+bxy+cy^2$; $\quad w_{xx}=2a, \quad w_{yy}=2c.$

$$w_{xx}+w_{yy}=0 \quad\Rightarrow\quad 2a+2c=0, \text{ or } c=-a.$$

Therefore all quadratic polynomials satisfying the Laplace equation are of the form

$$ax^2+bxy-ay^2 = a(x^2-y^2)+bxy;$$

i.e., linear combinations of the two polynomials $f(x,y)=x^2-y^2$ and $g(x,y)=xy$.

2K-4 The one-dimensional wave equation is $w_{xx} = \frac{1}{c^2}w_{tt}$. So

$$w = f(x+ct)+g(x-ct) \quad\Rightarrow\quad w_{xx} = f''(x+ct)+g''(x-ct)$$
$$\Rightarrow\quad w_t = cf'(x+ct)+-cg'(x-ct).$$
$$\Rightarrow\quad w_{tt} = c^2f''(x+ct)+c^2g''(x-ct) = c^2w_{xx},$$

which shows w satisfies the wave equation.

2K-5 The one-dimensional heat equation is $w_{xx} = \frac{1}{\alpha^2}w_t$. So if $w(x,t) = \sin kxe^rt$, then

$$w_{xx} = e^{rt}\cdot k^2(-\sin kx) = -k^2\,w.$$
$$w_t = re^{rt}\sin kx = r\,w.$$

Therefore, we must have $-k^2w = \frac{1}{\alpha^2}\,r\,w$, or $r=-\alpha^2k^2$.

However, from the additional condition that $w=0$ at $x=1$, we must have

$$\sin k\; e^{rt} = 0\;;$$

Therefore $\sin k = 0$, and so $k=n\pi$, where n is an integer.

To see what happens to w as $t\to\infty$, we note that since $|\sin kx| \le 1$,

$$|w| = e^{rt}|\sin kx| \le e^{rt}.$$

Now, if $k\neq 0$, then $r=-\alpha^2k^2$ is negative and $e^{rt}\to 0$ as $t\to\infty$; therefore $|w|\to 0$.

Thus w will be a solution satisfying the given side conditions if $k=n\pi$, where n is a non-zero integer, and $r=-\alpha^2k^2$.

3. Double Integrals

3A. Double integrals in rectangular coordinates

3A-1

a) Inner: $6x^2y + y^2\Big]_{y=-1}^{1} = 12x^2$; Outer: $4x^3\Big]_0^2 = 32$.

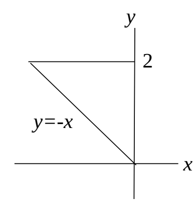
b) Inner: $-u\cos t + \frac{1}{2}t^2\cos u\Big]_{t=0}^{\pi} = 2u + \frac{1}{2}\pi^2\cos u$

Outer: $u^2 + \frac{1}{2}\pi^2\sin u\Big]_0^{\pi/2} = (\frac{1}{2}\pi)^2 + \frac{1}{2}\pi^2 = \frac{3}{4}\pi^2.$

c) Inner: $x^2y^2\Big]_{\sqrt{x}}^{x^2} = x^6 - x^3$; Outer: $\frac{1}{7}x^7 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}$

d) Inner: $v\sqrt{u^2+4}\Big]_0^u = u\sqrt{u^2+4}$; Outer: $\frac{1}{3}(u^2+4)^{3/2}\Big]_0^1 = \frac{1}{3}(5\sqrt{5}-8)$

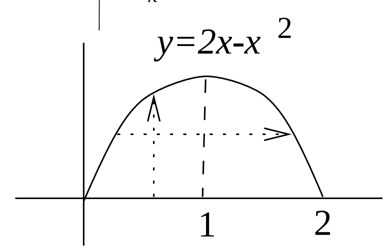
3A-2



a) (i) $\displaystyle\iint_R dy\,dx = \int_{-2}^{0}\int_{-x}^{2} dy\,dx$ (ii) $\displaystyle\iint_R dx\,dy = \int_0^2\int_{-y}^{0} dx\,dy$

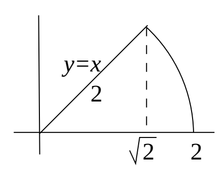
b) i) The ends of R are at 0 and 2, since $2x - x^2 = 0$ has 0 and 2 as roots.

$$\iint_R dydx = \int_0^2\int_0^{2x-x^2} dydx$$



ii) We solve $y = 2x - x^2$ for x in terms of y: write the equation as $x^2 - 2x + y = 0$ and solve for x by the quadratic formula, getting $x = 1 \pm \sqrt{1-y}$. Note also that the maximum point of the graph is $(1,1)$ (it lies midway between the two roots 0 and 2). We get

$$\iint_R dxdy = \int_0^1\int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dxdy,$$



c) (i) $\displaystyle\iint_R dy\,dx = \int_0^{\sqrt{2}}\int_0^x dy\,dx + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} dy\,dx$

(ii) $\displaystyle\iint_R dx\,dy = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dx\,dy$

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously $y^2 = x$ and $y = x - 2$ (eliminate x).

The integral $\displaystyle\iint_R dy\,dx$ requires two pieces; $\displaystyle\iint_R dx\,dy$ only one.

3A-3 a) $\displaystyle\iint_R x\,dA = \int_0^2\int_0^{1-x/2} x\,dy\,dx;$

Inner: $x(1-\frac{1}{2}x)$ Outer: $\frac{1}{2}x^2 - \frac{1}{6}x^3\Big]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}.$

b) $\displaystyle\iint_R (2x+y^2)\,dA = \int_0^1 \int_0^{1-y^2} (2x+y^2)\,dx\,dy$

Inner: $x^2+y^2x\Big]_0^{1-y^2} = 1-y^2;$ Outer: $y-\frac{1}{3}y^3\Big]_0^1 = \frac{2}{3}.$

c) $\displaystyle\iint_R y\,dA = \int_0^1 \int_{y-1}^{1-y} y\,dx\,dy$

Inner: $xy\Big]_{y-1}^{1-y} = y[(1-y)-(y-1)] = 2y-2y^2$ Outer: $y^2-\frac{2}{3}y^3\Big]_0^1 = \frac{1}{3}.$

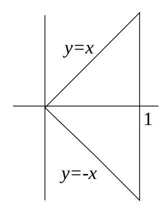
3A-4 a) $\displaystyle\iint_R \sin^2 x\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} \sin^2 x\,dy\,dx$

Inner: $y\,\sin^2 x\Big]_0^{\cos x} = \cos x\sin^2 x$ Outer: $\frac{1}{3}\sin^3 x\Big]-\pi/2^{\pi/2} = \frac{1}{3}(1-(-1)) = \frac{2}{3}.$

b) $\displaystyle\iint_R xy\,dA = \int_0^1\int_{x^2}^{x}(xy)\,dy\,dx.$

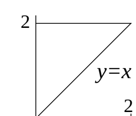
Inner: $\frac{1}{2}xy^2\Big]_{x^2}^{x} = \frac{1}{2}(x^3-x^5)$ Outer: $\displaystyle\frac{1}{2}\left(\frac{x^4}{4}-\frac{x^6}{6}\right)_0^1 = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}.$

c) The function x^2-y^2 is zero on the lines $y=x$ and $y=-x$, and positive on the region R shown, lying between $x=0$ and $x=1$. Therefore

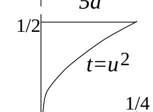


Volume $= \displaystyle\iint_R (x^2-y^2)\,dA = \int_0^1\int_{-x}^{x}(x^2-y^2)\,dy\,dx.$

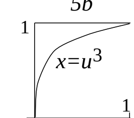
Inner: $x^2y-\frac{1}{3}y^3\Big]_{-x}^{x} = \frac{4}{3}x^3;$ Outer: $\frac{1}{3}x^4\Big]_0^1 = \frac{1}{3}.$



5a



5b



5c

3A-5 a) $\displaystyle\int_0^2\int_x^2 e^{-y^2}\,dy\,dx = \int_0^2\int_0^y e^{-y^2}\,dx\,dy = \int_0^2 e^{-y^2}\,y\,dy = -\frac{1}{2}e^{-y^2}\Big]_0^2 = \frac{1}{2}(1-e^{-4})$

b) $\displaystyle\int_0^{\frac{1}{4}}\int_{\sqrt{t}}^{\frac{1}{2}}\frac{e^u}{u}\,du\,dt = \int_0^{\frac{1}{2}}\int_0^{u^2}\frac{e^u}{u}\,dt\,du = \int_0^{\frac{1}{2}} u\,e^u\,du = (u-1)e^u\Big]_0^{\frac{1}{2}} = 1-\frac{1}{2}\sqrt{e}$

c) $\displaystyle\int_0^1\int_{x^{1/3}}^1\frac{1}{1+u^4}\,du\,dx = \int_0^1\int_0^{u^3}\frac{1}{1+u^4}\,dx\,du = \int_0^1\frac{u^3}{1+u^4}\,du = \frac{1}{4}\ln(1+u^4)\Big]_0^1 = \frac{\ln 2}{4}.$

3A-6 0; $\quad 2\displaystyle\iint_S e^x\,dA$, $S =$ upper half of R; $\quad 4\displaystyle\iint_Q x^2\,dA$, $Q =$ first quadrant

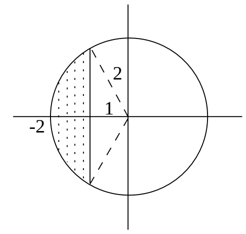
0; $\quad 4\displaystyle\iint_Q x^2\,dA$; $\quad$ 0

3A-7 a) $x^4+y^4\ge 0 \;\Rightarrow\; \dfrac{1}{1+x^4+y^4}\le 1$

b) $\displaystyle\iint_R \frac{x\,dA}{1+x^2+y^2} \le \int_0^1\int_0^1\frac{x}{1+x^2}\,dx\,dy = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{\ln 2}{2} < \frac{.7}{2}.$

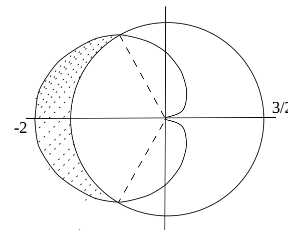
3B. Double Integrals in polar coordinates

3B-1



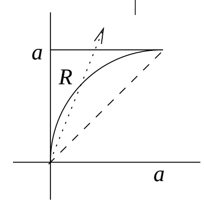
a) In polar coordinates, the line $x = -1$ becomes $r\cos\theta = -1$, or $r = -\sec\theta$. We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

$$\iint_R dr\,d\theta = \int_{2\pi/3}^{4\pi/3}\int_{-\sec\theta}^{2} dr\,d\theta.$$



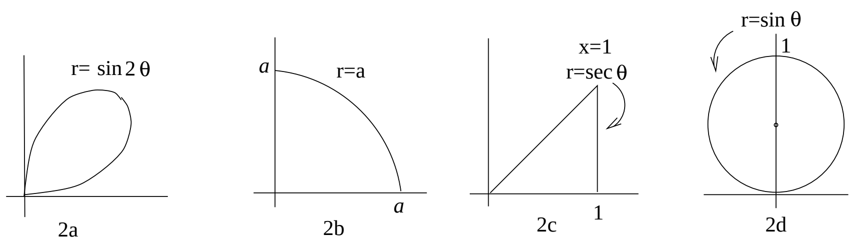
c) We need the polar angle of the intersection points. To find it, we solve the two equations $r = \frac{3}{2}$ and $r = 1 - \cos\theta$ simultanously. Eliminating r, we get $\frac{3}{2} = 1 - \cos\theta$, from which $\theta = 2\pi/3$ and $4\pi/3$. Thus the limits are (no integrand is given):

$$\iint_R dr\,d\theta = \int_{2\pi/3}^{4\pi/3}\int_{3/2}^{1-\cos\theta} dr\,d\theta.$$



d) The circle has polar equation $r = 2a\cos\theta$. The line $y = a$ has polar equation $r\sin\theta = a$, or $r = a\csc\theta$. Thus the limits are (no integrand):

$$\iint_R dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_{2a\cos\theta}^{a\csc\theta} dr\,d\theta.$$



