Unit 5. Integration techniques

5A. Inverse trigonometric functions; Hyperbolic functions

5A-1 a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ b) $\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

c) $\tan\theta = 5$ implies $\sin\theta = 5/\sqrt{26}$, $\cos\theta = 1/\sqrt{26}$, $\cot\theta = 1/5$, $\csc\theta = \sqrt{26}/5$, $\sec\theta = \sqrt{26}$ (from triangle)

d) $\sin^{-1}\cos(\frac{\pi}{6}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ e) $\tan^{-1}\tan(\frac{\pi}{3}) = \frac{\pi}{3}$

f) $\tan^{-1}\tan(\frac{2\pi}{3}) = \tan^{-1}\tan(\frac{-\pi}{3}) = \frac{-\pi}{3}$ g) $\lim_{x\to-\infty}\tan^{-1}x = \frac{-\pi}{2}$.

5A-2 a) $\int_1^2 \frac{dx}{x^2+1} = \tan^{-1}x\Big|_1^2 = \tan^{-1}2 - \frac{\pi}{4}$

b) $\int_b^{2b} \frac{dx}{x^2+b^2} = \int_b^{2b} \frac{d(by)}{(by)^2+b^2}$ (put $x = by$) $= \int_1^2 \frac{dy}{b(y^2+1)} = \frac{1}{b}(\tan^{-1}2 - \frac{\pi}{4})$

c) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_{-1}^1 = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$

5A-3 a) $y = \frac{x-1}{x+1}$, so $1-y^2 = 4x/(x+1)^2$, and $\frac{1}{\sqrt{1-y^2}} = \frac{(x+1)}{2\sqrt{x}}$. Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{(x+1)^2}\\ \frac{d}{dx}\sin^{-1}y &= \frac{dy/dx}{\sqrt{1-y^2}}\\ &= \frac{2}{(x+1)^2}\cdot\frac{(x+1)}{2\sqrt{x}}\\ &= \frac{1}{(x+1)\sqrt{x}}\end{aligned}$$

b) $\text{sech}^2 x = 1/\cosh^2 x = 4/(e^x+e^{-x})^2$

c) $y = x + \sqrt{x^2+1}$, $dy/dx = 1 + x/\sqrt{x^2+1}$.

$$\frac{d}{dx}\ln y = \frac{dy/dx}{y} = \frac{1+x/\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

d) $\cos y = x \implies (-\sin y)(dy/dx) = 1$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$$

e) Chain rule:

$$\frac{d}{dx}\sin^{-1}(x/a) = \frac{1}{\sqrt{1-(x/a)^2}}\cdot\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

f) Chain rule:

$$\frac{d}{dx}\sin^{-1}(a/x) = \frac{1}{\sqrt{1-(a/x)^2}} \cdot \frac{-a}{x^2} = \frac{-a}{x\sqrt{x^2-a^2}}$$

g) $y = x/\sqrt{1-x^2}$, $dy/dx = (1-x^2)^{-3/2}$, $1+y^2 = 1/(1-x^2)$. Thus

$$\frac{d}{dx}\tan^{-1} y = \frac{dy/dx}{1+y^2} = (1-x^2)^{-3/2}(1-x^2) = \frac{1}{\sqrt{1-x^2}}$$

Why is this the same as the derivative of $\sin^{-1} x$?

h) $y = \sqrt{1-x}$, $dy/dx = -1/2\sqrt{1-x}$, $1-y^2 = x$. Thus,

$$\frac{d}{dx}\sin^{-1} y = \frac{dy/dx}{\sqrt{1-y^2}} = \frac{-1}{2\sqrt{x(1-x)}}$$

5A-4 a) $y' = \sinh x$. A tangent line through the origin has the equation $y = mx$. If it meets the graph at $x = a$, then $ma = \cosh(a)$ and $m = \sinh(a)$. Therefore, $a\sinh(a) = \cosh(a)$.

b) Take the difference:

$$F(a) = a\sinh(a) - \cosh(a)$$

Newton's method for finding $F(a) = 0$, is the iteration

$$a_{n+1} = a_n - F(a_n)/F'(a_n) = a_n - \tanh(a_n) + 1/a_n$$

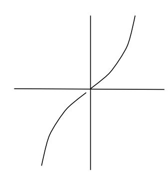
With $a_1 = 1$, $a_2 = 1.2384$, $a_3 = 1.2009$, $a_4 = 1.19968$. A serviceable approximation is

$$a \approx 1.2$$

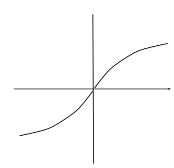
(The slope is $m = \sinh(a) \approx 1.5$.) The functions F and y are even. By symmetry, there is another solution $-a$ with slope $-\sinh a$.

