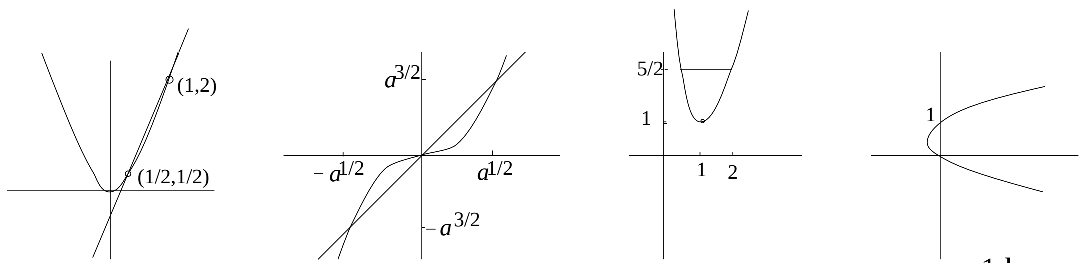
Unit 4. Applications of integration

4A. Areas between curves.

4A-1 a) $\displaystyle\int_{1/2}^{1}(3x-1-2x^2)dx = (3/2)x^2 - x - (2/3)x^3\Big|_{1/2}^{1} = 1/24$

b) $x^3 = ax \implies x = \pm a$ or $x = 0$. There are two enclosed pieces ($-a < x < 0$ and $0 < x < a$) with the same area by symmetry. Thus the total area is:

$$2\int_0^{\sqrt{a}}(ax - x^3)dx = ax^2 - (1/2)x^4\Big|_0^{\sqrt{a}} = a^2/2$$

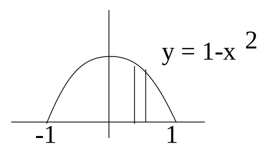


1a 1b 1c 1d

c) $x + 1/x = 5/2 \implies x^2 + 1 = 5x/2 \implies x = 2$ or $1/2$. Therefore, the area is

$$\int_{1/2}^{2}[5/2 - (x + 1/x)]dx = 5x/2 - x^2/2 - \ln x\Big|_{1/2}^{2} = 15/8 - 2\ln 2$$

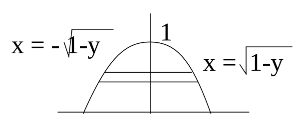
d) $\displaystyle\int_0^1 (y - y^2)dy = y^2/2 - y^3/3\Big|_0^1 = 1/6$



4A-2 First way (dx):

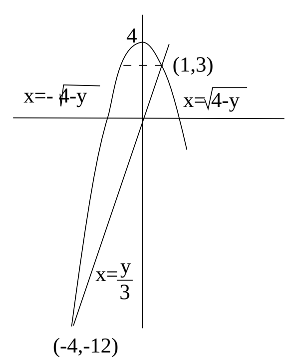
$$\int_{-1}^{1}(1-x^2)dx = 2\int_0^1(1-x^2)dx = 2x - 2x^3/3\Big|_0^1 = 4/3$$

Second way (dy): ($x = \pm\sqrt{1-y}$)



$$\int_0^1 2\sqrt{1-y}dy = (4/3)(1-y)^{3/2}\Big|_0^1 = 4/3$$

4A-3 $4 - x^2 = 3x \implies x = 1$ or -4. First way (dx):



$$\int_{-4}^{1}(4 - x^2 - 3x)dx = 4x - x^3/3 - 3x^2/2\Big|_{-4}^{1} = 125/6$$

Second way (dy): Lower section has area

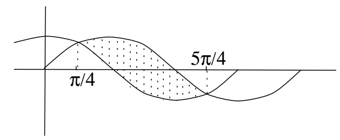
$$\int_{-12}^{3}(y/3 + \sqrt{4-y}dy = y^2/6 - (2/3)(4-y)^{3/2}\Big|_{-12}^{3} = 117/6$$

Upper section has area

$$\int_3^4 2\sqrt{4-y}dy = -(4/3)(4-y)^{3/2}\Big|_3^4 = 4/3$$

(See picture for limits of integration.) Note that $117/6 + 4/3 = 125/6$.