2a 2b 2c 2d

3B-2 a) $\displaystyle\int_0^{\pi/2}\int_0^{\sin 2\theta}\frac{r\,dr\,d\theta}{r} = \int_0^{\pi/2}\sin 2\theta\,d\theta = -\frac{1}{2}\cos 2\theta\Big]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1.$

b) $\displaystyle\int_0^{\pi/2}\int_0^{a}\frac{r}{1+r^2}\,dr\,d\theta = \frac{\pi}{2}\cdot\frac{1}{2}\ln(1+r^2)\Big]_0^{a} = \frac{\pi}{4}\ln(1+a^2).$

c) $\displaystyle\int_0^{\pi/4}\int_0^{\sec\theta}\tan^2\theta\cdot r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\tan^2\theta\,\sec^2\theta\,d\theta = \frac{1}{6}\tan^3\theta\Big]_0^{\pi/4} = \frac{1}{6}.$

d) $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$

Inner: $\displaystyle -\sqrt{1-r^2}\Big]_0^{\sin\theta} = 1-\cos\theta$ Outer: $\displaystyle \theta - \sin\theta\Big]_0^{\pi/2} = \pi/2 - 1.$

3B-3 a) the hemisphere is the graph of $z = \sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}$, so we get

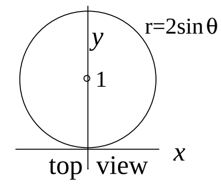
$$\iint_R \sqrt{a^2-r^2}\,dA = \int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\pi\cdot -\frac{1}{3}(a^2-r^2)^{3/2}\Big]_0^a = 2\pi\cdot\frac{1}{3}a^3 = \frac{2}{3}\pi a^3.$$

b) $$\int_0^{\pi/2}\int_0^a (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^a r^3\,dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{a^4}{4}\cdot\frac{1}{2} = \frac{a^4}{8}.$$

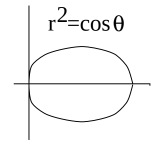
c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the y-axis to compute the volume of just the right side, and double the answer.

$$\iint_R \sqrt{x^2+y^2}\,dA = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{3}(2\sin\theta)^3\,d\theta$$

$= 2\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{32}{9}$, by the integral formula at the beginning of **3B**.



d) $$2\int_0^{\pi/2}\int_0^{\sqrt{\cos\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{4}\cos^2\theta\,d\theta = 2\cdot\frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$

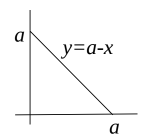


3C. Applications of Double Integration

3C-1 Placing the figure so its legs are on the positive x- and y-axes,

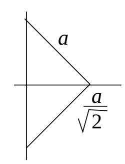
a) M.I. $= \int_0^a\int_0^{a-x} x^2\,dy\,dx$ Inner: $x^2y\Big]_0^{a-x} = x^2(a-x)$; Outer: $\frac{1}{3}x^3a - \frac{1}{4}x^4\Big]_0^a = \frac{1}{12}a^4$.

b) $$\iint_R (x^2+y^2)\,dA = \iint_R x^2\,dA + \iint_R y^2\,dA = \frac{1}{12}a^4 + \frac{1}{12}a^4 = \frac{1}{6}a^4.$$



c) Divide the triangle symmetrically into two smaller triangles, their legs are $\dfrac{a}{\sqrt{2}}$;

Using the result of part (a), M.I. of R about hypotenuse $= 2\cdot\frac{1}{12}\left(\frac{a}{\sqrt{2}}\right)^4 = \frac{a^4}{24}$



3C-2 In both cases, $\bar{x}$ is clear by symmetry; we only need $\bar{y}$.

a) Mass is $\iint_R dA = \int_0^\pi \sin x\,dx = 2$

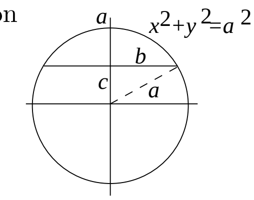
y-moment is $\iint_R y\,dA = \int_0^\pi\int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{\pi}{4}$; therefore $\bar{y} = \frac{\pi}{8}$.

b) Mass is $\iint_R y\,dA = \frac{\pi}{4}$, by part (a). Using the formulas at the beginning of **3B**,

y-moment is $\iint_R y^2\,dA = \int_0^\pi\int_0^{\sin x} y^2\,dy\,dx = 2\int_0^{\pi/2}\frac{\sin^3 x}{3}\,dx = 2\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9}$,

Therefore $\bar{y} = \frac{4}{9}\cdot\frac{4}{\pi} = \frac{16}{9\pi}$.

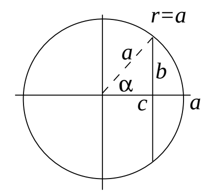
3C-3 Place the segment either horizontally or vertically, so the diameter is respectively on the x or y axis. Find the moment of half the segment and double the answer.



(a) (Horizontally, using rectangular coordinates) Note that $a^2 - c^2 = b^2$.

$$\int_0^b \int_c^{\sqrt{a^2-x^2}} y\,dy\,dx = \int_0^b \frac{1}{2}(a^2 - x^2 - c^2)\,dx = \frac{1}{2}\Big[b^2x - \frac{x^3}{3}\Big]_0^b = \frac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$

(b) (Vertically, using polar coordinates). Note that $x = c$ becomes $r = c\sec\theta$.

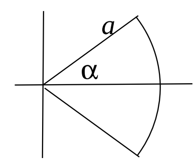


$$\text{Moment} = \int_0^\alpha \int_{c\sec\theta}^a (r\cos\theta)\,r\,dr\,d\theta \qquad \text{Inner: } \tfrac{1}{3}r^3\cos\theta\Big]_{c\sec\theta}^a = \tfrac{1}{3}(a^3\cos\theta - c^3\sec^2\theta)$$

$$\text{Outer: } \tfrac{1}{3}\Big[a^3\sin\theta - c^3\tan\theta\Big]_0^\alpha = \tfrac{1}{3}(a^2b - c^2b) = \tfrac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$

3C-4 Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive x-axis. By symmetry, the center of mass lies on the x-axis, so we only need find $\bar{x}$.

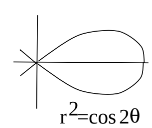
Since $\delta = 1$, the area and mass of the disc are the same: $\pi a^2 \cdot \dfrac{2\alpha}{2\pi} = a^2\alpha$.



$$x\text{-moment: } 2\int_0^\alpha \int_0^a r\cos\theta \cdot r\,dr\,d\theta \qquad \text{Inner: } \tfrac{2}{3}r^3\cos\theta\Big]_0^a;$$

$$\text{Outer: } \tfrac{2}{3}a^3\sin\theta\Big]_0^\alpha = \tfrac{2}{3}a^3\sin\alpha \qquad \bar{x} = \frac{\frac{2}{3}a^3\sin\alpha}{a^2\alpha} = \frac{2}{3}\cdot a \cdot \frac{\sin\alpha}{\alpha}.$$

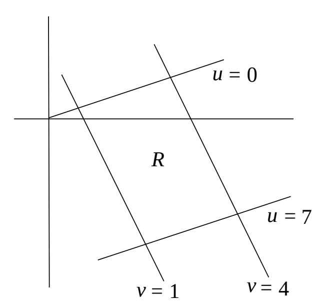
3C-5 By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between $\theta = 0$ and $\theta = \pi/4$.



$$2\int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/4} \frac{1}{4}a^4\cos^2 2\theta\,d\theta$$

$$\text{Putting } u = 2\theta, \text{ the above} = \frac{a^4}{2\cdot 2}\int_0^{\pi/2}\cos^2 u\,du = \frac{a^4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^4}{16}.$$

3D. Changing Variables



3D-1 Let $u = x - 3y$, $v = 2x + y$; $\quad \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7; \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{7}.$

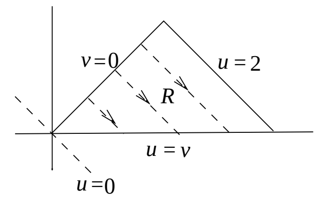
$$\iint_R \frac{x-3y}{2x+y}\,dx\,dy \quad = \quad \frac{1}{7}\int_0^7 \int_1^4 \frac{u}{v}\,dv\,du$$

$$\text{Inner: } u\ln v\Big]_1^4 = u\ln 4; \quad \text{Outer: } \tfrac{1}{2}u^2\ln 4\Big]_0^7 = \frac{49\ln 4}{2}; \quad \text{Ans: } \frac{1}{7}\frac{49\ln 4}{2} \quad = \quad 7\ln 2$$

3D-2 Let $u = x + y$, $v = x - y$. Then $\dfrac{\partial(u,v)}{\partial(x,y)} = 2$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$.

To get the uv-equation of the bottom of the triangular region:

$y = 0 \quad \Rightarrow \quad u = x,\ v = x \quad \Rightarrow \quad u = v.$



$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy \;=\; \frac{1}{2}\int_0^2\int_0^u \cos\frac{v}{u}\,dv\,du$$

Inner: $u\sin\dfrac{v}{u}\Big]_0^u = u\sin 1$ Outer: $\frac{1}{2}u^2\sin 1\Big]_0^2 = 2\sin 1$ Ans: $\sin 1$

3D-3 Let $u = x$, $v = 2y$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2}\end{vmatrix} = \dfrac{1}{2}$

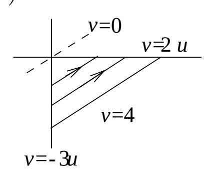
Letting R be the elliptical region whose boundary is $x^2 + 4y^2 = 16$ in xy-coordinates, and $u^2 + v^2 = 16$ in uv-coordinates (a circular disc), we have

$$\iint_R (16 - x^2 - 4y^2)\,dy\,dx \;=\; \frac{1}{2}\iint_R (16 - u^2 - v^2)\,dv\,du$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 (16 - r^2)\,r\,dr\,d\theta \;=\; \pi\left(16\frac{r^2}{2} - \frac{r^4}{4}\right)_0^4 \;=\; 64\pi.$$

3D-4 Let $u = x + y$, $v = 2x - 3y$; then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3\end{vmatrix} = -5$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5}$.

We next express the boundary of the region R in uv-coordinates.
For the x-axis, we have $y = 0$, so $u = x$, $v = 2x$, giving $v = 2u$.
For the y-axis, we have $x = 0$, so $u = y$, $v = -3y$, giving $v = -3u$.



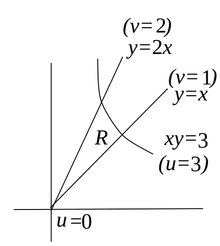
It is best to integrate first over the lines shown, $v = c$; this means v is held constant, i.e., we are integrating first with respect to u. This gives

$$\iint_R (2x - 3y)^2(x+y)^2 dx\,dy \;=\; \int_0^4\int_{-v/3}^{v/2} v^2u^2\frac{du\,dv}{5}.$$

Inner: $\dfrac{v^2}{15}u^3\Big]_{-v/3}^{v/2} = \dfrac{v^2}{15}v^3\left(\dfrac{1}{8} - \dfrac{-1}{27}\right)$ Outer: $\dfrac{v^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right)_0^4 = \dfrac{4^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right).$

3D-5 Let $u = xy$, $v = y/x$; in the other direction this gives $y^2 = uv$, $x^2 = u/v$.