5A-5 a)

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$
$$y' = \cosh x = \frac{e^x + e^{-x}}{2}$$
$$y'' = \sinh x$$



y = sinh x



y = sinh^{-1}x

y' is never zero, so no critical points. Inflection point $x = 0$; slope of y is 1 there. y is an odd function, like $e^x/2$ for $x >> 0$.

b) $y = \sinh^{-1} x \iff x = \sinh y$. Domain is the whole x-axis.

c) Differentiate $x = \sinh y$ implicitly with respect to x:

$$1 = \cosh y \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$
$$\frac{d\sinh^{-1} x}{dx} = \frac{1}{\sqrt{x^2+1}}$$

d)

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{dx}{a\sqrt{x^2+a^2/a^2}}\\ &= \int \frac{d(x/a)}{\sqrt{(x/a)^2+1}}\\ &= \sinh^{-1}(x/a)+c\end{aligned}$$

5A-6 a) $\frac{1}{\pi}\int_0^{\pi} \sin\theta d\theta = 2/\pi$

b) $y=\sqrt{1-x^2} \implies y'=-x/\sqrt{1-x^2} \implies 1+(y')^2 = 1/(1-x^2)$. Thus

$$ds = w(x)dx = dx/\sqrt{1-x^2}.$$

Therefore the average is

$$\int_{-1}^{1} \sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} \Big/ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

The numerator is $\int_{-1}^{1} dx = 2$. To see that these integrals are the same as the ones in part (a), take $x=\cos\theta$ (as in polar coordinates). Then $dx = -\sin\theta d\theta$ and the limits of integral are from $\theta=\pi$ to $\theta=0$. Reversing the limits changes the minus back to plus:

$$\begin{aligned}\int_{-1}^{1} \sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} &= \int_0^{\pi} \sin\theta d\theta\\ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} &= \int_0^{\pi} d\theta = \pi\end{aligned}$$

(The substitution $x=\sin t$ works similarly, but the limits of integration are $-\pi/2$ and $\pi/2$.)

c) ($x=\sin t$, $dx=\cos t dt$)

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2 t dt &= \int_0^{\pi/2}\cos^2 t dt\\ &= \int_0^{\pi/2}\frac{1+\cos 2t}{2}dt\\ &= \pi/4\end{aligned}$$

5B. Integration by direct substitution

Do these by guessing and correcting the factor out front. The substitution used implicitly is given alongside the answer.

5B-1 $\int x\sqrt{x^2-1}dx = \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c$ ($u=x^2-1$, $du=2xdx$)

5B-2 $\int e^{8x}dx = \frac{1}{8}e^{8x} + c \; (u = 8x,\; du = 8dx)$

5B-3 $\int \frac{\ln x dx}{x} = \frac{1}{2}(\ln x)^2 + c \; (u = \ln x,\; du = dx/x)$

5B-4 $\int \frac{\cos x dx}{2 + 3\sin x} = \frac{\ln(2 + 3\sin x)}{3} + c \; (u = 2 + 3\sin x,\; du = 3\cos x dx)$

5B-5 $\int \sin^2 x \cos x dx = \frac{\sin x^3}{3} + c \; (u = \sin x,\; du = \cos x dx)$

5B-6 $\int \sin 7x dx = \frac{-\cos 7x}{7} + c \; (u = 7x,\; du = 7dx)$

5B-7 $\int \frac{6x dx}{\sqrt{x^2 + 4}} = 6\sqrt{x^2 + 4} + c \; (u = x^2 + 4,\; du = 2x dx)$

5B-8 Use $u = \cos(4x)$, $du = -4\sin(4x)dx$,

$$\int \tan 4x dx = \int \frac{\sin(4x)dx}{\cos(4x)} = \int \frac{-du}{4u}$$
$$= -\frac{\ln u}{4} + c = -\frac{\ln(\cos 4x)}{4} + c$$

5B-9 $\int e^x(1 + e^x)^{-1/3}dx = \frac{3}{2}(1 + e^x)^{2/3} + c \; (u = 1 + e^x,\; du = e^x dx)$

5B-10 $\int \sec 9x dx = \frac{1}{9}\ln(\sec(9x) + \tan(9x)) + c \; (u = 9x,\; du = 9dx)$

5B-11 $\int \sec^2 9x dx = \frac{\tan 9x}{9} + c \; (u = 9x,\; du = 9dx)$

5B-12 $\int x e^{-x^2} dx = \frac{-e^{-x^2}}{2} + c \; (u = x^2,\; du = 2x dx)$

5B-13 $u = x^3$, $du = 3x^2 dx$ implies

$$\int \frac{x^2 dx}{1 + x^6} = \int \frac{du}{3(1 + u^2)} = \frac{\tan^{-1} u}{3} + c$$
$$= \frac{\tan^{-1}(x^3)}{3} + c$$

5B-14 $\int_0^{\pi/3} \sin^3 x \cos x dx = \int_{\sin 0}^{\sin \pi/3} u^3 du \; (u = \sin x,\; du = \cos x dx)$