4A-4 $\sin x = \cos x \implies x = \pi/4 + k\pi$. So the area is



$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx = (-\cos x - \sin x)|_{\pi/4}^{5\pi/4} = 2\sqrt{2}$$

4B. Volumes by slicing; volumes of revolution

4B-1 a) $\displaystyle\int_{-1}^{1} \pi y^2 dx = \int_{-1}^{1} \pi(1-x^2)^2 dx = 2\pi\int_0^1 (1-2x^2+x^4)dx$

$$= 2\pi(x - 2x^3/3 + x^5/5)\Big|_0^1 = 16\pi/15$$

b) $\int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi(a^2-x^2)^2 dx = 2\pi\int_0^a (a^4 - 2a^2x^2 + x^4)dx$

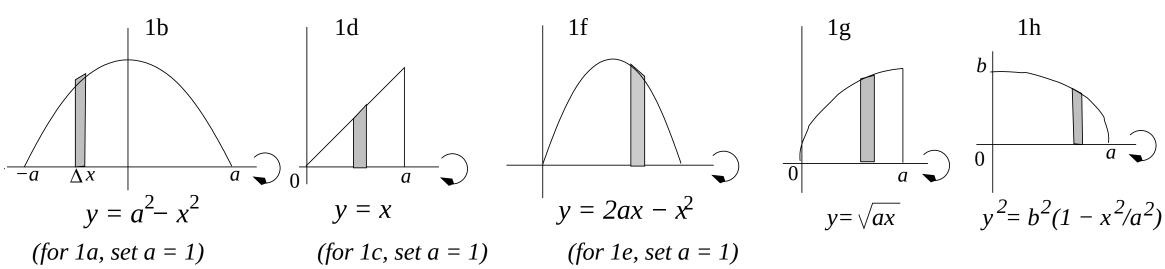
$$= 2\pi(a^4x - 2a^2x^3/3 + x^5/5)\Big|_0^a = 16\pi a^5/15$$

c) $\displaystyle\int_0^1 \pi x^2 dx = \pi/3$

d) $\displaystyle\int_0^a \pi x^2 dx = \pi a^3/3$

e) $\displaystyle\int_0^2 \pi(2x-x^2)^2 dx = \int_0^2 \pi(4x^2 - 4x^3 + x^4)dx = \pi(4x^3/3 - x^4 + x^5/5)\Big|_0^2 = 16\pi/15$

(Why (e) the same as (a)? Complete the square and translate.)



f) $\int_0^{2a} \pi(2ax - x^2)^2 dx = \int_0^{2a} \pi(4a^2x^2 - 4ax^3 + x^4)dx$

$$= \pi(4a^2x^3/3 - ax^4 + x^5/5)\Big|_0^2 = 16\pi a^5/15$$

(Why is (f) the same as (b)? Complete the square and translate.)

g) $\displaystyle\int_0^a axdx = \pi a^3/2$

h) $\displaystyle\int_0^a \pi y^2 dx = \int_0^a \pi b^2(1 - x^2/a^2)dx = \pi b^2(x - x^3/3a^2)\Big|_0^a = 2\pi b^2 a/3$

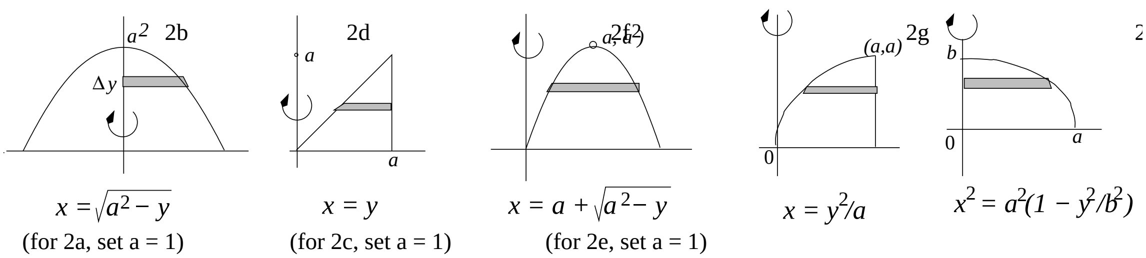
4B-2 a) $\displaystyle\int_0^1 \pi(1-y)dy = \pi/2$ b) $\displaystyle\int_0^{a^2} \pi(a^2-y)dy = \pi a^4/2$

c) $\displaystyle\int_0^1 \pi(1-y^2)dy = 2\pi/3$ d) $\displaystyle\int_0^a \pi(a^2-y^2)dy = 2\pi a^3/3$

e) $x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y}$. Using the method of washers:

$$\int_0^1 \pi[(1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2]dy = \int_0^1 4\pi\sqrt{1-y}dy$$
$$= -(8/3)\pi(1-y)^{3/2}\Big|_0^1 = 8\pi/3$$