We have $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x\end{vmatrix} = \dfrac{2y}{x} = 2v$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2v}$; this gives



$$\iint_R (x^2 + y^2)\,dx\,dy \;=\; \int_0^3\int_1^2 \left(\frac{u}{v} + uv\right)\frac{1}{2v}\,dv\,du.$$

Inner: $\dfrac{-u}{2v} + \dfrac{u}{2}v\Big]_1^2 = u\left(-\dfrac{1}{4} + 1 + \dfrac{1}{2} - \dfrac{1}{2}\right) = \dfrac{3u}{4}$; Outer: $\dfrac{3}{8}u^2\Big]_0^3 = \dfrac{27}{8}$.

3D-8 a) $y = x^2$; therefore $u = x^3$, $v = x$, which gives $u = v^3$.

b) We get $\dfrac{u}{v} + uv = 1$, or $u = \dfrac{v}{v^2 + 1}$; (cf. 3D-5)

4. Line Integrals in the Plane

4A. Plane Vector Fields

4A-1

a) All vectors in the field are identical; continuously differentiable everywhere.
b) The vector at P has its tail at P and head at the origin; field is cont. diff. everywhere.
c) All vectors have unit length and point radially outwards; cont. diff. except at $(0,0)$.
d) Vector at P has unit length, and the clockwise direction perpendicular to OP.

4A-2 a) $a\,\mathbf{i} + b\,\mathbf{j}$ b) $\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^2}$ c) $f'(r)\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r}$

4A-3 a) $\mathbf{i} + 2\,\mathbf{j}$ b) $-r(x\,\mathbf{i} + y\,\mathbf{j}\,)$ c) $\dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{r^3}$ d) $f(x,y)(\,\mathbf{i} + \mathbf{j}\,)$

4A-4 $k \cdot \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{r^2}$

4B. Line Integrals in the Plane

4B-1

a) On C_1: $y = 0,\ dy = 0$; therefore $\displaystyle\int_{C_1} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\bigg]_{-1}^{1} = \frac{2}{3}.$

On C_2: $y = 1 - x^2,\ dy = -2x\,dx$; $\displaystyle\int_{C_2} (x^2 - y)\,dx + 2x\,dy = \int_{-1}^{1} (2x^2 - 1)\,dx - 4x^2\,dx$

$$= \int_{-1}^{1} (-2x^2 - 1)\,dx = -\left[\frac{2}{3}x^3 + x\right]_{-1}^{1} = -\frac{4}{3} - 2 = -\frac{10}{3}.$$

b) C: use the parametrization $x = \cos t,\ y = \sin t$; then $dx = -\sin t\,dt,\ dy = \cos t\,dt$

$$\int_C xy\,dx - x^2\,dy = \int_{\pi/2}^{0} -\sin^2 t\cos t\,dt - \cos^2 t\cos t\,dt = -\int_{\pi/2}^{0} \cos t\,dt = -\sin t\bigg]_{\pi/2}^{0} = 1.$$

c) $C = C_1 + C_2 + C_3$; $C_1 : x = dx = 0$; $C_2 :\ y = 1 - x$; $C_3 : y = dy = 0$

$$\int_C y\,dx - x\,dy \ = \ \int_{C_1} 0 + \int_0^1 (1 - x)dx - x(-dx) + \int_{C_3} 0 \ = \ \int_0^1 dx \ = \ 1.$$

d) $C : x = 2\cos t,\ y = \sin t$; $dx = -2\sin t\,dt$ $\displaystyle\int_C y\,dx = \int_0^{2\pi} -2\sin^2 t\,dt = -2\pi.$

e) $C :\ x = t^2,\ y = t^3$; $dx = 2t\,dt,\ dy = 3t^2\,dt$

$$\int_C 6y\,dx + x\,dy = \int_1^2 6t^3(2t\,dt) + t^2(3t^2\,dt) = \int_1^2 (15t^4)\,dt = 3t^5\bigg]_1^2 = 3 \cdot 31.$$

f) $\displaystyle\int_C (x + y)dx + xy\,dy = \int_{C_1} 0 + \int_0^1 (x + 2)dx = \frac{x^2}{2} + 2x\bigg]_0^1 = \frac{5}{2}.$

4B-2 a) The field $\mathbf{F}$ points radially outward, the unit tangent $\mathbf{t}$ to the circle is always perpendicular to the radius; therefore $\mathbf{F} \cdot \mathbf{t} = 0$ and $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\,ds = 0$

b) The field $\mathbf{F}$ is always tangent to the circle of radius a, in the clockwise direction, and of magnitude a. Therefore $\mathbf{F} = -a\mathbf{t}$, so that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{t}\,ds = -\int_C a\,ds = -2\pi a^2$.

4B-3 a) maximum if C is in the direction of the field: $C = \dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

b) minimum if C is in the opposite direction to the field: $C = -\dfrac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$

c) zero if C is perpendicular to the field: $C = \pm\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$

d) max $= \sqrt{2}$, min $= -\sqrt{2}$: by (a) and (b), for the max or min $\mathbf{F}$ and C have respectively the same or opposite constant direction, so $\int_C \mathbf{F}\cdot d\mathbf{r} = \pm|\mathbf{F}|\cdot|C| = \pm\sqrt{2}$.

4C. Gradient Fields and Exact Differentials

4C-1 a) $\mathbf{F} = \nabla f = 3x^2y\,\mathbf{i} + (x^3+3y^2)\,\mathbf{j}$

b) (i) Using y as parameter, C_1 is: $x = y^2$, $y = y$; thus $dx = 2y\,dy$, and

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} 3(y^2)^2y\cdot 2y\,dy + [(y^2)^3+3y^2]\,dy = \int_{-1}^{1}(7y^6+3y^2)\,dy = (y^7+y^3)\Big]_{-1}^{1} = 4.$$

b) (ii) Using y as parameter, C_2 is: $x = 1$, $y = y$; thus $dx = 0$, and

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}(1+3y^2)\,dy = (y+y^3)\Big]_{-1}^{1} = 4.$$

b) (iii) By the Fundamental Theorem of Calculus for line integrals,

$$\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A).$$

Here $A = (1,-1)$ and $B = (1,1)$, so that $\displaystyle\int_C \nabla f\cdot d\mathbf{r} = (1+1) - (-1-1) = 4$.

4C-2 a) $\mathbf{F} = \nabla f = (xye^{xy} + e^{xy})\,\mathbf{i} + (x^2e^{xy})\,\mathbf{j}$.

b) (i) Using x as parameter, C is: $x = x$, $y = 1/x$, so $dy = -dx/x^2$, and so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (e+e)\,dx + (x^2e)(-dx/x^2) = (2ex - ex)\Big]_1^0 = -e.$$

b) (ii) Using the F.T.C. for line integrals, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = f(0,\infty) - f(1,1) = 0 - e = -e$.

4C-3 a) $\mathbf{F} = \nabla f = (\cos x\cos y)\,\mathbf{i} - (\sin x\sin y)\,\mathbf{j}$.

b) Since $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is path-independent, for any C connecting $A : (x_0, y_0)$ to $B : (x_1, y_1)$, we have by the F.T.C. for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin x_1\cos y_1 - \sin x_0\cos y_0$$

This difference on the right-hand side is maximized if $\sin x_1\cos y_1$ is maximized, and $\sin x_0\cos y_0$ is minimized. Since $|\sin x\cos y| = |\sin x||\cos y| \le 1$, the difference on the right hand side has a maximum of 2, attained when $\sin x_1\cos y_1 = 1$ and $\sin x_0\cos y_0 = -1$.

(For example, a C running from $(-\pi/2, 0)$ to $(\pi/2, 0)$ gives this maximum value.)

4C-5 a) $\mathbf{F}$ is a gradient field only if $M_y = N_x$, that is, if $2y = ay$, so $a = 2$.

By inspection, the potential function is $f(x, y) = xy^2 + x^2 + c$; you can check that $\mathbf{F} = \nabla f$.

b) The equation $M_y = N_x$ becomes $e^{x+y}(x+a) = xe^{x+y} + e^{x+y}$, which $= e^{x+y}(x+1)$. Therefore $a = 1$.

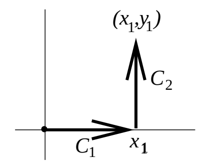
To find the potential function $f(x, y)$, using Method 2 we have

$f_x = e^y e^x(x + 1) \;\Rightarrow\; f(x, y) = e^y xe^x + g(y)$.

Differentiating, and comparing the result with N, we find

$f_y = e^y xe^x + g'(y) = xe^{x+y}$; therefore $g'(y) = 0$, so $g(y) = c$ and $f(x, y) = x\,e^{x+y} + c$.

4C-6 a) $ydx - xdy$ is not exact, since $M_y = 1$ but $N_x = -1$.



b) $y(2x + y)\,dx + x(2y + x)\,dy$ is exact, since $M_y = 2x + 2y = N_x$.

Using Method 1 to find the potential function $f(x, y)$, we calculate the line integral over the standard broken line path shown, $C = C_1 + C_2$.

$$f(x_1, y_1) = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0)}^{(x_1,y_1)} y(2x + y)\,dx + x(2y + x)\,dy.$$

On C_1 we have $y = 0$ and $dy = 0$, so $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$.

On C_2, we have $x = x_1$ and $dx = 0$, so $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{y_1} x_1(2y + x_1)\,dx = x_1y_1^2 + x_1^2y_1$.

Therefore, $f(x, y) = x^2y + xy^2$; to get all possible functions, add $+c$.

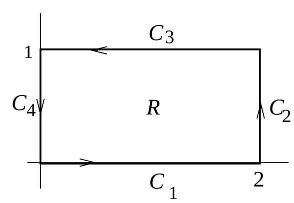
4D. Green's Theorem

4D-1 a) Evaluating the line integral first, we have $C:\; x = \cos t,\; y = \sin t,$ so

$$\oint_C 2y\,dx + x\,dy = \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} (1 - 3\sin^2 t)\,dt = t - 3\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Bigg]_0^{2\pi} = -\pi.$$

For the double integral over the circular region R inside the C, we have

$$\iint_R (N_x - M_y)\,dA = \iint_R (1 - 2)\,dA = -\text{ area of } R = -\pi.$$



b) Evaluating the line integral, over the indicated path $C = C_1 + C_2 + C_3 + C_4$,

$$\oint_C x^2dx + x^2dy = \int_0^2 x^2dx + \int_0^1 4\,dy + \int_2^0 x^2dx + \int_1^0 0\,dy \;=\; 4,$$

since the first and third integrals cancel, and the fourth is 0.