$= \int_0^{\sqrt{3}/2} u^3 du = u^4/4 \Big|_0^{\sqrt{3}/2} = \frac{9}{64}$

5B-15 $\int_1^e \frac{(\ln x)^{3/2} dx}{x} = \int_{\ln 1}^{\ln e} u^{3/2} du \; (u = \ln x,\; du = dx/x)$

$$= \int_0^1 y^{3/2}dy = (2/5)y^{5/2}\Big|_0^1 = \frac{2}{5}$$

5B-16 $\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x dx}{1+x^2} = \int_{\tan^{-1}(-1)}^{\tan^{-1} 1} udu$ $(u = \tan^{-1} x,\ du = dx/(1+x^2))$

$$= \int_{-\pi/4}^{\pi/4} udu = \frac{u^2}{2}\Big|_{-\pi/4}^{\pi/4} = 0$$

($\tan x$ is odd and hence $\tan^{-1} x$ is also odd, so the integral had better be 0)

5C. Trigonometric integrals

5C-1 $\displaystyle\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

5C-2 $\displaystyle\int \sin^3(x/2)dx = \int (1-\cos^2(x/2))\sin(x/2)dx = \int -2(1-u^2)du$
(put $u = \cos(x/2)$, $du = (-1/2)\sin(x/2)dx$)

$$= -2u + \frac{2u^3}{3} + c = -2\cos(x/2) + \frac{2\cos(x/2)^3}{3} + c$$

5C-3 $\displaystyle\int \sin^4 x dx = \int (\frac{1-\cos 2x}{2})^2 dx = \int \frac{1 - 2\cos 2x + \cos^2 2x}{4}dx$

$$\int \frac{\cos^2(2x)}{4}dx = \int \frac{1+\cos 4x}{8}dx = \frac{x}{8} + \frac{\sin 4x}{32} + c$$

Adding together all terms:

$$\int \sin^4 x dx = \frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$$

5C-4 $\displaystyle\int \cos^3(3x)dx = \int (1-\sin^2(3x))\cos(3x)dx = \int \frac{1-u^2}{3}du$ $(u = \sin(3x),\ du = 3\cos(3x)dx)$

$$= \frac{u}{3} - \frac{u^3}{9} + c = \frac{\sin(3x)}{3} - \frac{\sin(3x)^3}{9} + c$$

5C-5 $\displaystyle\int \sin^3 x\cos^2 x dx = \int (1-\cos^2 x)\cos^2 x \sin x dx = \int -(1-u^2)u^2 dy$ $(u = \cos x,\ du = -\sin x dx)$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos x^3}{3} + \frac{\cos x^5}{5} + c$$

5C-6 $\displaystyle\int \sec^4 x dx = \int (1+\tan^2 x)\sec^2 x dx = \int (1+u^2)du$ $(u = \tan x,\ du = \sec^2 x dx)$

$$= u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

5C-7 $\displaystyle\int \sin^2(4x)\cos^2(4x)dx = \int \frac{\sin^2 8x dx}{4} = \int \frac{(1-\cos 16x)dx}{8} = \frac{1}{8} - \frac{\sin 16x}{128} + c$

A slower way is to use

$$\sin^2(4x)\cos^2(4x) = \left(\frac{1-\cos(8x)}{2}\right)\left(\frac{1+\cos(8x)}{2}\right)$$

multiply out and use a similar trick to handle $\cos^2(8x)$.

5C-8

$$\begin{aligned}\int \tan^2(ax)\cos(ax)dx &= \int \frac{\sin^2(ax)}{\cos(ax)}dx\\ &= \int \frac{1-\cos^2(ax)}{\cos(ax)}dx\\ &= \int (\sec(ax)-\cos(ax))dx\\ &= \frac{1}{a}\ln(\sec(ax)+\tan(ax)) - \frac{1}{a}\sin(ax)+c\end{aligned}$$

5C-9

$$\begin{aligned}\int \sin^3 x\sec^2 x dx &= \int \frac{1-\cos^2 x}{\cos^2 x}\sin x dx\\ &= \int -\frac{1-u^2}{u^2}du \qquad (u=\cos x,\ du=-\sin x dx)\\ &= u+\frac{1}{u}+c = \cos x+\sec x+c\end{aligned}$$

5C-10

$$\begin{aligned}\int (\tan x+\cot x)^2 dx = \int \tan^2 x + 2 + \cot^2 x dx &= \int \sec^2 x + \csc^2 x dx\\ = \tan x - \cot x + c\end{aligned}$$

5C-11 $\int \sin x\cos(2x)dx$

$$= \int \sin x(2\cos^2 x-1)dx = \int (1-2u^2)du \ (u=\cos x,\ du - \sin x dx)$$

$$= u - \frac{2}{3}u^3 + c = \cos x - \frac{2}{3}\cos^3 x + c$$

5C-12 $\int_0^\pi \sin x\cos(2x)dx = \cos x - \frac{2}{3}\cos^3 x\Big|_0^\pi = \frac{-2}{3}$ (See 27.)