(In contrast with 1(e) and 1(a), rotation around the y-axis makes the solid in 2(e) different from 2(a).)

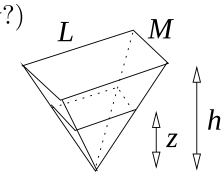


f) $x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2 - y}$. Using the method of washers:

$$\int_0^{a^2} \pi[(a+\sqrt{a^2-y})^2 - (a-\sqrt{a^2-y})^2]dy = \int_0^{a^2} 4\pi a\sqrt{a^2-y}dy$$
$$= -(8/3)\pi a(a^2-y)^{3/2}\Big|_0^1 = 8\pi a^4/3$$

g) Using washers: $\displaystyle\int_0^a \pi(a^2 - (y^2/a)^2)dy = \pi(a^2y - y^5/5a^2)\Big|_0^a = 4\pi a^3/5.$

h) $\displaystyle\int_{-b}^b \pi x^2 dy = 2\pi\int_0^b a^2(1-y^2/b^2)dy = 2\pi(a^2y - a^2y^3/3b^2)\Big|_0^b = 4\pi a^2 b/3$ (The answer in 2(h) is double the answer in 1(h), with a and b reversed. Can you see why?)



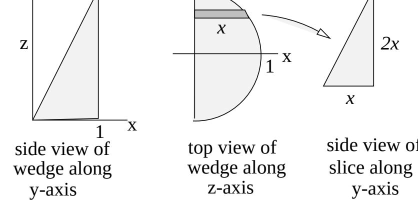
4B-3 Put the pyramid upside-down. By similar triangles, the base of the smaller bottom pyramid has sides of length $(z/h)L$ and $(z/h)M$.

The base of the big pyramid has area $b = LM$; the base of the smaller pyramid forms a cross-sectional slice, and has area

$$(z/h)L \cdot (z/h)M = (z/h)^2 LM = (z/h)^2 b$$

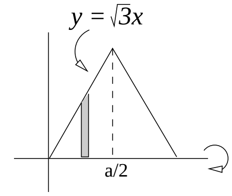
Therefore, the volume is

$$\int_0^h (z/h)^2 b dz = bz^3/3h^2\Big|_0^h = bh/3$$



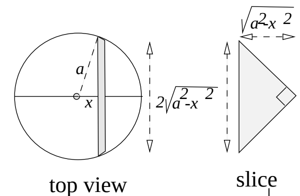
4B-4 The slice perpendicular to the xz-plane are right triangles with base of length x and height $z = 2x$. Therefore the area of a slice is x^2. The volume is

$$\int_{-1}^{1} x^2 dy = \int_{-1}^{1} (1 - y^2) dy = 4/3$$



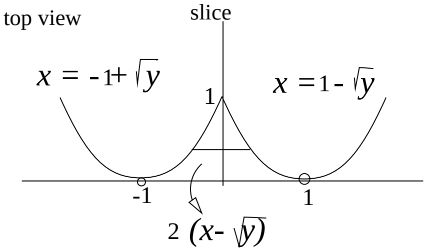
4B-5 One side can be described by $y = \sqrt{3}x$ for $0 \le x \le a/2$. Therefore, the volume is

$$2\int_0^{a/2} \pi y^2 dx 2 \int_0^{a/2} \pi(\sqrt{3}x)^2 dx = \pi a^3/4$$



4B-6 If the hypotenuse of an isosceles right triangle has length h, then its area is $h^2/4$. The endpoints of the slice in the xy-plane are $y = \pm\sqrt{a^2 - x^2}$, so $h = 2\sqrt{a^2 - x^2}$. In all the volume is

$$\int_{-a}^{a} (h^2/4) dx = \int_{-a}^{a} (a^2 - x^2) dx = 4a^3/3$$



4B-7 Solving for x in $y = (x-1)^2$ and $y = (x+1)^2$ gives the values

$$x = 1 \pm \sqrt{y} \quad \text{and} \quad x = -1 \pm \sqrt{y}$$

The hard part is deciding which sign of the square root representing the endpoints of the square.