For the double integral over the rectangle R,

$$\iint_R 2x\,dA = \int_0^2\int_0^1 2x\,dydx = x^2\Big]_0^2 = 4.$$

c) Evaluating the line integral over $C = C_1 + C_2$, we have

$$C_1: \; x = x, \; y = x^2; \quad \int_{C_1} xy\,dx + y^2 dy = \int_0^1 x \cdot x^2 dx + x^4 \cdot 2x\,dx = \frac{x^4}{4} + \frac{x^6}{3}\bigg]_0^1 = \frac{7}{12}$$

$$C_2: \; x = x, \; y = x; \quad \int_{C_2} xy\,dx + y^2 dy = \int_1^0 (x^2 dx + x^2 dx) = \frac{2}{3}x^3\bigg]_1^0 = -\frac{2}{3}.$$

Therefore, $\displaystyle\oint_C xy\,dx + y^2 dy = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}.$

Evaluating the double integral over the interior R of C, we have

$$\iint_R -x\,dA = \int_0^1 \int_{x^2}^x -x\,dydx;$$

evaluating: Inner: $\displaystyle -xy\bigg]_{y=x^2}^{y=x} = -x^2 + x^3$; Outer: $\displaystyle -\frac{x^3}{3} + \frac{x^4}{4}\bigg]_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}.$

4D-2 By Green's theorem, $\displaystyle\oint_C 4x^3 y\,dx + x^4\,dy = \iint (4x^3 - 4x^3)\,dA = 0.$

This is true for every closed curve C in the plane, since M and N have continuous derivatives for all x, y.

4D-3 We use the symmetric form for the integrand since the parametrization of the curve does not favor x or y; this leads to the easiest calculation.

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t\,dt + 3\sin^2 t\cos^4 t\,dt = \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt$$

Using $\sin^2 t\cos^2 t = \frac{1}{4}(\sin 2t)^2 = \frac{1}{4}\cdot\frac{1}{2}(1 - \cos 4t)$, the above $\displaystyle = \frac{3}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)_0^{2\pi} = \frac{3\pi}{8}.$

4D-4 By Green's theorem, $\displaystyle\oint_C -y^3 dx + x^3 dy = \iint_R (3x^2 + 3y^2)\,dA > 0$, since the integrand is always positive outside the origin.

4D-5 Let C be a square, and R its interior. Using Green's theorem,

$$\oint_C xy^2 dx + (x^2 y + 2x)\,dy = \iint_R (2xy + 2 - 2xy)\,dA = \iint_R 2\,dA = \; 2(\text{area of } R).$$

4E. Two-dimensional Flux

4E-1 The vector $\mathbf{F}$ is the velocity vector for a rotating disc; it is at each point tangent to the circle centered at the origin and passing through that point.

a) Since $\mathbf{F}$ is tangent to the circle, $\mathbf{F}\cdot\mathbf{n} = 0$ at every point on the circle, so the flux is 0.

b) $\mathbf{F} = x\,\mathbf{j}$ at the point $(x, 0)$ on the line. So if $x_0 > 0$, the flux at x_0 has the same magnitude as the flux at $-x_0$ but the opposite sign, so the net flux over the line is 0.

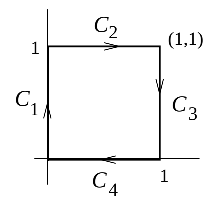
c) $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = x\,\mathbf{j}\cdot -\mathbf{j} = -x$. Thus $\displaystyle\int \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{1}{2}.$

4E-2 All the vectors of $\mathbf{F}$ have length $\sqrt{2}$ and point northeast. So the flux across a line segment C of length 1 will be

a) maximal, if C points northwest;
b) minimal, if C point southeast;
c) zero, if C points northeast or southwest;
d) -1, if C has the direction and magnitude of $\mathbf{i}$ or $-\mathbf{j}$; the corresponding normal vectors are then respectively $-\mathbf{j}$ and $-\mathbf{i}$, by convention, so that $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i}+\mathbf{j})\cdot -\mathbf{j} = -1.$ or $(\mathbf{i}+\mathbf{j})\cdot -\mathbf{i} = -1.$
e) respectively $\sqrt{2}$ and $-\sqrt{2}$, since the angle θ between $\mathbf{F}$ and n is respectively 0 and π, so that respectively $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}|\cos\theta = \pm\sqrt{2}$.

4E-3 $\displaystyle\int_C M\,dy - N\,dx = \int_C x^2dy - xy\,dx = \int_0^1 (t+1)^2 2t\,dt - (t+1)t^2\,dt$

$$= \int_0^1 (t^3+3t^2+2t)\,dt = \frac{t^4}{4} + t^3 + t^2\Big]_0^1 = \frac{9}{4}.$$



4E-4 Taking the curve $C = C_1 + C_2 + C_3 + C_4$ as shown,

$$\int_C x\,dy - y\,dx = \int_{C_1} 0 + \int_0^1 -dx + \int_1^0 dy + \int_{C_4} 0 = -2.$$

4E-5 Since $\mathbf{F}$ and $\mathbf{n}$ both point radially outwards, $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = a^m$, at every point of the circle C of radius a centered at the origin.

a) The flux across C is $a^m \cdot 2\pi a = 2\pi a^{m+1}$.
b) The flux will be independent of a if $m = -1$.

4F. Green's Theorem in Normal Form

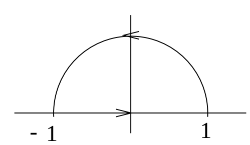
4F-1 a) both are 0 b) div $\mathbf{F} = 2x + 2y$; curl $\mathbf{F} = 0$ c) div $\mathbf{F} = x + y$; curl $\mathbf{F} = y - x$

4F-2 a) div $\mathbf{F} = (-\omega y)_x + (\omega x)_y = 0$; curl $\mathbf{F} = (\omega x)_x - (-\omega y)_y = 2\omega$.

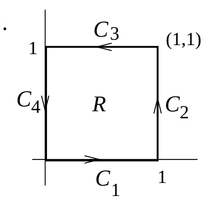
b) Since $\mathbf{F}$ is the velocity field of a fluid rotating with constant angular velocity (like a rigid disc), there are no sources or sinks: fluid is not being added to or subtracted from the flow at any point.

c) A paddlewheel placed at the origin will clearly spin with the same angular velocity ω as the rotating fluid, so by Notes V4,(11), the curl should be 2ω at the origin. (It is much less clear that the curl is 2ω at all other points as well.)

4F-3 The line integral for flux is $\displaystyle\int_C x\,dy - y\,dx$; its value is 0 on any segment of the x-axis since $y = dy = 0$; on the upper half of the unit semicircle (oriented counterclockwise), $\mathbf{F}\cdot\mathbf{n} = 1$, so the flux is the length of the semicircle: π.



Letting R be the region inside C, $\displaystyle\iint_R \text{div}\,\mathbf{F}\,dA = \iint_R 2\,dA = 2(\pi/2) = \pi.$



4F-4 For the flux integral $\displaystyle\oint_C x^2dy - xy\,dx$ over $C = C_1 + C_2 + C_3 + C_4$,

we get for the four sides respectively $\displaystyle\int_{C_1} 0 + \int_0^1 dy + \int_1^0 -x\,dx + \int_{C_4} 0 = \frac{3}{2}.$

For the double integral, $\displaystyle\iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R 3x\,dA = \int_0^1\int_0^1 3x\,dydx = \frac{3}{2}x^2\Big]_0^1 = \frac{3}{2}.$

4F-5 $\quad r = (x^2+y^2)^{1/2} \;\Rightarrow\; r_x = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \dfrac{x}{r};\;$ by symmetry, $r_y = \dfrac{y}{r}$.

To calculate div $\mathbf{F}$, we have $M = r^n x$ and $N = r^n y$; therefore by the chain rule, and the above values for r_x and r_y, we have

$$M_x = r^n + nr^{n-1}x\cdot\frac{x}{r} = r^n + n\,r^{n-2}x^2; \quad \text{similarly (or by symmetry),}$$

$$N_y = r^n + nr^{n-1}y\cdot\frac{y}{r} = r^n + n\,r^{n-2}y^2, \qquad \text{so that}$$

$$\operatorname{div}\mathbf{F} = M_x + N_y = 2r^n + nr^{n-2}(x^2+y^2) = r^n(2+n), \;\text{ which } = 0 \text{ if } n = -2.$$

To calculate curl $\mathbf{F}$, we have by the chain rule

$$N_x = nr^{n-1}\cdot\frac{x}{r}\cdot y; \quad M_y = nr^{n-1}\cdot\frac{y}{r}\cdot x, \quad \text{so that curl } \mathbf{F} = N_x - M_y = 0, \text{ for all } n.$$

4G. Simply-connected Regions

4G-1 Hypotheses: the region R is simply connected, $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ has continuous derivatives in R, and curl $\mathbf{F} = 0$ in R.

Conclusion: $\mathbf{F}$ is a gradient field in R (or, $M\,dx + N\,dy$ is an exact differential).

a) curl $\mathbf{F} = 2y - 2y = 0$, and R is the whole xy-plane. Therefore $\mathbf{F} = \nabla f$ in the plane.

b) curl $\mathbf{F} = -y\sin x - x\sin y \neq 0$, so the differential is not exact.

c) curl $\mathbf{F} = 0$, but R is the exterior of the unit circle, which is not simply-connected; criterion fails.

d) curl $\mathbf{F} = 0$, and R is the interior of the unit circle, which is simply-connected, so the differential is exact.

e) curl $\mathbf{F} = 0$ and R is the first quadrant, which is simply-connected, so $\mathbf{F}$ is a gradient field.

4G-2 a) $f(x,y) = xy^2 + 2x$ $\qquad$ b) $f(x,y) = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}$

c) Using Method 1, we take the origin as the starting point and use the straight line to (x_1, y_1) as the path C. In polar coordinates, $x_1 = r_1\cos\theta_1,\; y_1 = r_1\sin\theta_1$; we use r as the parameter, so the path is $C: x = r\cos\theta_1,\; y = r\sin\theta_1,\; 0 \le r \le r_1$. Then

$$f(x_1,y_1) = \int_C \frac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \int_0^{r_1}\frac{r\cos^2\theta_1 + r\sin^2\theta_1}{\sqrt{1-r^2}}\,dr$$

$$= \int_0^{r_1}\frac{r}{\sqrt{1-r^2}}dr = -\sqrt{1-r^2}\Big]_0^{r_1} = -\sqrt{1-{r_1}^2} + 1.$$

Therefore, $\qquad \dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$

Another approach: $x\,dx + y\,dy = \frac{1}{2}d(r^2)$; therefore $\dfrac{x\,dx + y\,dy}{\sqrt{1-r^2}} = \dfrac{1}{2}\dfrac{d(r^2)}{\sqrt{1-r^2}} = d(-\sqrt{1-r^2}).$
(Think of r^2 as a new variable u, and integrate.)

4G-3 By Example 3 in Notes V5, we know that $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{r^3} = \nabla\left(-\dfrac{1}{r}\right)$.

Therefore, $\displaystyle\int_{(1,1)}^{(3,4)} = -\frac{1}{r}\bigg]_{\sqrt{2}}^{5} = \frac{1}{\sqrt{2}} - \frac{1}{5}$.

4G-4 By Green's theorem $\displaystyle\oint_C xy\,dx + x^2\,dy = \iint_R x\,dA$.

For any plane region of density 1, we have $\iint_R x\,dA = \bar{x}\cdot$(area of R), where $\bar{x}$ is the x-component of its center of mass. Since our region is symmetric with respect to the y-axis, its center of mass is on the y-axis, hence $\bar{x} = 0$ and so $\iint_R x\,dA = 0$.

4G-5

a) yes
b) no (a circle surrounding the line segment lies in R, but its interior does not)
c) yes (no finite curve could surround the entire positive x-axis)
d) no (the region does not consist of one connected piece)
e) yes if $\theta_0 < 2\pi$; no if $\theta_0 \geq 2\pi$, since then R is the plane with $(0,0)$ removed
f) no (a circle between the two boundary circles lies in R, but its interior does not)
g) yes

4G-6

a) continuously differentiable for $x, y > 0$; thus R is the first quadrant without the two axes, which is simply-connected.

b) continuous differentiable if $r < 1$; thus R is the interior of the unit circle, and is simply-connected.

c) continuously differentiable if $r > 1$; thus R is the exterior of the unit circle, and is not simply-connected.

d) continuously differentiable if $r \neq 0$; thus R is the plane with the origin removed, and is not simply-connected.

e) continuously differentiable if $r \neq 0$; same as (d).