5C-13 $ds = \sqrt{1+(y')^2}dx = \sqrt{1+\cot^2 x}dx = \csc x dx.$

$$\text{arclength} = \int_{\pi/4}^{\pi/2} \csc x dx = -\ln(\csc x+\cot x)\Big|_{\pi/4}^{\pi/2} = \ln(1+\sqrt{2})$$

5C-14 $\int_0^{\pi/a} \pi\sin^2(ax)dx = \pi\int_0^{\pi/a}(1/2)(1-\cos(2ax))dx = \pi^2/2a$

5D. Integration by inverse substitution

5D-1 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{1}{a^2}\int \sec^2\theta d\theta = \frac{1}{a^2}\tan\theta + c = \frac{x}{a^2\sqrt{a^2-x^2}} + c$$

5D-2 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\begin{aligned}\int \frac{x^3dx}{\sqrt{a^2-x^2}} &= a^3\int \sin^3\theta d\theta = a^3\int (1-\cos^2\theta)\sin\theta d\theta \\ &= a^3(-\cos\theta + (1/3)\cos^3\theta) + c \\ &= -a^2\sqrt{a^2-x^2} + (a^2-x^2)^{3/2}/3 + c\end{aligned}$$

5D-3 By direct substitution ($u = 4 + x^2$),

$$\int \frac{xdx}{4+x^2} = (1/2)\ln(4+x^2) + c$$

Put $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$,

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\int d\theta = \theta/2 + c$$

In all,

$$\int \frac{(x+1)dx}{4+x^2} = (1/2)\ln(4+x^2) + (1/2)\tan^{-1}(x/2) + c$$

5D-4 Put $x = a\sinh y$, $dx = a\cosh y dy$. Since $1 + \sinh^2 y = \cosh^2 y$,

$$\begin{aligned}\int \sqrt{a^2+x^2}dx = a^2\int \cosh^2 y dy &= \frac{a^2}{2}\int (\cosh(2y)-1)dy \\ &= (a^2/4)\sinh(2y) - a^2y/2 + c = (a^2/2)\sinh y\cosh y - a^2y/2 + c \\ &= x\sqrt{a^2+x^2}/2 - a^2\sinh^{-1}(x/a) + c\end{aligned}$$

5D-5 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\begin{aligned}\int \frac{\sqrt{a^2-x^2}dx}{x^2} &= \int \cot^2\theta d\theta \\ &= \int (\csc^2\theta - 1)d\theta = -\ln(\csc\theta + \cot\theta) - \theta + c \\ &= -\ln(a/x + \sqrt{a^2-x^2}/x) - \sin^{-1}(x/a) + c\end{aligned}$$

5D-6 Put $x = a\sinh y$, $dx = a\cosh y dy$.

$$\begin{aligned}\int x^2\sqrt{a^2+x^2}dx &= a^4\int \sinh^2 y\cosh^2 y dy \\ &= (a^4/2)\int \sinh^2(2y)dy = a^4/4\int (\cosh(4y)-1)dy \\ &= (a^4/16)\sinh(4y) - a^4y/4 + c \\ &= (a^4/8)\sinh(2y)\cosh(2y) - a^4y/4 + c \\ &= (a^4/4)\sinh y\cosh y(\cosh^2 y + \sinh^2 y) - a^4y/4 + c \\ &= (1/4)x\sqrt{a^2+x^2}(2x^2+a^2) - (a^4/4)\sinh^{-1}(x/a) + c\end{aligned}$$

5D-7 Put $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$:

$$\begin{aligned}
\int \frac{\sqrt{x^2-a^2}dx}{x^2} &= \int \frac{\tan^2\theta d\theta}{\sec\theta} \\
&= \int \frac{(\sec^2\theta - 1)d\theta}{\sec\theta} = \int (\sec\theta - \cos\theta)d\theta \\
&= \ln(\sec\theta + \tan\theta) - \sin\theta + c \\
&= \ln(x/a + \sqrt{x^2-a^2}/a) - \sqrt{x^2-a^2}/x + c \\
&= \ln(x + \sqrt{x^2-a^2}) - \sqrt{x^2-a^2}/x + c_1 \quad (c_1 = c - \ln a)
\end{aligned}$$

5D-8 Short way: $u = x^2 - 9$, $du = 2xdx$,

$$\int x\sqrt{x^2-9}dx = (1/3)(x^2-9)^{3/2} + c \qquad \text{direct substitution}$$

Long way (method of this section): Put $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta d\theta$.

$$\begin{aligned}
\int x\sqrt{x^2-9}dx &= 27\int \sec^2\theta\tan^2\theta d\theta \\
&= 27\int \tan^2\theta d(\tan\theta) = 9\tan^3\theta + c \\
&= (1/3)(x^2-9)^{3/2} + c
\end{aligned}$$

($\tan\theta = \sqrt{x^2-9}/3$). The trig substitution method does not lead to a dead end, but it's not always fastest.