Method 1: The point $(0, 1)$ has to be on the two curves. Plug in $y = 1$ and $x = 0$ to see that the square root must have the opposite sign from 1: $x = 1 - \sqrt{y}$ and $x = -1 + \sqrt{y}$.

Method 2: Look at the picture. $x = 1 + \sqrt{y}$ is the wrong choice because it is the right half of the parabola with vertex $(1, 0)$. We want the left half: $x = 1 - \sqrt{y}$. Similarly, we want $x = -1 + \sqrt{y}$, the right half of the parabola with vertex $(-1, 0)$. Hence, the side of the square is the interval $-1 + \sqrt{y} \le x \le 1 - \sqrt{y}$, whose length is $2(1 - \sqrt{y})$, and the

$$\text{Volume} = \int_0^1 (2(1 - \sqrt{y}))^2 dy = 4\int_0^1 (1 - 2\sqrt{y} + y) dy = 2/3 \, .$$

4C. Volumes by shells

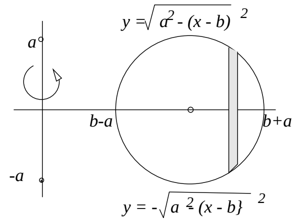
4C-1 a)

$$\text{Shells:} \quad \int_{b-a}^{b+a} (2\pi x)(2y) dx = \int_{b-a}^{b+a} 4\pi x \sqrt{a^2 - (x-b)^2} dx$$

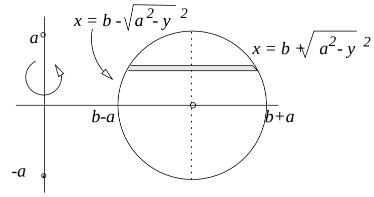
b) $(x - b)^2 = a^2 - y^2 \implies x = b \pm \sqrt{a^2 - y^2}$

$$\text{Washers:} \quad \int_{-a}^{a} \pi(x_2^2 - x_1^2) dy = \int_{-a}^{a} \pi((b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2) dy$$

$$= \pi \int_{-a}^{a} 4b\sqrt{a^2 - y^2} dy$$



Shells



Washers

c) $\displaystyle\int_{-a}^{a} \sqrt{a^2 - y^2}dy = \pi a^2/2$, because it's the area of a semicircle of radius a.

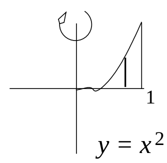
$$\text{Thus (b)} \implies \text{Volume of torus } = 2\pi^2 a^2 b$$

d) $z = x - b$, $dz = dx$

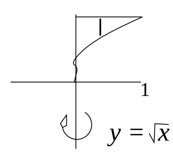
$$\int_{b-a}^{b+a} 4\pi x\sqrt{a^2 - (x-b)^2}dx = \int_{-a}^{a} 4\pi(z+b)\sqrt{a^2 - z^2}dz = \int_{-a}^{a} 4\pi b\sqrt{a^2 - z^2}dz$$

because the part of the integrand with the factor z is odd, and so it integrates to 0.

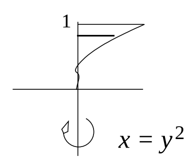
4C-2 $\displaystyle\int_0^1 2\pi xy dx = \int_0^1 2\pi x^3 dx = \pi/2$



4C−2 (shells)



4C−3a (shells)



4C−3b (discs)

4C-3 Shells: $\displaystyle\int_0^1 2\pi x(1-y)dx = \int_0^1 2\pi x(1-\sqrt{x})dx = \pi/5$

