Unit 7. Infinite Series

7A: Basic Definitions

7A-1

a) Sum the geometric series: $\displaystyle\sum_0^\infty \frac{1}{4^n} = \sum_0^\infty \left(\frac{1}{4}\right)^n = \frac{1}{1-(1/4)} = \frac{4}{3}$.

b) $1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$ diverges, since the partial sums s_n are successively $1, 0, 1, 0, \ldots$, and therefore do not approach a limit.

c) Diverges, since the n-th term $\frac{n-1}{n}$ does not tend to 0 (using the n-th term test for divergence).

d) The given series $= \ln 2 + \frac{1}{2}\ln 2 + \frac{1}{3}\ln 2 + \ldots = \ln 2(1 + \frac{1}{2} + \frac{1}{3} + \ldots)$; but $\displaystyle\sum_1^\infty 1/n$ diverges; therefore the given series diverges.

e) $\displaystyle\sum_1^\infty \frac{2^{n-1}}{3^n} = \frac{1}{3}\sum_1^\infty \frac{2^{n-1}}{3^{n-1}}$, geometric series with sum $\displaystyle\frac{1}{3}\left(\frac{1}{1-(2/3)}\right) = \frac{1}{3}\cdot 3 = 1.$

f) series $\displaystyle = \sum_0^\infty \left(\frac{-1}{3}\right)^n = \frac{1}{1-(-1/3)} = \frac{3}{4}$ (sum of a geometric series)

7A-2 $.21111\ldots = .2 + .01 + .001 + \ldots = .2 + .01(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots = .2 + .01(\left(\frac{1}{1-1/10}\right) = \frac{19}{90}$.

7A-3 Geometric series; converges if $|x/2| < 1$, i.e., if $|x| < 2$, or equivalently, $-2 < x < 2$.

7A-4

a) Partial sum: $\displaystyle s_m = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \ldots + \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}}\right)$

$\displaystyle = 1 - \frac{1}{\sqrt{m+1}} \to 1$ as $m \to \infty$. Therefore the sum is 1.

b) $\displaystyle\frac{1}{n(n+2)} = \frac{1/2}{n} + \frac{-1/2}{n+2}$; therefore $\displaystyle\sum_1^\infty \frac{1}{n(n+2)} = \frac{1}{2}\left(\sum_0^\infty \left(\frac{1}{n} - \frac{1}{n+2}\right)\right).$

The m-th partial sum of the series is

$$s_m = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots + \frac{1}{m} - \frac{1}{m+2}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{m+1} - \frac{1}{m+2}\right),$$

since all other terms cancel.

Therefore $s_m \to \frac{3}{4}$ as $m \to \infty$, so the sum is $3/4$.

7A-5 The distance the ball travels is $\displaystyle h + \frac{2}{3}h + \frac{2}{3}h + \frac{2}{3}\left(\frac{2}{3}h\right) + \frac{2}{3}\left(\frac{2}{3}h\right) + \ldots$;
the successive terms give the first down, the first up, the second down, and so on. Add h to the series to make the terms uniform; you get a geometric series to sum:

$$2\left(h + 2h/3 + (2/3)^2h + \ldots\right) = 2h(1 + 2/3 + (2/3)^2 + \ldots) = 2h\left(\frac{1}{1-2/3}\right) = 6h.$$

Subtracting the h that we added on gives: the total distance traveled $= 5h$.

7B: Convergence Tests

7B-1

a) $\displaystyle\int_0^\infty \frac{x}{x^2+4} = \frac{1}{2}\ln(x^2+4)\Big]_0^\infty = \infty$; divergent

b) $\displaystyle\int_0^\infty \frac{1}{x^2+1} = \tan^{-1} x\Big]_0^\infty = \frac{\pi}{2}$; convergent

c) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+1}} = 2(x+1)^{1/2}\Big]_0^\infty = \infty$; divergent

d) $\displaystyle\int_1^\infty \frac{\ln x}{x} = \frac{1}{2}(\ln x)^2\Big]_1^\infty = \infty$; divergent

e) $\displaystyle\int_2^\infty \frac{1}{(\ln x)^p \cdot x} = \frac{(\ln x)^{1-p}}{1-p}\Big]_2^\infty$, if $p \neq 1$: divergent if $p < 1$, convergent if $p > 1$

If $p = 1$, $\displaystyle\int_2^\infty \frac{dx}{\ln x} = \ln(\ln x)\Big]_2^\infty = \infty$. Thus series converges if $p > 1$, diverges if $p \leq 1$.

f) $\displaystyle\int_1^\infty \frac{1}{x^p} = \frac{x^{1-p}}{1-p}\Big]_1^\infty$, if $p \neq 1$; diverges if $p < 1$, converges if $p > 1$.