5D-9 $y' = 1/x$, $ds = \sqrt{1+1/x^2}dx$, so

$$\text{arclength} = \int_1^b \sqrt{1+1/x^2}dx$$

Put $x = \tan\theta$, $dx = \sec^2\theta d\theta$,

$$\begin{aligned}
\int \frac{\sqrt{x^2+1}dx}{x} &= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta \\
&= \int \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}d\theta \\
&= \int (\csc\theta + \sec\theta\tan\theta)d\theta \\
&= -\ln(\csc\theta + \cot\theta) + \sec\theta + c \\
&= -\ln(\sqrt{x^2+1}/x + 1/x) + \sqrt{x^2+1} + c \\
&= -\ln(\sqrt{x^2+1} + 1) + \ln x + \sqrt{x^2+1} + c
\end{aligned}$$

$$\text{arclength} = -\ln(\sqrt{b^2+1}+1) + \ln b + \sqrt{b^2+1} + \ln(\sqrt{2}+1) - \sqrt{2}$$

Completing the square

5D-10 $\displaystyle\int \frac{dx}{(x^2+4x+13)^{3/2}} = \int \frac{dx}{((x+2)^2+3^2)^{3/2}} \quad (x+2 = 3\tan\theta,\ dx = 3\sec^2\theta d\theta)$

$$= \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + c = \frac{(x+2)}{9\sqrt{x^2+4x+13}} + c$$

5D-11

$$\begin{aligned}
\int x\sqrt{-8+6x-x^2}dx &= \int x\sqrt{1-(x-3)^2}dx \quad (x-3=\sin\theta,\ dx = \cos\theta d\theta)\\
&= \int (\sin\theta+3)\cos^2\theta d\theta\\
&= (-1/3)\cos^3\theta + (3/2)\int(\cos 2\theta + 1)d\theta\\
&= -(1/3)\cos^3\theta + (3/4)\sin 2\theta + (3/2)\theta + c\\
&= -(1/3)\cos^3\theta + (3/2)\sin\theta\cos\theta + (3/2)\theta + c\\
&= -(1/3)(-8+6x-x^2)^{3/2}\\
&\quad + (3/2)(x-3)\sqrt{-8+6x-x^2} + (3/2)\sin^{-1}(x-3) + c
\end{aligned}$$

5D-12

$$\begin{aligned}
\int \sqrt{-8+6x-x^2}dx &= \int \sqrt{1-(x-3)^2}dx \quad (x-3=\sin\theta,\ dx = \cos\theta d\theta)\\
&= \int \cos^2\theta d\theta\\
&= \frac{1}{2}\int(\cos 2\theta+1)d\theta\\
&= \frac{1}{4}\sin 2\theta + \frac{\theta}{2} + c\\
&= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c\\
&= \frac{(x-3)\sqrt{-8+6x-x^2}}{2} + \frac{\sin^{-1}(x-3)}{2} + c
\end{aligned}$$

5D-13 $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$. Put $x-1=\sin\theta,\ dx = \cos\theta d\theta$.

$$= \int d\theta = \theta + c = \sin^{-1}(x-1) + c$$

5D-14 $\displaystyle\int \frac{xdx}{\sqrt{x^2+4x+13}} = \int \frac{xdx}{\sqrt{(x+2)^2+3^2}}$. Put $x+2 = 3\tan\theta,\ dx = 3\sec^2\theta$.

$$\begin{aligned}
&= \int(3\tan\theta - 2)\sec\theta d\theta = 3\sec\theta - 2\ln(\sec\theta+\tan\theta) + c\\
&= \sqrt{x^2+4x+13} - 2\ln(\sqrt{x^2+4x+13}/3 + (x+2)/3) + c\\
&= \sqrt{x^2+4x+13} - 2\ln(\sqrt{x^2+4x+13} + (x+2)) + c_1 \quad (c_1 = c - \ln 3)
\end{aligned}$$

5D-15 $\displaystyle\int \frac{\sqrt{4x^2-4x+17}dx}{2x-1} = \int \frac{\sqrt{(2x-1)^2+4^2}dx}{2x-1}$
(put $2x-1 = 4\tan\theta$, $dx = 2\sec^2\theta d\theta$ as in Problem 9)

$$
\begin{aligned}
&= 2\int \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta \\
&= 2\int \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}d\theta \\
&= 2\int (\csc\theta + \sec\theta\tan\theta)d\theta \\
&= -2\ln(\csc\theta+\cot\theta) + 2\sec\theta + c \\
&= -2\ln(\sqrt{4x^2-4x+17}/(2x-1) + 4/(2x-1)) + \sqrt{4x^2-4x+17}/2 + c \\
&= -2\ln(\sqrt{4x^2-4x+17}+4) + 2\ln(2x-1) + \sqrt{4x^2-4x+17}/2 + c
\end{aligned}
$$

5E. Integration by partial fractions

5E-1 $\displaystyle\frac{1}{(x-2)(x+3)} = \frac{1/5}{x-2} + \frac{-1/5}{x+3}$ (cover up)

$$\int \frac{dx}{(x-2)(x+3)} = (1/5)\ln(x-2) - (1/5)\ln(x+3) + c$$

5E-2 $\displaystyle\frac{x}{(x-2)(x+3)} = \frac{2/5}{x-2} + \frac{3/5}{x+3}$ (cover up)

$$\int \frac{xdx}{(x-2)(x+3)} = (2/5)\ln(x-2) + (3/5)\ln(x+3) + c$$

5E-3 $\displaystyle\frac{x}{(x-2)(x+2)(x+3)} = \frac{1/10}{x-2} + \frac{1/2}{x+2} + \frac{-3/5}{x+3}$ (cover up)

$$\int \frac{xdx}{(x^2-4)(x+3)} = (1/10)\ln(x-2) + (1/2)\ln(x+2) - (3/5)\ln(x+3)$$

5E-4 $\displaystyle\frac{3x^2+4x-11}{(x^2-1)(x-2)}dx = \frac{2}{x-1} + \frac{-2}{x+1} + \frac{3}{x-2}$ (cover-up)

$$\int \frac{2dx}{x-1} + \frac{-2dx}{x+1} + \frac{3dx}{x-2} = 2\ln(x-1) - 2\ln(x+1) + 3\ln(x-2) + c$$

.