4H. Multiply-connected Regions

4H-1 a) 0; 0 b) 2; 4π c) -1; -2π d) -2; -4π

4H-2 In each case, the winding number about each of the points is given, then the value of the line integral of $\mathbf{F}$ around the curve.

a) $(1, -1, 1)$; $2 - \sqrt{2} + \sqrt{3}$
b) $(-1, 0, 1)$; $-2 + \sqrt{3}$
c) $(-1, 0, 0)$; -2
d) $(-1, -2, 1)$; $-2 - 2\sqrt{2} + \sqrt{3}$

5. Triple Integrals

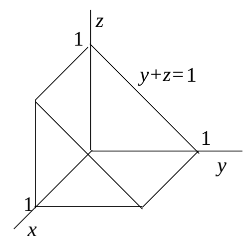
5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 a) $\displaystyle\int_0^2\int_{-1}^1\int_0^1 (x+y+z)dx\,dy\,dz$ Inner: $\frac{1}{2}x^2 + x(y+z)\Big]_{x=0}^{1} = \frac{1}{2} + y + z$

Middle: $\frac{1}{2}y + \frac{1}{2}y^2 + yz\Big]_{y=-1}^{1} = 1 + z - (-z) = 1 + 2z$ Outer: $z + z^2\Big]_0^2 = 6$

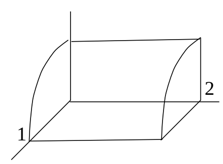
b) $\displaystyle\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy$ Inner: $xy^2z^2\Big]_0^{xy} = x^3y^4$

Middle: $\frac{1}{4}x^4y^4\Big]_0^{\sqrt{y}} = \frac{1}{4}y^6$ Outer: $\frac{1}{28}y^7\Big]_0^2 = \frac{32}{7}$.



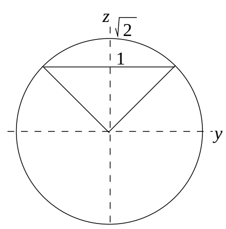
5A-2

a) (i) $\displaystyle\int_0^1\int_0^1\int_0^{1-y} dz\,dy\,dx$ (ii) $\displaystyle\int_0^1\int_0^{1-y}\int_0^1 dx\,dz\,dy$ (iii) $\displaystyle\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz$



c) In cylindrical coordinates, with the polar coordinates r and θ in xz-plane, we get

$$\iiint_R r\,dy\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^2 r\,dy\,dr\,d\theta$$



cross-section view

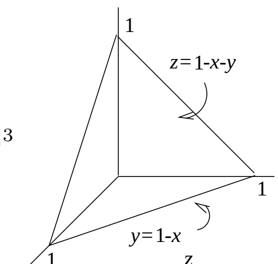
d) The sphere has equation $x^2 + y^2 + z^2 = 2$, or $r^2 + z^2 = 2$ in cylindrical coordinates.

The cone has equation $z^2 = r^2$, or $z = r$. The circle in which they intersect has a radius r found by solving the two equations $z = r$ and $z^2+r^2 = 2$ simultaneously; eliminating z we get $r^2 = 1$, so $r = 1$. Putting it all together, we get

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta.$$

5A-3 By symmetry, $\bar{x} = \bar{y} = \bar{z}$, so it suffices to calculate just one of these, say $\bar{z}$. We have

$$z\text{-moment} = \iiint_D z\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$$

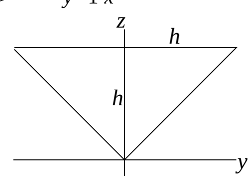


Inner: $\frac{1}{2}z^2\Big]_0^{1-x-y} = \frac{1}{2}(1-x-y)^2$ Middle: $-\frac{1}{6}(1-x-y)^3\Big]_0^{1-x} = \frac{1}{6}(1-x)^3$

Outer: $-\frac{1}{24}(1-x)^4\Big]_0^1 = \frac{1}{24} = \bar{z}$ moment.

mass of D = volume of $D = \frac{1}{3}$(base)(height) $= \frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}$.

Therefore $\bar{z} = \frac{1}{24}/\frac{1}{6} = \frac{1}{4}$; this is also $\bar{x}$ and $\bar{y}$, by symmetry.



vertical cross-section

5A-4 Placing the cone as shown, its equation in cylindrical coordinates is $z = r$ and the density is given by $\delta = r$. By the geometry, its projection onto the xy-plane is the interior R of the origin-centered circle of radius h.

a) Mass of solid $D = \iiint_D \delta\, dV = \int_0^{2\pi} \int_0^h \int_r^h r \cdot r\, dz\, dr\, d\theta$

Inner: $(h-r)r^2$; Middle: $\displaystyle \frac{hr^3}{3} - \frac{r^4}{4}\bigg]_0^h = \frac{h^4}{12}$; Outer: $\displaystyle \frac{2\pi h^4}{12}$

b) By symmetry, the center of mass is on the z-axis, so we only have to compute its z-coordinate, $\bar{z}$.

z-moment of $D = \iiint_D z\,\delta\, dV = \int_0^{2\pi} \int_0^h \int_r^h zr \cdot r\, dz\, dr\, d\theta$

Inner: $\frac{1}{2}z^2 r^2\Big]_r^h = \frac{1}{2}(h^2r^2 - r^4)$ Middle: $\displaystyle \frac{1}{2}\left(h^2\frac{r^2}{3} - \frac{r^5}{5}\right)_0^h = \frac{1}{2}h^5 \cdot \frac{2}{15}$

Outer: $\displaystyle \frac{2\pi h^5}{15}$. Therefore, $\displaystyle \bar{z} = \frac{\frac{2}{15}\pi h^5}{\frac{2}{12}\pi h^4} = \frac{4}{5}h.$

5A-5 Position S so that its base is in the xy-plane and its diagonal D lies along the x-axis (the y-axis would do equally well). The octants divide S into four tetrahedra, which by symmetry have the same moment of inertia about the x-axis; we calculate the one in the first octant. (The picture looks like that for 5A-3, except the height is 2.)

The top of the tetrahedron is a plane intersecting the x- and y-axes at 1, and the z-axis at 2. Its equation is therefore $x + y + \frac{1}{2}z = 1$.

The square of the distance of a point (x, y, z) to the axis of rotation (i.e., the x-axis) is given by $y^2 + z^2$. We therefore get:

$$\text{moment of inertia} = 4\int_0^1 \int_0^{1-x} \int_0^{2(1-x-y)} (y^2 + z^2)\, dz\, dy\, dx.$$

5A-6 Placing D so its axis lies along the positive z-axis and its base is the origin-centered disc of radius a in the xy-plane, the equation of the hemisphere is $z = \sqrt{a^2 - x^2 - y^2}$, or $z = \sqrt{a^2 - r^2}$ in cylindrical coordinates. Doing the inner and outer integrals mentally:

$$z\text{-moment of inertia of } D = \iiint_D r^2\, dV = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} r^2\, dz\, r\, dr\, d\theta = 2\pi \int_0^a r^3 \sqrt{a^2 - r^2} dr.$$

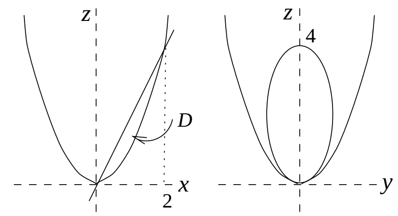
The integral can be done using integration by parts (write the integrand $r^2 \cdot r\sqrt{a^2 - r^2}$), or by substitution; following the latter course, we substitute $r = a \sin u$, $dr = a \cos u\, du$, and get (using the formulas at the beginning of exercises 3B)

$$\int_0^a r^3 \sqrt{a^2 - r^2} dr = \int_0^{\pi/2} a^3 \sin^3 u \cdot a^2 \cos^2 u\, du$$

$$= a^5 \int_0^{\pi/2} \left(\sin^3 u - \sin^5 u\right) du = a^5 \left(\frac{2}{3} - \frac{2 \cdot 4}{1 \cdot 3 \cdot 5}\right) = \frac{2}{15}a^5. \qquad \text{Ans: } \frac{4\pi}{15}a^5.$$

5A-7 The solid D is bounded below by $z = x^2 + y^2$ and above by $z = 2x$. The main problem is determining the projection R of D to the xy-plane, since we need to know this before we can put in the limits on the iterated integral.

The outline of R is the projection (i.e., vertical shadow) of the curve in which the paraboloid and plane intersect. This curve is made up of the points in which the graphs of $z = 2x$ and $z = x^2 + y^2$ intersect, i.e., the simultaneous solutions of the two equations. To project the curve, we omit the z-coordinates of the points on it. Algebraically, this amounts to solving the equations simultaneously by eliminating z from the two equations; doing this, we get as the outline of R the curve



cross-section of D view of D along x axis

$$x^2 + y^2 = 2x \quad \text{or, completing the square,} \quad (x-1)^2 + y^2 = 1.$$

This is a circle of radius 1 and center at $(1,0)$, whose polar equation is therefore $r = 2\cos\theta$.

We use symmetry to calculate just the right half of D and double the answer:

$$z\text{-moment of inertia of } D = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{x^2+y^2}^{2x} r^2\, dz\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} r^3\, dz\, dr\, d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3(2r\cos\theta - r^2)\, dr\, d\theta$$

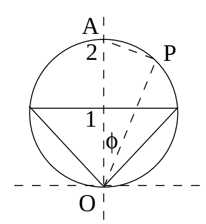
Inner: $\displaystyle \frac{2}{5}r^5\cos\theta - \frac{1}{6}r^6\Big]_0^{2\cos\theta} = \frac{2}{5}\cdot 32\cos^6\theta - \frac{1}{3}\cdot 32\cos^6\theta$

Outer: $\displaystyle \cdot\frac{32}{15}\int_0^{\pi/2}\cos^6\theta\, d\theta = \cdot\frac{32}{15}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2} = \frac{\pi}{3}.$ Ans: $\displaystyle \frac{2\pi}{3}$

5B. Triple Integrals in spherical coordinates

5B-1 a) The angle between the central axis of the cone and any of the lines on the cone is $\pi/4$; the sphere is $\rho = \sqrt{2}$; so the limits are (no integrand given):: $\displaystyle \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} d\rho\, d\phi\, d\theta.$

b) The limits are (no integrand is given): $\displaystyle \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} d\rho\, d\phi\, d\theta$



cross-section

c) To get the equation of the sphere in spherical coordinates, we note that AOP is always a right triangle, for any position of P on the sphere. Since $AO = 2$ and $OP = \rho$, we get according to the definition of the cosine, $\cos\phi = \rho/2$, or $\rho = 2\cos\phi$. (The picture shows the cross-section of the sphere by the plane containing P and the central axis AO.)