If $p = 1$, $\displaystyle\int_1^\infty \frac{dx}{x} = \ln x\Big]_1^\infty = \infty$; thus series converges if $p > 1$, diverges if $p \leq 1$.

7B-2

a) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^2}$: $\displaystyle\frac{n^2}{n^2+3n} = \frac{1}{1+3/n} \to 1$ as $n \to \infty$

b) Divergent; compare with $\displaystyle\sum \frac{1}{n}$: $\displaystyle\frac{n}{n+\sqrt{n}} = \frac{1}{1+1/\sqrt{n}} \to 1$, as $n \to \infty$

c) Divergent; compare with $\displaystyle\sum \frac{1}{n}$: $\displaystyle\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+1/n}} \to 1$, as $n \to \infty$

d) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^2}$: $\displaystyle\lim_{n\to\infty} n^2 \sin\left(\frac{1}{n^2}\right) = \lim_{h\to 0} \frac{\sin h}{h} = 1$

e) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^{3/2}}$: $\displaystyle\frac{n^{3/2}\sqrt{n}}{n^2+1} = \frac{n^2}{n^2+1} = \frac{1}{1+1/n^2} \to 1$ as $n \to \infty$

f) Divergent, by comparison test : $\displaystyle\frac{\ln n}{n} > \frac{1}{n}$; $\displaystyle\sum_1^\infty \frac{1}{n}$ diverges

g) Convergent; compare with $\displaystyle\sum \frac{1}{n^2}$: $\displaystyle\frac{n^2 \cdot n^2}{n^4-1} = \frac{n^4}{n^4-1} \to 1$ as $n \to \infty$

h) Divergent; compare with $\displaystyle\sum \frac{1}{4n}$: $\displaystyle\frac{4n \cdot n^3}{4n^4+n^2} = \frac{1}{1+1/4n^2} \to 1$

7B-3 By the mean-value theorem, $\sin x < x$, if $x > 0$; therefore $\sum_0^\infty \sin a_n < \sum_0^\infty a_n$; so the series converges by the comparison test.

7B-4

a) By ratio test, $\dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \left(\dfrac{n+1}{n}\right) \cdot \dfrac{1}{2} \to \dfrac{1}{2}$ as $n \to \infty$; convergent

b) By ratio test, $\dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n} = \dfrac{2}{n+1} \to 0$ as $n \to \infty$; convergent

c) By ratio test, $\dfrac{2^{n+1}}{1 \cdot 3 \cdot \cdots \cdot 2n+1} \cdot \dfrac{1 \cdot 3 \cdot \cdots \cdot 2n-1}{2^n} = \dfrac{2}{2n+1} \to 0$ as $n \to \infty$; convergent

d) By ratio test, $\dfrac{(n+1)!^2}{(2n+2)!} \cdot \dfrac{(2n)!}{n!^2} = \dfrac{(n+1)^2}{(2n+2)(2n+1)} \to \dfrac{1}{4}$ as $n \to \infty$; convergent

e) Ratio test fails: $\dfrac{1}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{1} \to 1$ as $n \to \infty$; but $\sum \dfrac{1}{\sqrt{n}}$ diverges; therefore the series is not absolutely convergent.

f) By ratio test, $\dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!} = \dfrac{n^n}{(n+1)^n} = \dfrac{1}{(1+1/n)^n} \to \dfrac{1}{e} < 1$ as $n \to \infty$; convergent

g) Ratio test fails: $\dfrac{1}{(n+1)^2} \cdot \dfrac{n^2}{1} \to 1$ as $n \to \infty$; but $\sum \dfrac{1}{n^2}$ converges; therefore the series is absolutely convergent.

h) Ratio test fails: $\sum \dfrac{1}{\sqrt{n^2+1}}$ diverges, by limit comparison with $\sum \dfrac{1}{n}$; therefore the series is not absolutely convergent.

i) Ratio test fails: $\sum \dfrac{n}{n+1}$ diverges by the n-th term test; therefore the series is not absolutely convergent

7B-5

e) conditionally convergent: terms alternate in sign, $\dfrac{1}{\sqrt{n}} \to 0$, decreasing;

h) conditionally convergent: terms alternate in sign, $\dfrac{1}{\sqrt{n^2+1}} \to 0$, decreasing;

i) divergent, by the n-th term test: $\lim\limits_{n\to\infty} \dfrac{(-1)^n n}{n+1} \neq 0$.