Disks: $\displaystyle\int_0^1 \pi x^2 dy = \int_0^1 \pi y^4 dy = \pi/5$

4C-4 a) $\displaystyle\int_0^1 2\pi y(2x)dy = 4\pi\int_0^1 y\sqrt{1-y}dy$

b) $\displaystyle\int_0^{a^2} 2\pi y(2x)dy = 4\pi\int_0^{a^2} y\sqrt{a^2-y}dy$

c) $\displaystyle\int_0^1 2\pi y(1-y)dy$

d) $\displaystyle\int_0^a 2\pi y(a-y)dy$

e) $$x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y}.$$

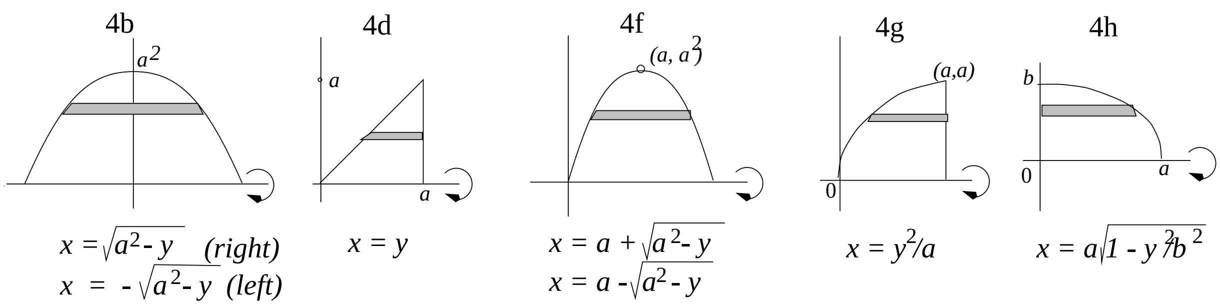
$$\text{The interval} \quad 1 - \sqrt{1-y} \le x \le 1 + \sqrt{1-y} \quad \text{has length } 2\sqrt{1-y}$$

$$\implies V = \int_0^1 2\pi y(2\sqrt{1-y})dy = 4\pi \int_0^1 y\sqrt{1-y}dy$$

f) $$x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2 - y}.$$

$$\text{The interval} \quad a - \sqrt{a^2-y} \le x \le a + \sqrt{a^2-y} \quad \text{has length } 2\sqrt{a^2-y}$$

$$\implies V = \int_0^{a^2} 2\pi y(2\sqrt{a^2-y}dy = 4\pi \int_0^{a^2} y\sqrt{a^2-y}dy$$



g) $\displaystyle\int_0^a 2\pi y(a - y^2/a)dy$

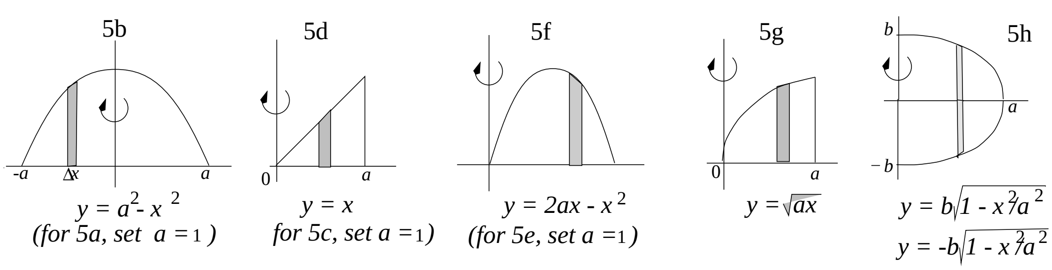
h) $\displaystyle\int_0^b 2\pi yx dy = \int_0^b 2\pi y(a^2(1 - y^2/b^2)dy$

(Why is the lower limit of integration 0 rather than $-b$?)

4C-5 a) $\displaystyle\int_0^1 2\pi x(1-x^2)dx$ c) $\displaystyle\int_0^1 2\pi xy dx = \int_0^1 2\pi x^2 dx$

b) $\displaystyle\int_0^a 2\pi x(a^2 - x^2)dx$ d) $\displaystyle\int_0^a 2\pi xy dx = \int_0^a 2\pi x^2 dx$

e) $\displaystyle\int_0^2 2\pi xy dx = \int_0^2 2\pi x(2x - x^2)dx$

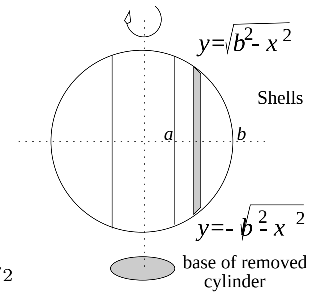


f) $\int_0^{2a} 2\pi xy dx = \int_0^{2a} 2\pi x(ax - x^2)dx$ g) $\int_0^a 2\pi xy dx = \int_0^a 2\pi x\sqrt{ax}dx$

h) $\int_0^a 2\pi x(2y)dx = \int_0^a 2\pi x(2b^2(1 - x^2/a^2))dx$

(Why did y get doubled this time?)