The plane $z = 1$ has in spherical coordinates the equation $\rho\cos\phi = 1$, or $\rho = \sec\phi$. It intersects the sphere in a circle of radius 1; this shows that $\pi/4$ is the maximum value of ϕ for which the ray having angle ϕ intersects the region.. Therefore the limits are (no integrand is given):

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\phi}^{2\cos\phi} d\rho\, d\phi\, d\theta.$$

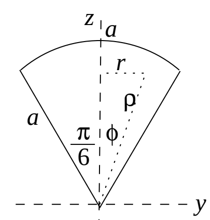
5B-2 Place the solid hemisphere D so that its central axis lies along the positive z-axis and its base is in the xy-plane. By symmetry, $\bar{x} = 0$ and $\bar{y} = 0$, so we only need $\bar{z}$. The integral for it is the product of three separate one-variable integrals, since the integrand is the product of three one-variable functions and the limits of integration are all constants.

$$\bar{z}\text{-moment} = \iiint_D z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\left(\frac{1}{4}\rho^4\right)_0^a\cdot\left(\frac{1}{2}\sin^2\phi\right)_0^{\pi/2} = 2\pi\cdot\frac{1}{4}a^4\cdot\frac{1}{2} = \frac{\pi a^4}{4}.$$

Since the mass is $\frac{2}{3}\pi a^3$, we have finally $\bar{z} = \dfrac{\pi a^4/4}{2\pi a^3/3} = \dfrac{3}{8}a.$

5B-3 Place the solid so the vertex is at the origin, and the central axis lies along the positive z-axis. In spherical coordinates, the density is given by $\delta = z = \rho\cos\phi$, and referring to the picture, we have



cross-section

$$\text{M. of I.} = \iiint_D r^2\cdot z\,dV = \iiint_D (\rho\sin\phi)^2(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^a \rho^5\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\sin^4\phi\Big]_0^{\pi/6} = 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4}\left(\frac{1}{2}\right)^4 = \frac{\pi a^6}{2^6\cdot 3}.$$

5B-4 Place the sphere so its center is at the origin. In each case the iterated integral can be expressed as the product of three one-variable integrals (which are easily calculated), since the integrand is the product of one-variable functions and the limits are constants.

a) $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot 2\cdot\frac{1}{4}a^4 = \pi a^4;$ $\quad$ average $= \dfrac{\pi a^4}{4\pi a^3/3} = \dfrac{3a}{4}.$

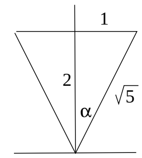
b) Use the z-axis as diameter. The distance of a point from the z-axis is $r = \rho\sin\phi$.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\sin\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{\pi}{2}\cdot\frac{1}{4}a^4 = \frac{\pi^2 a^4}{4}; \qquad \text{average} = \frac{\pi^2a^4/4}{4\pi a^3/3} = \frac{3\pi a}{16}.$$

c) Use the xy-plane and the upper solid hemisphere. The distance is $z = \rho\cos\phi$.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\cdot\frac{1}{2}\cdot\frac{1}{4}a^4 = \frac{\pi a^4}{4}; \qquad \text{average} = \frac{\pi a^4/4}{2\pi a^3/3} = \frac{3a}{8}.$$

5C. Gravitational Attraction



5C-2 The top of the cone is given by $z = 2$; in spherical coordinates: $\rho\cos\phi = 2$. Let α be the angle between the axis of the cone and any of its generators. The density $\delta = 1$. Since the cone is symmetric about its axis, the gravitational attraction has only a k-component, and is

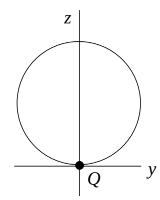
$$G\int_0^{2\pi}\int_0^{\alpha}\int_0^{2/\cos\phi} \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta.$$

Inner: $\dfrac{2}{\cos\phi}\sin\phi\cos\phi$ Middle: $-2\cos\phi\Big]_0^{\alpha} = -2\cos\alpha + 2$ Outer: $2\pi\cdot 2(1-\cos\alpha)$

Ans: $4\pi G\left(1 - \dfrac{2}{\sqrt{5}}\right)$.

5C-3 Place the sphere as shown so that Q is at the origin. Since it is rotationally symmetric about the z-axis, the force will be in the $\mathbf{k}$-direction.

Equation of sphere: $\rho = 2\cos\phi$ Density: $\delta = \rho^{-1/2}$



$$F_z = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi} \rho^{-1/2}\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

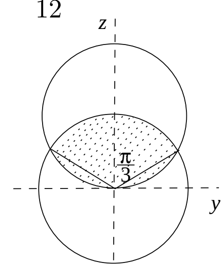
Inner: $\cos\phi\sin\phi\ 2\rho^{1/2}\Big]_0^{2\cos\phi} = 2\sqrt{2}\ \cos^{3/2}\phi\ \sin\phi$

Middle: $2\sqrt{2}\left[-\dfrac{2}{5}\cos^{5/2}\phi\right]_0^{\pi/2} = \dfrac{4\sqrt{2}}{5}$ Outer: $2\pi G\dfrac{4\sqrt{2}}{5} = \dfrac{8\sqrt{2}}{5}\pi G$.

5C-4 Referring to the figure, the total gravitational attraction (which is in the $\mathbf{k}$ direction, by rotational symmetry) is the sum of the two integrals

$$G\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \cos\phi\ \sin\phi\, d\rho\, d\phi\, d\theta \quad + \quad G\int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi} \cos\phi\ \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi G\cdot\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + 2\pi G\cdot\frac{2}{3}\left(\frac{1}{2}\right)^3 = \frac{3}{4}\pi G + \frac{1}{6}\pi G = \frac{11}{12}\pi G.$$



The two spheres are shown in cross-section. The spheres intersect at the points where $\phi = \pi/3$.

The first integral respresents the gravitational attraction of the top part of the solid, bounded below by the cone $\phi = \pi/3$ and above by the sphere $\rho = 1$.

The second integral represents the bottom part of the solid, bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone.

6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 a) the vectors are all unit vectors, pointing radially outward.
b) the vector at P has its head on the y-axis, and is perpendicular to it

6A-2 $\frac{1}{2}(-x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k})$

6A-3 $\omega(-z\,\mathbf{j} + y\,\mathbf{k})$

6A-4 A vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is parallel to the plane $3x - 4y + z = 2$ if it is perpendicular to the normal vector to the plane, $3\,\mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$: the condition on M, N, P therefore is $3M - 4N + P = 0$, or $P = 4N - 3M$.

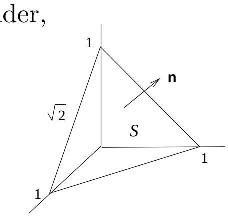
The most general such field is therefore $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + (4N - 3M)\,\mathbf{k}$, where M and N are functions of x, y, z.

6B. Surface Integrals and Flux

6B-1 We have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; therefore $\mathbf{F}\cdot\mathbf{n} = a$.

Flux through $S = \displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = a(\text{area of } S) = 4\pi\,a^3$.

6B-2 Since $\mathbf{k}$ is parallel to the surface, the field is everywhere tangent to the cylinder, hence the flux is 0.



6B-3 $\dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$ is a normal vector to the plane, so $\mathbf{F}\cdot\mathbf{n} = \dfrac{1}{\sqrt{3}}$.

Therefore, flux $= \dfrac{\text{area of region}}{\sqrt{3}} = \dfrac{\frac{1}{2}(\text{base})(\text{height})}{\sqrt{3}} = \dfrac{\frac{1}{2}(\sqrt{2})(\frac{\sqrt{3}}{2}\sqrt{2})}{\sqrt{3}} = \dfrac{1}{2}$.

6B-4 $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{y^2}{a}$. Calculating in spherical coordinates,

flux $= \displaystyle\iint_S \frac{y^2}{a}\,dS = \frac{1}{a}\int_0^{\pi}\int_0^{\pi} a^4 \sin^3\phi \sin^2\theta\,d\phi\,d\theta = a^3 \int_0^{\pi}\int_0^{\pi} \sin^3\phi\sin^2\theta\,d\phi d\theta.$

Inner integral: $\sin^2\theta(-\cos\phi + \frac{1}{3}\cos^3\phi)\Big]_0^{\pi} = \frac{4}{3}\sin^2\theta;$

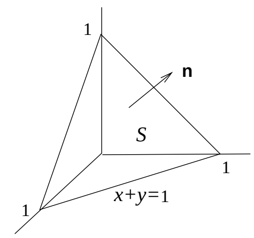
Outer integral: $\frac{4}{3}a^3(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\Big]_0^{\pi} = \frac{2}{3}\pi a^3.$

6B-5 $\mathbf{n} = \dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$; $\mathbf{F}\cdot\mathbf{n} = \dfrac{z}{\sqrt{3}}$.

$$\text{flux } = \iint_S \frac{z}{\sqrt{3}}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{1}{\sqrt{3}}\iint_S (1-x-y)\frac{dx\,dy}{1/\sqrt{3}} = \int_0^1\int_0^{1-y}(1-x-y)\,dx\,dy.$$

$$\text{Inner integral: } = x - \tfrac{1}{2}x^2 - xy\Big]_0^{1-y} = \tfrac{1}{2}(1-y)^2.$$

$$\text{Outer integral: } = \int_0^1 \frac{1}{2}(1-y)^2dy = \frac{1}{2}\cdot -\frac{1}{3}\cdot(1-y)^3\Big]_0^1 = \frac{1}{6}.$$



6B-6 $z = f(x,y) = x^2+y^2$ (a paraboloid). By (13) in Notes V9,

$$d\mathbf{S} = (-2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k})\,dx\,dy.$$

(This points generally "up", since the $\mathbf{k}$ component is positive.) Since $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_R (-2x^2 - 2y^2 + z)\,dxdy \qquad ,$$

where R is the interior of the unit circle in the xy-plane, i.e., the projection of S onto the xy-plane). Since $z = x^2+y^2$, the above integral

$$= -\iint_R (x^2+y^2)\,dxdy = -\int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = -2\pi\cdot\frac{1}{4} = -\frac{\pi}{2}\ .$$

The answer is negative since the positive direction for flux is that of $\mathbf{n}$, which here points into the inside of the paraboloidal cup, whereas the flow $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is generally from the inside toward the outside of the cup, i.e., in the opposite direction.

6B-8 On the cylindrical surface, $\mathbf{n} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{a}$, $\mathbf{F}\cdot\mathbf{n} = \dfrac{y^2}{a}$.
In cylindrical coordinates, since $y = a\sin\theta$, this gives us $\mathbf{F}\cdot d\mathbf{S} = \mathbf{F}\cdot\mathbf{n}\,dS = a^2\sin^2\theta\,dz\,d\theta$.

$$\text{Flux } = \int_{-\pi/2}^{\pi/2}\int_0^k a^2\sin^2\theta\,dz\,d\theta = a^2h\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta = a^2h\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_{-\pi/2}^{\pi/2} = \frac{\pi}{2}a^2h\ .$$

6B-12 Since the distance from a point $(x,y,0)$ up to the hemispherical surface is z,

$$\text{average distance } = \frac{\iint_S z\,dS}{\iint_S dS}.$$

In spherical coordinates, $\displaystyle\iint_S z\,dS = \int_0^{2\pi}\int_0^{\pi/2} a\cos\phi\cdot a^2\sin\phi\,d\phi\,d\theta.$

$$\text{Inner: } = a^3\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = a^3(\frac{\sin^2\phi}{2}\Big]_0^{\pi/2} = \frac{a^3}{2}. \qquad \text{Outer: } = \frac{a^3}{2}\int_0^{2\pi}d\theta = \pi a^3.$$

Finally, $\displaystyle\iint_S dS =$ area of hemisphere $= 2\pi a^2$, so average distance $= \dfrac{\pi a^3}{2\pi a^2} = \dfrac{a}{2}$.

6C. Divergence Theorem

6C-1a div $\mathbf{F} = M_x + N_y + P_z = 2xy + x + x = 2x(y+1)$.