7B-6 In all of these, we are using the ratio test.

a) $\dfrac{|x|^{n+1}}{n+1} \cdot \dfrac{n}{|x|^n} = |x| \cdot \left(\dfrac{n}{n+1}\right) \to |x|$ as $n \to \infty$; converges for $|x| < 1$; $R = 1$

b) $\dfrac{2^{n+1}|x|^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{2^n|x|^n} = 2|x| \cdot \left(\dfrac{n}{n+1}\right)^2 \to 2|x|$ as $n \to \infty$;

converges for $2|x| < 1$ or $|x| < 1/2$; $R = 1/2$

c) $\dfrac{(n+1)!|x|^{n+1}}{n!|x|^n} = (n+1)|x| \to \infty$ as $n \to \infty$; converges only for $|x| = 0$; $R = 0$

d) $\dfrac{|x|^{2(n+1)}}{3^{n+1}} \cdot \dfrac{3^n}{|x|^{2n}} = \dfrac{|x|^2}{3} \to \dfrac{|x|^2}{3}$ as $n \to \infty$; converges for $\dfrac{|x|^2}{3} < 1$,
that is, for $|x| < \sqrt{3}$; $R = \sqrt{3}$

e) $\dfrac{|x|^{2n+3}}{2^{n+1}\sqrt{n+1}} \cdot \dfrac{2^n\sqrt{n}}{|x|^{2n+1}} = \dfrac{|x|^2}{2} \cdot \sqrt{\dfrac{n}{n+1}} \to \dfrac{|x|^2}{2}$ as $n \to \infty$; converges for $\dfrac{|x|^2}{2} < 1$ or $|x| < \sqrt{2}$; $R = \sqrt{2}$

f) $\dfrac{(2n+2)!|x|^{2n+2}}{(n+1)!^2} \cdot \dfrac{n!^2}{(2n)!|x|^{2n}} = |x|^2 \cdot \dfrac{(2n+2)(2n+1)}{(n+1)^2} \to 4|x|^2$ as $n \to \infty$;
converges for $4|x|^2 < 1$, or $|x| < 1/2$; $R = 1/2$

g) $\dfrac{|x|^{n+1}}{\ln(n+1)} \cdot \dfrac{\ln n}{|x|^n} = |x| \cdot \dfrac{\ln n}{\ln(n+1)} \to |x|$ as $n \to \infty$; converges for $|x| < 1$; $R = 1$

(By L'Hospital's rule, $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/x}{1/(x+1)} = 1.$)

h) $\dfrac{2^{2n+2}|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^{2n}|x|^n} = \dfrac{2^2|x|}{n+1} \to 0$ as $n \to \infty$; converges for all x; $R = \infty$

7C: Taylor Approximations and Series

7C-1

(a)

$y = \cos x$	$y' = -\sin x$	$y'' = -\cos x$	$y^{(3)} = \sin x$	$y^{(4)} = \cos x, \ \dots$
$y(0) = 1$	$y'(0) = 0$	$y''(0) = -1$	$y^{(3)}(0) = 0$	$y^{(4)}(0) = 1, \ \dots$
$a_0 = 1$	$a_1 = 0$	$a_2 = -1/2!$	$a_3 = 0$	$a_4 = 1/4! \ \dots$

The pattern then repeats with the higher coefficients, so we get finally

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

(b)

$y = \ln(1+x)$	$y' = (1+x)^{-1}$	$y'' = -(1+x)^{-2}$	$y^{(3)} = 2!(1+x)^{-3}$	$y^{(4)} = -3!(1+x)^{-4}, \ \dots$
$y(0) = 0$	$y'(0) = 1$	$y''(0) = -1$	$y^{(3)}(0) = 2!$	$y^{(4)}(0) = -3!, \ \dots$
$a_0 = 0$	$a_1 = 1$	$a_2 = -1/2$	$a_3 = 1/3$	$a_4 = -1/4 \ \dots$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}x^n}{n} + \dots$$

(c) Typical terms in the calculation are given.

$$y = (1+x)^{1/2} \qquad y'' = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(1+x)^{-3/2} \qquad y^{(4)} = \frac{(-1)(-3)(-5)}{2^4}(1+x)^{-7/2}$$

$$y(0) = 1 \qquad y''(0) = \frac{-1}{2^2} \qquad y^{(4)}(0) = \frac{(-1)^3(1\cdot 3\cdot 5)}{2^4}$$

$$a_0 = 1 \qquad a_2 = -1/8 \qquad a_4 = -\frac{1\cdot 3\cdot 5}{2^4 4!}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^n + \ldots$$

One gets the same answer by using the binomial formula; this is the way to remember the series:

$$(1+x)^{1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \ldots$$

7C-2 $\sin x \;=\; x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + R_6(x).$