5E-5 $\dfrac{3x+2}{x(x+1)^2} = \dfrac{2}{x} + \dfrac{B}{x+1} + \dfrac{1}{(x+1)^2}$ (coverup); to get B, put say $x = 1$:

$$\frac{5}{4} = 2 + \frac{B}{2} + \frac{1}{4} \implies B = -2$$

$$\int \frac{3x+2}{x(x+1)^2}dx = 2\ln x - 2\ln(x+1) - \frac{1}{x+1} + c$$

5E-6 $\dfrac{2x-9}{(x^2+9)(x+2)} = \dfrac{Ax+B}{x^2+9} + \dfrac{C}{x+2}$
By cover-up, $C = -1$. To get B and A,

$$x = 0 \implies \frac{-9}{9\cdot 2} = \frac{B}{9} - \frac{1}{2} \implies B = 0$$

$$x = 1 \implies \frac{-7}{10\cdot 3} = \frac{A}{10} - \frac{1}{3} \implies A = 1$$

$$\int \frac{2x-9}{(x^2+9)(x+2)}dx = \frac{1}{2}\ln(x^2+9) - \ln(x+2) + c$$

5E-7 Instead of thinking of (4) as arising from (1) by multiplication by $x-1$, think of it as arising from

$$x - 7 = A(x+2) + B(x-1)$$

by division by $x+2$; since this new equation is valid for all x, the line (4) will be valid for $x \neq -2$, in particular it will be valid for $x = 1$.

5E-8 Long division:

a) $\dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$

b) $\dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

c) $\dfrac{x^2}{3x-1} = x/3 + 1/9 + \dfrac{1/9}{3x-1}$

d) $\dfrac{x+2}{3x-1} = \dfrac{1}{3} + \dfrac{7/3}{3x-1}$

e) $\dfrac{x^8}{(x+2)^2(x-2)^2} = A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 + \dfrac{B_3x^3 + B_2x^2 + B_1x + B_0}{(x+2)^2(x-2)^2}$

5E-9 a) Cover-up gives

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

From 8a,

$$\frac{x^2}{x^2-1} = 1 + \frac{1/2}{x-1} + \frac{-1/2}{x+1} \quad \text{and}$$

$$\int \frac{x^2dx}{x^2-1} = x + (1/2)\ln(x-1) - (1/2)\ln(x+1) + c$$

b) Cover-up gives

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{1/2}{x+1}$$

From 8b,

$$\frac{x^3}{x^2-1} = x + \frac{1/2}{x-1} + \frac{1/2}{x+1} \quad \text{and}$$

$$\int \frac{x^3 dx}{x^2-1} = x^2/2 + (1/2)\ln(x-1) + (1/2)\ln(x+1) + c$$

c) From 8c,

$$\int \frac{x^2}{3x-1} dx = x^2/6 + x/9 + (1/27)\ln(3x-1) + c$$

d) From 8d,

$$\int \frac{x+2}{3x-1} dx = x/3 + (7/9)\ln(3x-1)$$

e) Cover-up says that the proper rational function will be written as

$$\frac{a_1}{x-2} + \frac{a_2}{(x-2)^2} + \frac{b_1}{x+2} + \frac{b_2}{(x+2)^2}$$

where the coefficients a_2 and b_2 can be evaluted from the B's using cover-up and the coefficients a_1 and b_1 can then be evaluated using $x = 0$ and $x = 1$, say. Therefore, the integral has the form

$$A_4x^5/5 + A_3x^4/4 + A_2x^3/3 + A_1x^2/2 + A_0x + c$$
$$+ a_1 \ln(x-2) - \frac{a_2}{x-2} + b_1 \ln(x+2) - \frac{b_2}{x+2}$$

5E-10 a) By cover-up,

$$\frac{1}{x^3-x} = \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$$

$$\int \frac{dx}{x^3-x} = -\ln x + \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + c$$

b) By cover-up, $\dfrac{(x+1)}{(x-2)(x-3)} = \dfrac{-3}{x-2} + \dfrac{4}{x-3}$. Therefore,

$$\int \frac{(x+1)}{(x-2)(x-3)} dx = -3\ln(x-2) + 4\ln(x-3) + c$$

c) $\dfrac{(x^2+x+1)}{x^2+8x} = 1 + \dfrac{-7x+1}{x^2+8x}$. By cover-up,

$$\frac{-7x+1}{x^2+8x} = \frac{-7x+1}{x(x+8)} = \frac{1/8}{x} + \frac{-57/8}{x+8} \quad \text{and}$$

$$\int \frac{(x^2+x+1)}{x^2+8x} = x + (1/8)\ln x - (57/8)\ln(x+8) + c$$

d) Seeing double? It must be late.