6C-2 Using the product and chain rules for the first, symmetry for the others,

$$(\rho^n x)_x = n\rho^{n-1}\,\frac{x}{\rho}\,x + \rho^n, \quad (\rho^n y)_y = n\rho^{n-1}\,\frac{y}{\rho}\,y + \rho^n, \quad (\rho^n z)_z = n\rho^{n-1}\,\frac{z}{\rho}\,z + \rho^n;$$

adding these three, we get $\quad \text{div}\,\mathbf{F} = n\rho^{n-1}\dfrac{x^2+y^2+z^2}{\rho} \;+\; 3\rho^n \;=\; \rho^n(n+3)$.

Therefore, div $\mathbf{F} = 0 \;\Leftrightarrow\; n = -3$.

6C-3 Evaluating the triple integral first, we have div $\mathbf{F} = 3$, therefore

$$\iiint_D \text{div}\,\mathbf{F}\,dV = 3(\text{vol.of } D) = 3\,\frac{2}{3}\pi a^3 = 2\pi a^3.$$

To evaluate the double integral over the closed surface $S_1 + S_2$, the normal vectors are:

$$\mathbf{n}_1 = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a} \quad (\text{hemisphere } S_1), \qquad \mathbf{n}_2 = -\,\mathbf{k} \quad (\text{disc } S_2);$$

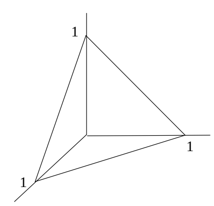
using these, the surface integral for the flux through S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \frac{x^2+y^2+z^2}{a}\,dS + \iint_{S_2} -z\,dS = \iint_{S_1} a\,dS,$$

since $x^2+y^2+z^2 = \rho^2 = a^2$ on S_1, and $z = 0$ on S_2. So the value of the surface integral is

$$a(\text{area of } S_1) = a(2\pi a^2) = 2\pi a^3,$$

which agrees with the triple integral above.



6C-5 The divergence theorem says $\quad \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV$.

Here div $\mathbf{F} = 1$, so that the right-hand integral is just the volume of the tetrahedron, which is $\frac{1}{3}$(base)(height)$= \frac{1}{3}(\frac{1}{2})(1) = \frac{1}{6}$.

6C-6 The divergence theorem says $\quad \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \text{div}\,\mathbf{F}\,dV$.

Here div $\mathbf{F} = 1$, so the right-hand integral is the volume of the solid cone, which has height 1 and base radius 1; its volume is $\frac{1}{3}$(base)(height)$= \pi/3$.

6C-7a Evaluating the triple integral first, over the cylindrical solid D, we have

$$\text{div}\,\mathbf{F} = 2x + x = 3x; \qquad \iiint_D 3x\,dV = 0,$$

since the solid is symmetric with respect to the yz-plane. (Physically, assuming the density is 1, the integral has the value $\bar{x}$(mass of D), where $\bar{x}$ is the x-coordinate of the center of mass; this must be in the yz plane since the solid is symmetric with respect to this plane.)

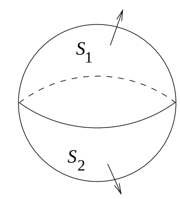
To evaluate the double integral, note that $\mathbf{F}$ has no $\mathbf{k}$-component, so there is no flux across the two disc-like ends of the solid. To find the flux across the cylindrical side,

$$\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j}, \qquad \mathbf{F}\cdot\mathbf{n} = x^3 + xy^2 = x^3 + x(1-x^2) = x,$$

since the cylinder has radius 1 and equation $x^2 + y^2 = 1$. Thus

$$\iint_S x\,dS = \int_0^{2\pi}\int_0^1 \cos\theta\,dz\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = 0.$$

6C-8 a) Reorient the lower hemisphere S_2 by reversing its normal vector; call the reoriented surface S_2'. Then $S = S_1 + S_2'$ is a closed surface, with the normal vector pointing outward everywhere, so by the divergence theorem,



$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV = 0,$$

since by hypothesis div $\mathbf{F} = 0$. The above shows

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = -\iint_{S_2'}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S},$$

since reversing the orientation of a surface changes the sign of the flux through it.

b) The same statement holds if S_1 and S_2 are two oriented surfaces having the same boundary curve, but not intersecting anywhere else, and oriented so that S_1 and S_2' (i.e., S_2 with its orientation reversed) together make up a closed surface S with outward-pointing normal.

6C-10 If div $\mathbf{F} = 0$, then for any closed surface S, we have by the divergence theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV = 0.$$

Conversely: $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface $S \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = 0$.

For suppose there were a point P_0 at which $(\operatorname{div}\mathbf{F})_0 \neq 0$ — say $(\operatorname{div}\mathbf{F})_0 > 0$. Then by continuity, div $\mathbf{F} > 0$ in a very small spherical ball D surrounding P_0, so that by the divergence theorem (S is the surface of the ball D),

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV \quad > \quad 0.$$

But this contradicts our hypothesis that $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$ for every closed surface S.

6C-11 flux of $\mathbf{F} = \displaystyle\iint_S \mathbf{F}\cdot d\mathbf{n} = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iiint_D 3\,dV = 3(\text{vol. of } D)$.

6D. Line Integrals in Space

6D-1 a) $C:\quad x = t,\ dx = dt;\quad y = t^2,\ dy = 2t\,dt;\quad z = t^3,\ dz = 3t^2\,dt;$

$$\int_C y\,dx + z\,dy - x\,dz \;=\; \int_0^1 (t^2)dt + t^3(2t\,dt) - t(3t^2\,dt)$$

$$= \int_0^1 (t^2 + 2t^4 - 3t^3)dt \;=\; \frac{t^3}{3} + \frac{2t^5}{5} - \frac{3t^4}{4}\Big]_0^1 \;=\; \frac{1}{3} + \frac{2}{5} - \frac{3}{4} \;=\; -\frac{1}{60}.$$

b) $C:\quad x = t,\ y = t,\ z = t;\qquad \displaystyle\int_C y\,dx + z\,dy - x\,dz \;=\; \int_0^1 t\,dt \;=\; \frac{1}{2}.$

c) $C = C_1 + C_2 + C_3$; $\quad C_1 : y = z = 0$; $\quad C_2 : x = 1,\ z = 0$; $\quad C_3 : x = 1,\ y = 1$

$$\int_C y\,dx + z\,dy - x\,dz \;=\; \int_{C_1} 0 + \int_{C_2} 0 + \int_0^1 -dz \;=\; -1.$$

d) $C : x = \cos t,\ y = \sin t,\ z = t$; $\quad \int_C zx\,dx + zy\,dy + x\,dz$

$$= \int_0^{2\pi} t\cos t(-\sin t\,dt) + t\sin t(\cos t\,dt) + \cos t\,dt \;=\; \int_0^{2\pi} \cos t\,dt \;=\; 0.$$

6D-2 The field $\mathbf{F}$ is always pointed radially outward; if C lies on a sphere centered at the origin, its unit tangent $\mathbf{t}$ is always tangent to the sphere, therefore perpendicular to the radius; this means $\mathbf{F}\cdot\mathbf{t} = 0$ at every point of C. Thus $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{t}\,ds = 0$.

6D-4 a) $\mathbf{F} = \nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

b) (i) Directly, letting C be the helix: $x = \cos t,\ y = \sin t,\ z = t$, from $t = 0$ to $t = 2n\pi$,

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2\cos t(-\sin t)dt + 2\sin t(\cos t)dt + 2t\,dt = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (ii) Choose the vertical path $x = 1,\ y = 0,\ z = t$; then

$$\int_C M dx + N dy + P dz = \int_0^{2n\pi} 2t\,dt = (2n\pi)^2.$$

b) (iii) By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,0,2n\pi) - f(1,0,0) = 91^2 + (2n\pi)^2) - 1^2 = (2n\pi)^2$$

6D-5 By the First Fundamental Theorem for line integrals,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \sin(xyz)\Big|_Q - \sin(xyz)\Big|_P,$$

where C is any path joining P to Q. The maximum value of this difference is $1 - (-1) = 2$, since $\sin(xyz)$ ranges between -1 and 1.

For example, any path C connecting $P : (1, 1, -\pi/2)$ to $Q : (1, 1, \pi/2)$ will give this maximum value of 2 for $\int_C \mathbf{F}\cdot d\mathbf{r}$.

6E. Gradient Fields in Space

6E-1 a) Since $M = x^2,\ N = y^2,\ P = z^2$ are continuously differentiable, the differential is exact because $N_z = P_y = 0,\quad M_z = P_x = 0,\quad M_y = N_x = 0;\ f(x,y,z) = (x^3 + y^3 + z^3)/3$.

b) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = 2xy, \quad M_z = P_x = y^2, \quad M_y = N_x = 2yz; \quad f(x,y,z) = xy^2\ .$$

c) Exact: M, N, P are continuously differentiable for all x, y, z, and

$$N_z = P_y = x, \quad M_z = P_x = y, \quad M_y = N_x = 6x^2 + z;\ f(x,y,z) = 2x^3y + xyz.$$

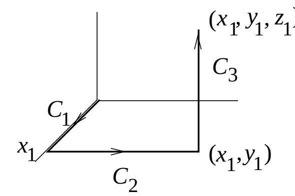
6E-2 $\quad \operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & yz & xyz^2 \end{vmatrix} = (xz^2 - y)\,\mathbf{i} - yz^2\,\mathbf{j} - x^2\,\mathbf{k}$.

6E-3 a) It is easily checked that curl $\mathbf{F} = 0$.

b) (i) using method I:

$$f(x_1, y_1, z_1) = \int_{(0,0,0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{x_1} x\,dx + \int_0^{y_1} y\,dy + \int_0^{z_1} z\,dz = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + \frac{1}{2}z_2^2.$$



Therefore $f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + c$.

(ii) Using method II: We seek $f(x, y, z)$ such that $f_x = 2xy + z$, $f_y = x^2$, $f_z = x$.

$f_x = 2xy + z \quad \Rightarrow \quad f = x^2y + xz + g(y, z)$.
$f_y = x^2 + g_y = x^2 \quad \Rightarrow \quad g_y = 0 \quad \Rightarrow \quad g = h(z)$
$f_z = x + h'(z) = x \quad \Rightarrow \quad h' = 0 \quad \Rightarrow \quad h = c$

Therefore $f(x, y, z) = x^2y + xz + c$.

(iii) If $f_x = yz$, $f_y = xz$, $f_z = xy$, then by inspection, $f(x, y, z) = xyz + c$.

6E-4 Let $F = f - g$. Since ∇ is a linear operator, $\nabla F = \nabla f - \nabla g = \mathbf{0}$

We now show: $\nabla F = \mathbf{0} \Rightarrow F = c$.

Fix a point $P_0 : (x_0, y_0, z_0)$. Then by the Fundamental Theorem for line integrals,

$$F(P) - F(P_0) = \int_{P_0}^{P} \nabla F \cdot d\mathbf{r} = 0.$$

Therefore $F(P) = F(P_0)$ for all P, i.e., $F(x, y, z) = F(x_0, y_0, z_0) = c$.

6E-5 $\mathbf{F}$ is a gradient field only if these equations are satisfied:

$$N_z = P_y : \ 2xz + ay = bxz + 2y \qquad M_z = P_x : \ 2yz = byz \qquad M_y = N_x : \ z^2 = z^2.$$

Thus the conditions are: $a = 2$, $b = 2$.