(We could use either $R_5(x)$ or $R_6(x)$, since the above polynomial is both $T_5(x)$ and $T_6(x)$, but $R_6(x)$ gives a smaller error estimation if $|x| < 1$, since it contains a higher power of x.)

$$R_6(1) \;=\; \frac{\sin^{(7)} c}{7!}\cdot 1^7 \;=\; \frac{-\cos c}{7!}, \text{ for some } 0 < c < 1. \text{ Therefore}$$

$$|R_6(1) \;\le\; \frac{1}{7!} \;=\; \frac{1}{5040} \;<\; .0002$$

Thus $\sin 1 \;\approx\; 1 - \dfrac{1}{3!} + \dfrac{1}{5!} \;\approx\;$ .84166; the true value is $\sin 1 = .84147$, which is within the error predicted by the Taylor remainder.

7C-3 Since $f(x) = e^x$, the n-th remainder term is given by

$$R_n(1) \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}\cdot 1^{n+1} \;=\; \frac{e^c}{(n+1)!} \;<\; \frac{3}{(n+1)!} \;<\; \frac{5}{10^5} \quad \text{if } n+1 = 8.$$

Therefore we want $n = 7$, i.e., we should use the Taylor polynomial of degree 7; calculation gives $e \approx 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 + 1/720 + 1/5040 = 2.71825\ldots$, which is indeed correct to 3 decimal places.

7C-4 Using as in 7C-2 the remainder $R_3(x)$, rather than $R_2(x)$, we have

$$|R_3(x)| \;=\; \left|\frac{\cos^{(4)}(c)}{4!}x^4\right| \;=\; \left|\frac{\cos c}{4!}x^4\right| \le \frac{|x|^4}{4!} \;\le\; \frac{(.5)^4}{24} \;=\; .0026.$$

So the answer is no, if $|x| < .5$. (If the interval is shrunk to $|x| < .3$, the answer will be yes, since $(.3)^4/24 < .001$.)

7C-5 By Taylor's formula for e^x, substituting $-x^2$ for x,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \frac{e^c(-x^2)^3}{3!}, \quad 0 < c < .5$$

Since $0 < e^c < 2$, the remainder term is $< \dfrac{x^6}{3}$; integrating,

$$\int_0^{.5} e^{-x^2} dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{.5} + \text{ error } = .461 + \text{error};$$

where $|\text{error}| < \displaystyle\int_0^{.5} \frac{x^6}{3} = \frac{x^7}{21}\Big]_0^{.5} = .00028 < .0003$; thus the answer .461 is good to 3 decimal places.

7D: Power Series

7D-1

(a) $$e^{-2x} = 1 - 2x + \frac{2^2}{2!}x^2 + \ldots + (-1)^n \frac{2^n}{n!}x^n + \ldots,$$

by substituting $-2x$ for x in the series for e^x.

(b) $$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^n x^n}{(2n)!} + \ldots$$

(c) $$\begin{aligned}\sin^2 x &= \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left(1 - \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right]\right) \\ &= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \ldots + \frac{(-1)^{n-1}(2x)^{2n}}{(2n)!} + \ldots\right)\end{aligned}$$

(d) Write the series for $1/(1+x)$, differentiate and multiply both sides by -1:

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots + (-1)^{n+1}x^{n+1} + \ldots \\ \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 + \ldots + (-1)^n(n+1)x^n + \ldots\end{aligned}$$

(e) $$D\tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots,$$

by substituting x^2 for x in the series for $1/(1+x)$; (cf. (d) above). Now integrate both sides of the above equation:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + \ldots + C;$$

Evaluate the constant of integration by putting $x = 0$, one gets $0 = 0 + C$, so $C = 0$.

(f)
$$D\ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^{n+1}x^{n+1} + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^n x^{n+1}}{n+1} + \ldots + C,$$

by integrating both sides. Find C by putting $x = 0$, one gets $C = 0$.

(g)
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Adding and dividing by 2 gives: $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