e) $\dfrac{1}{x^3+x^2} = \dfrac{1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

Use the cover-up method to get $B = 1$ and $C = 1$. For A,

$$x = 1 \implies \frac{1}{2} = A + 1 + \frac{1}{2} \implies A = -1$$

In all,

$$\int \frac{dx}{x^3+x^2} = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx = -\ln x + \ln(x+1) - \frac{1}{x} + c$$

f) $\dfrac{x^2+1}{x^3+2x^2+x} = \dfrac{x^2+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

By cover-up, $A = 1$ and $C = -2$. For B,

$$x = 1 \implies \frac{2}{4} = 1 + \frac{B}{2} - \frac{2}{4} \implies B = 0 \quad \text{and}$$

$$\int \frac{x^2+1}{x^3+2x^2+x} dx = \int \left(\frac{1}{x} - \frac{2}{(x+1)^2}\right) dx = \ln x + \frac{2}{x+1} + c$$

g) Multiply out denominator: $(x+1)^2(x-1) = x^3+x^2-x-1$. Divide into numerator:

$$\frac{x^3}{x^3+x^2-x-1} = 1 + \frac{-x^2+x+1}{x^3+x^2-x-1}$$

Write the proper rational function as

$$\frac{-x^2+x+1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

By cover-up, $B = 1/2$ and $C = 1/4$. For A,

$$x = 0 \implies -1 = A + \frac{1}{2} - \frac{1}{4} \implies A = -\frac{5}{4} \quad \text{and}$$

$$\int \frac{x^3}{(x+1)^2(x-1)} dx = \int \left(1 + \frac{-5/4}{x+1} + \frac{1/2}{(x+1)^2} + \frac{1/4}{x-1}\right) dx$$
$$= x - (5/4)\ln(x+1) - \frac{1}{2(x+1)} + (1/4)\ln(x-1) + c$$

h) $\displaystyle\int \frac{(x^2+1)dx}{x^2+2x+2} = \int (1 - \frac{1+2x}{x^2+2x+2}) dx = x - \int \frac{(2y-1)dy}{y^2+1}$ (put $y = x+1$)

$$= x - \ln(y^2+1) + \tan^{-1} y + c$$
$$= x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + c$$

5E-11 Separate:

$$\frac{dy}{y(1-y)} = dx$$

Expand using partial fractions and integrate

$$\int (\frac{1}{y} - \frac{1}{y-1})dy = \int dx$$

Hence,

$$\ln y - \ln(y-1) = x + c$$

Exponentiate:

$$\frac{y}{y-1} = e^{x+c} = Ae^x \quad (A = e^c)$$

$$y = \frac{Ae^x}{Ae^x - 1}$$

(If you integrated $1/(1-y)$ to get $-\ln(1-y)$ then you arrive at

$$y = \frac{Ae^x}{Ae^x + 1}$$

This is the same family of answers with A and $-A$ traded.)

5E-12 a) $1 + z^2 = 1 + \tan^2(\theta/2) = \sec^2(\theta/2)$. Therefore,

$$\cos^2(\theta/2) = \frac{1}{1+z^2} \quad \text{and} \quad \sin^2(\theta/2) = 1 - \frac{1}{1+z^2} = \frac{z^2}{1+z^2}$$

Next,

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-z^2}{1+z^2} \quad \text{and}$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2\sqrt{\frac{1}{1+z^2}}\sqrt{\frac{z^2}{1+z^2}} = \frac{2z}{1+z^2}$$

Finally,

$$dz = (1/2)\sec^2(\theta/2)d\theta = (1/2)(1+z^2)d\theta \implies d\theta = \frac{2dz}{1+z^2}$$

b)

$$\begin{aligned}
\int_0^\pi \frac{d\theta}{1+\sin\theta} &= \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{1+2z/(1+z^2)} \\
&= \int_0^\infty \frac{2dz}{z^2+1+2z} = \int_0^\infty \frac{2dz}{(z+1)^2} \\
&= \left.\frac{-2}{1+z}\right|_0^\infty = 2
\end{aligned}$$

c)

$$\int_0^\pi \frac{d\theta}{(1+\sin\theta)^2} = \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{(1+2z/(1+z^2))^2} = \int_0^\infty \frac{2(1+z^2)dz}{(1+z)^4}$$
$$= \int_1^\infty \frac{2(1+(y-1)^2)dy}{y^4} \qquad \text{(put } y = z+1\text{)}$$
$$= \int_1^\infty \frac{(2y^2-4y+4)dy}{y^4} = \int_1^\infty (2y^{-2} - 4y^{-3} + 4y^{-4})dy$$
$$= -2y^{-1} + 2y^{-2} - (4/3)y^{-3}\Big|_1^\infty = 4/3$$