Using these values of a and b we employ Method 2 to find the potential function f:

$f_x = yz^2 \quad \Rightarrow \quad f = xyz^2 + g(y, z)$;
$f_y = xz^2 + g_y = xz^2 + 2yz \quad \Rightarrow \quad g_y = 2yz \quad \Rightarrow \quad g = y^2z + h(z)$
$f_z = 2xyz + y^2 + h'(z) = 2xyz + y^2 \quad \Rightarrow \quad h = c$;

therefore, $f(x, y, z) = xyz^2 + y^2z + c$.

6E-6 a) $Mdx + Ndy + Pdz$ is an exact differential if there exists some function $f(x, y, z)$ for which $df = Mdx + Ndy + Pdz$; that, is, for which $f_x = M$, $f_y = N$, $f_z = P$.

b) The given differential is exact if the following equations are satisfied:

$N_z = P_y : \quad (a/2)x^2 + 6xy^2z + 3byz^2 = 3x^2 + 3cxy^2z + 12yz^2$;
$M_z = P_x : \quad axy + 2y^3z = 6xy + cy^3z$
$M_y = N_x : \quad axz + 3y^2z^2 = axz + 3y^2z^2$.

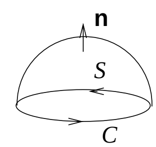
Solving these, we find that the differential is exact if $a = 6$, $b = 4$, $c = 2$.

c) We find $f(x, y, z)$ using method 2:

$$\begin{aligned}
&f_x = 6xyz + y^3z^2 \quad \Rightarrow \quad f = 3x^2yz + xy^3z^2 + g(y, z);\\
&f_y = 3x^2z + 3xy^2z^2 + g_y = 3x^2z + 3xy^2z^2 + 4yz^3 \quad \Rightarrow \quad g_y = 4yz^3 \quad \Rightarrow \quad g = 2y^2z^3 + h(z)\\
&f_z = 3x^2y + 2xy^3z + 6y^2z^2 + h'(z) = 3x^2y + 2xy^3z + 6y^2z^2 \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h = c.
\end{aligned}$$

Therefore, $\quad f(x, y, z) = 3x^2yz + xy^3z^2 + 2y^2z^3 + c.$

6F. Stokes' Theorem



6F-1 a) For the line integral, $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C x\,dx + y\,dy + z\,dz = 0,$ since the differential is exact.

For the surface integral, $\quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \mathbf{0}$, and therefore $\displaystyle\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = 0.$

b) Line integral: $\displaystyle\oint_C y\,dx + z\,dy + x\,dz = \oint_C y\,dx,$ since $z = 0$ and $dz = 0$ on C.

Using $x = \cos t, \quad y = \sin t, \quad \displaystyle\int_0^{2\pi} -\sin^2 t\,dt = -\int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = -\pi.$

Surface integral: curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\,\mathbf{i} - \mathbf{j} - \mathbf{k}\,; \qquad \mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$$\iint_S \nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = -\iint_S (x + y + z)\,dS.$$

To evaluate, we use $x = r\cos\theta, \quad y = r\sin\theta, \quad z = \rho\cos\phi. \quad r = \rho\sin\phi, \quad dS = \rho^2\sin\phi\,d\phi d\theta$; note that $\rho = 1$ on S. The integral then becomes

$$-\int_0^{2\pi}\int_0^{\pi/2} [\sin\phi(\cos\theta + \sin\theta) + \cos\phi]\sin\phi\,d\phi\,d\theta$$

Inner: $\displaystyle -\left[(\cos\theta + \sin\theta)(\frac{\phi}{2} - \frac{\sin 2\phi}{4}) + \frac{1}{2}\sin^2\phi\right]_0^{\pi/2} \quad = \quad -\left[(\cos\theta + \sin\theta)\frac{\pi}{4} + \frac{1}{2}\right]\;;$

Outer: $\displaystyle\int_0^{2\pi}\left(-\frac{1}{2} - (\cos\theta + \sin\theta)\frac{\pi}{4}\right)d\theta = -\pi.$

6F-2 The surface S is: $\;z = -x - y$, so that $f(x, y) = -x - y$.

$$\mathbf{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy \;=\; \langle 1, 1, 1\rangle\,dx\,dy.$$

(Note the signs: $\mathbf{n}$ points upwards, and therefore should have a positive k-component.)

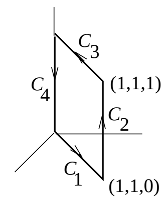
$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\,\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Therefore $\displaystyle\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S'} 3\,dA = -3\pi$, where S' is the projection of S, i.e., the interior of the unit circle in the xy-plane.

As for the line integral, we have $C: \quad x = \cos t, \quad y = \sin t \quad z = -\cos t - \sin t, \;$ so that

$$\oint_C ydx + zdy + xdz = \int_0^{2\pi} \Big[-\sin^2 t - (\cos^2 t + \sin t\cos t) + \cos t(\sin t - \cos t)\Big]\,dt$$
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t - \cos^2 t)\,dt = \int_0^{2\pi} \Big[-1 - \frac{1}{2}\,(1 + \cos 2t)\Big]\,dt = -\frac{3}{2}\cdot 2\pi = -3\pi.$$

6F-3 Line integral: $\oint_C yz\,dx + xz\,dy + xy\,dz$ over the path $C = C_1 + \ldots + C_4$:

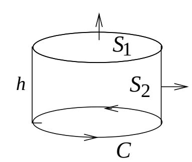


$$\int_{C_1} = 0, \quad \text{since } z = dz = 0 \text{ on } C_1;$$
$$\int_{C_2} = \int_0^1 1\cdot 1\,dz = 1, \quad \text{since } x = 1,\ y = 1,\ dx = 0,\ dy = 0 \text{ on } C_2;$$
$$\int_{C_3} ydx + xdy = \int_1^0 xdx + xdx = -1, \quad \text{since } y = x,\ z = 1,\ dz = 0 \text{ on } C_3;$$
$$\int_{C_4} = 0, \quad \text{since } x = 0,\ y = 0 \text{ on } C_4.$$

Adding up, we get $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0.$ For the surface integral,

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = \mathbf{i}\,(x - x) - \mathbf{j}\,(y - y) + \mathbf{k}\,(z - z) = \mathbf{0}; \text{ thus } \iint \text{curl }\mathbf{F}\cdot d\mathbf{S} = 0.$$

6F-5 Let S_1 be the top of the cylinder (oriented so $\mathbf{n} = \mathbf{k}$), and S_2 the side.



a) We have $\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & x^2 \end{vmatrix} = -2x\,\mathbf{j} + 2\,\mathbf{k}\,.$

For the top: $\iint_{S_1} \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} 2\,dS = 2(\text{area of } S_1) = 2\pi a^2.$

For the side: we have $\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{a}$, and $dS = dz\cdot a\,d\theta$, so that

$$\iint_{S_2} \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^h \frac{-2xy}{a}\,a\,dz\,d\theta = \int_0^{2\pi} -2h(a\cos\theta)(a\sin\theta)\,d\theta = -ha^2\sin^2\theta\Big]_0^{2\pi} = 0.$$

Adding, $\iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} + \iint_{S_2} = 2\pi a^2.$

b) Let C be the circular boundary of S, parameterized by $x = a\cos\theta,\ y = a\sin\theta,\ z = 0.$ Then using Stokes' theorem,

$$\iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S} = \oint_C -y\,dx + x\,dy + x^2\,dz = \int_0^{2\pi} (a^2\sin^2\theta + a^2\cos^2\theta)\,d\theta = 2\pi a^2.$$

6G. Topological Questions

6G-1 a) yes b) no c) yes d) no; yes; no; yes; no; yes

6G-2 Recall that $\rho_x = x/\rho$, etc. Then, using the chain rule,

$$\text{curl }\mathbf{F} = (n\rho^{n-1}z\,\frac{y}{\rho} - n\rho^{n-1}y\,\frac{z}{\rho})\,\mathbf{i} + (n\rho^{n-1}z\,\frac{x}{\rho} - n\rho^{n-1}x\,\frac{z}{\rho})\,\mathbf{j} + (n\rho^{n-1}y\,\frac{x}{\rho} - n\rho^{n-1}x\,\frac{y}{\rho})\,\mathbf{k}\,.$$

Therefore curl $\mathbf{F} = \mathbf{0}$. To find the potential function, we let P_0 be any convenient starting point, and integrate along some path to $P_1 : (x_1, y_1, z_1)$. Then, if $n \neq -2$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{P_0}^{P_1} \rho^n (x\,dx + y\,dy + z\,dz) = \int_{P_0}^{P_1} \rho^n \,\frac{1}{2}\, d(\rho^2)$$
$$= \int_{P_0}^{P_1} \rho^{n+1} d\rho = \frac{\rho^{n+2}}{n+2}\Big]_{P_0}^{P_1} = \frac{\rho_1^{n+2}}{n+2} - \frac{\rho_0^{n+2}}{n+2} = \frac{\rho_1^{n+2}}{n+2} + c, \text{ since } P_0 \text{ is fixed.}$$

Therefore, we get $\mathbf{F} = \nabla \dfrac{\rho^{n+2}}{n+2}$, if $n \neq -2$.

If $n = -2$, the line integral becomes $\displaystyle\int_{P_0}^{P_1} \frac{d\rho}{\rho} = \ln \rho_1 + c$, so that $\mathbf{F} = \nabla(\ln \rho)$.

6H. Applications and Further Exercises

6H-1 Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$. By the definition of curl $\mathbf{F}$, we have

$$\nabla \times \mathbf{F} = (P_y - N_z)\,\mathbf{i} + (M_z - P_x)\,\mathbf{j} + (N_x - M_y)\,\mathbf{k}\,,$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$

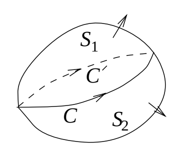
If all the mixed partials exist and are continuous, then $P_{xy} = P_{yx}$, etc. and the right-hand side of the above equation is zero: div (curl $\mathbf{F}$) $= 0$.

6H-2 a) Using the divergence theorem, and the previous problem, (D is the interior of S),

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iiint_D \text{div curl } \mathbf{F}\, dV = \iiint_D 0\, dV = 0.$$

b) Draw a closed curve C on S that divides it into two pieces S_1 and S_2 both having C as boundary. Orient C compatibly with S_1, then the curve C' obtained by reversing the orientation of C will be oriented compatibly with S_2. Using Stokes' theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl } \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{C'} \mathbf{F} \cdot d\mathbf{r} = 0,$$



since the integral on C' is the negative of the integral on C.

Or more simply, consider the limiting case where C has been shrunk to a point; even as a point, it can still be considered to be the boundary of S. Since it has zero length, the line integral around it is zero, and therefore Stokes' theorem gives

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \;=\; \oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; 0.$$

6H-10 Let C be an oriented closed curve, and S a compatibly-oriented surface having C as its boundary. Using Stokes' theorem and the Maxwell equation, we get respectively

$$\iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \oint_C \mathbf{B}\cdot d\mathbf{r} \qquad \text{and} \qquad \iint_S \nabla\times\mathbf{B}\cdot d\mathbf{S} = \iint_S \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S} = \frac{1}{c}\frac{d}{dt}\iint_S E\cdot d\mathbf{S}.$$

Since the two left sides are the same, we get $\displaystyle\oint_C \mathbf{B} \cdot d\mathbf{r} = \frac{1}{c}\frac{d}{dt}\iint_S \mathbf{E} \cdot d\mathbf{S}.$

In words: for the magnetic field $\mathbf{B}$ produced by a moving electric field $\mathbf{E}(t)$, the magneto-motive force around a closed loop C is, up to a constant factor depending on the units, the time-rate at which the electric flux through C is changing.