7D-2

a) $$\frac{1}{x+9} = \frac{1/9}{1+x/9} = \frac{1}{9}\left(1 - \frac{x}{9} + \frac{x^2}{9^2} - \frac{x^3}{9^3} + \ldots\right) = \frac{1}{9} - \frac{x}{9^2} + \frac{x^2}{9^3} - \ldots$$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$; substituting $-x^2$ for x gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \ldots + \frac{(-1)^n x^{2n}}{n!} + \ldots$$

c) $$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{x^2}{2} + \ldots\right) = 1 + x + \left(\frac{x^3}{6} - \frac{x^3}{2} + \ldots\right)$$
$$= 1 + x - \frac{x^3}{3} + \ldots \text{ ; the terms in } x^2 \text{ cancel.}$$

d) $$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} + \ldots + \frac{(-1)^n t^{2n}}{(2n+1)!} + \ldots$$
$$\int_0^x \frac{\sin t}{t}\,dt = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot((2n+1)!} + \ldots$$

e) $$e^{-t^2/2} = 1 - \frac{t^2}{2} + \frac{t^4}{2^2\cdot 2!} - \frac{t^6}{2^3\cdot 3!} + \ldots$$
$$\int_0^x e^{-t^2/2}\,dt = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 2^2\cdot 2!} - \frac{x^7}{7\cdot 2^3\cdot 3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 2^n\cdot n!} + \ldots$$

f) $$\frac{1}{x^3-1} = \frac{-1}{1-x^3} = -1 - x^3 - x^6 - \ldots - x^{3n} - \ldots$$

g) $y = \cos^2 x \Rightarrow y' = -2\cos x \sin x = -\sin 2x$; substituting $2x$ into the series for $\sin x$,

$$y' = -2x + \frac{2^3x^3}{3!} - \frac{2^5x^5}{5!} + \ldots ; \quad \text{integrating,}$$
$$y = \cos^2 x = -x^2 + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \ldots + \frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!} + \ldots + C ;$$

Since $y(0) = 1$, we see that $C = 1$, so $\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \ldots$

h) Method 1:
$$\frac{\sin x}{1-x} = (\sin x)\left(\frac{1}{1-x}\right) = \left(x - \frac{x^3}{6} + \ldots\right)(1 + x + x^2 + x^3 + \ldots)$$
$$= x + x^2 + \left(x^3 - \frac{x^3}{6} + \ldots\right) = x + x^2 + \frac{5}{6}x^3 + \ldots$$

Method 2: divide $1-x$ into $x - x^3/6 + \ldots$, as done on the left below:

$$\begin{array}{lllll} & x & + x^2 & + 5x^3/6 & + \ldots \\ 1-x & x & & - x^3/6 & \ldots \\ & x & - x^2 & & \\ & & x^2 & - x^3/6 & + \ldots \\ & & x^2 & - x^3 & \\ & & & 5x^3/6 & + \ldots \end{array} \qquad \begin{array}{llll} & x & + x^3/3 & + \ldots \\ 1 - x^2/2 & x & - x^3/6 & + \ldots \\ & x & - x^3/2 & \\ & & x^3/3 & + \ldots \end{array}$$

i) Method 1: Calculating successive derivatives gives:

$$y = \tan x, \quad y' = \sec^2 x, \quad y'' = 2\sec^2 x \tan x, \quad y^{(3)} = 2(2\sec^2 x \tan x \cdot \tan x + \sec^2 x \cdot \sec^2 x)$$
$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y^{(3)}(0) = 2,$$

so the Taylor series starts
$$\tan x = x + \frac{2x^3}{3!} + \ldots = x + \frac{x^3}{3} + \ldots$$

Method 2: $\tan x = \dfrac{\sin x}{\cos x}$; divide the $\cos x$ series into the $\sin x$ series (done on the right above) — this turns out to be easier here than taking derivatives!

7D-3

a) $$\frac{1-\cos x}{x^2} = \frac{1-(1-x^2/2+\ldots)}{x^2} = \frac{x^2/2+\ldots}{x^2} \to \frac{1}{2} \quad \text{as } x \to 0.$$

b) $$\frac{x - \sin x}{x^3} = \frac{x - (x - x^3/6 + \ldots)}{x^3} = \frac{x^3/6 + \ldots}{x^3} \to \frac{1}{6} \quad \text{as } x \to 0$$

c) $$(1+x)^{1/2} = 1 + x/2 - x^2/8 + \ldots \Rightarrow (1+x)^{1/2} - 1 - x/2 = -x^2/8 + \ldots$$
$$\sin x = x - x^3/6 + \ldots \Rightarrow \sin^2 x = x^2 + \ldots$$

Therefore, $$\frac{(1+x)^{1/2} - 1 - x/2}{\sin^2 x} = \frac{-x^2/8 + \ldots}{x^2 + \ldots} \to \frac{-1}{8} \quad \text{as } x \to 0.$$

d) $$\cos u - 1 = -u^2/2 + \ldots ; \qquad \ln(1+u) - u = -u^2/2 + \ldots;$$

Therefore, $$\frac{\cos u - 1}{\ln(1+u) - u} = \frac{-u^2/2 + \ldots}{-u^2/2 + \ldots} \to 1 \quad \text{as } u \to 0.$$