(d)

$$\int_0^\pi \sin\theta d\theta = \int_0^\infty \frac{2z}{1+z^2}\frac{2dz}{1+z^2} = \int_0^\infty \frac{4zdz}{(1+z^2)^2}$$
$$= \frac{-2}{1+z^2}\Big|_0^\infty = 2$$

5E-13 a) $z = \tan(\theta/2) \implies 1+\cos\theta = 2/(1+z^2)$ and $0 \le \theta \le \pi/2$ corresponds to $0 \le z \le 1$.

$$A = \int_0^{\pi/2} \frac{d\theta}{2(1+\cos\theta)^2} = \int_0^1 \frac{2dz/(1+z^2)}{8/(1+z^2)^2}$$
$$= \int_0^1 (1/4)(1+z^2)dz = (1/4)(z+z^3/3)\Big|_0^1 = 1/3$$

b) The curve $r = 1/(1+\cos\theta)$ is a parabola:

$$r + r\cos\theta = 1 \implies r + x = 1 \implies r^2 = (1-x)^2 \implies y^2 = 1-2x$$

This is the region under $y = \sqrt{1-2x}$ in the first quadrant:

$$A = \int_0^{1/2} \sqrt{1-2x}dx = -(1/3)(1-2x)^{3/2}\Big|_0^{1/2} = 1/3$$

5F. Integration by parts. Reduction formulas

5F-1 a) $\displaystyle\int x^a \ln x dx = \int \ln x d(\frac{x^{a+1}}{a+1}) = \ln x \cdot \frac{x^{a+1}}{a+1} - \int \frac{x^{a+1}}{a+1}\cdot\frac{1}{x}dx$

$$= \frac{x^{a+1}\ln x}{a+1} - \int \frac{x^a}{a+1}dx = \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + c \ (a \ne -1)$$

b) $\displaystyle\int x^{-1}\ln x dx = (\ln x)^2/2 + c \ \ (u = \ln x,\ du = dx/x)$

5F-2 a) $\displaystyle\int xe^x dx = \int xd(e^x) = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + c$

b) $\displaystyle\int x^2 e^x dx = \int x^2 d(e^x) = x^2\cdot e^x - \int e^x \cdot 2xdx$

$= x^2 \cdot e^x - 2\int xe^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x + c$

c) $\int x^3 e^x dx = \int x^3 d(e^x) = x^3 \cdot e^x - \int e^x \cdot 3x^2 dx$

$= x^3 \cdot e^x - 3\int x^2 e^x dx = x^3 \cdot e^x - 3x^2 \cdot e^x + 6x \cdot e^x - 6e^x + c$

d) $\int x^n e^{ax} dx = \int x^n d(\frac{e^{ax}}{a}) = \frac{e^{ax}}{a} \cdot x^n - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx$

$= \frac{e^{ax}}{a} \cdot x^n - \frac{n}{a}\int x^{n-1} e^{ax} dx$

5F-3

$$\int \sin^{-1}(4x)dx = x \cdot \sin^{-1}(4x) - \int xd(\sin^{-1}(4x)) = x \cdot \sin^{-1}(4x) - \int x \cdot \frac{4dx}{\sqrt{1-(4x)^2}}$$

$$= x \cdot \sin^{-1}(4x) + \int \frac{du}{8\sqrt{u}} \qquad (\text{put } u = 1 - 16x^2,\ du = -32xdx)$$

$$= x \cdot \sin^{-1}(4x) + \frac{1}{4}\sqrt{u} + c$$

$$= x \cdot \sin^{-1}(4x) + \frac{1}{4}\sqrt{1-16x^2} + c$$

5F-4

$$\int e^x \cos x dx = \int e^x d(\sin x) = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \int e^x d(-\cos x)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

Add $\int e^x \cos x dx$ to both sides to get

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + c$$

Divide by 2 and replace the arbitrary constant c by $c/2$:

$$\int e^x \cos x dx = (e^x \sin x + e^x \cos x)/2 + c$$

5F-5

$$\int \cos(\ln x)dx = x \cdot \cos(\ln x) - \int xd(\cos(\ln x))$$

$$= x \cdot \cos(\ln x) + \int \sin(\ln x)dx$$

$$= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int xd(\sin(\ln x))$$

$$= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int \cos(\ln x)dx$$

Add $\int \cos(\ln x)dx$ to both sides to get

$$2\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) + c$$

Divide by 2 and replace the arbitrary constant c by $c/2$:

$$\int \cos(\ln x)dx = (x\cos(\ln x) + x\sin(\ln x))/2 + c$$

5F-6 Put $t = e^x \implies dt = e^x dx$ and $x = \ln t$. Therefore

$$\int x^n e^x dx = \int (\ln t)^n dt$$

Integrate by parts:

$$\int (\ln t)^n dt = t \cdot (\ln t)^n - \int t d(\ln t)^n = t(\ln t)^n - n\int (\ln t)^{n-1} dt$$

because $d(\ln t)^n = n(\ln t)^{n-1}t^{-1}dt$.