4C-6



$$\int_a^b 2\pi x(2y)dx = \int_a^b 2\pi x(2\sqrt{b^2 - x^2})dx$$

$$= -(4/3)\pi(b^2 - x^2)^{3/2}\Big|_a^b = (4\pi/3)(b^2 - a^2)^{3/2}$$

4D. Average value

4D-1 Cross-sectional area at x is $= \pi y^2 = \pi \cdot (x^2)^2 = \pi x^4$. Therefore,

$$\text{average cross-sectional area } = \frac{1}{2}\int_0^2 \pi x^4\, dx = \frac{\pi x^5}{10}\Big|_0^2 = \frac{16\pi}{5}\ .$$

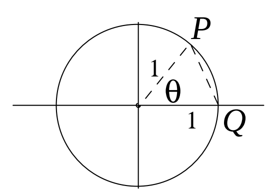
4D-2 Average $= \frac{1}{a}\int_a^{2a} \frac{dx}{x} = \frac{1}{a}\ln x\Big|_a^{2a} = \frac{1}{a}(\ln 2a - \ln a) = \frac{1}{a}\ln\left(\frac{2a}{a}\right) = \frac{\ln 2}{a}$.

4D-3 Let $s(t)$ be the distance function; then the velocity is $v(t) = s'(t)$

$$\text{Average value of velocity} = \frac{1}{b-a}\int_a^b s'(t)dt = \frac{s(b) - s(a)}{b-a} \text{ by FT1}$$

= average velocity over time interval [a,b]

4D-4 By symmetry, we can restrict P to the upper semicircle.



By the law of cosines, we have $|PQ|^2 = 1^2 + 1^2 - 2\cos\theta$. Thus

$$\text{average of } |PQ|^2 = \frac{1}{\pi}\int_0^\pi (2 - 2\cos\theta)d\theta = \frac{1}{\pi}\left[2\theta - 2\sin\theta\right]_0^\pi = 2$$

(This is the value of $|PQ|^2$ when $\theta = \pi/2$, so the answer is reasonable.))

4D-5 By hypothesis, $g(x) = \frac{1}{x}\int_0^x f(t)dt$ To express $f(x)$ in terms of $g(x)$, multiply thourgh by x and apply the Sec. Fund. Thm:

$$\int_0^x f(t)dt = xg(x) \;\Rightarrow\; f(x) = g(x) + xg'(x) \text{ , by FT2..}$$

4D-6 Average value of $A(t) = \frac{1}{T}\int_0^T A_0 e^{rt}dt = \frac{1}{T}\frac{A_0}{r}e^{rt}|_0^T = \frac{A_0}{rT}(e^{rT} - 1)$

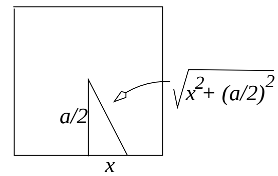
If rT is small, we can approximate: $e^{rT} \approx 1 + rT + \frac{(rT)^2}{2}$, so we get

$$A(t) \approx \frac{A_0}{rT}(rT + \frac{(rT)^2}{2}) = A_0(1 + \frac{rT}{2})\ .$$

(If $T \approx 0$, at the end of T years the interest added will be $A_0 rT$; thus the average is approximately what the account grows to in $T/2$ years, which seems reasonable.)

4D-7 $\frac{1}{b}\int_0^b x^2 dx = b^2/3$

4D-8 The average on each side is the same as the average over all four sides. Thus the average distance is



$$\frac{1}{a}\int_{-a/2}^{a/2}\sqrt{x^2+(a/2)^2}dx$$

Can't be evaluated by a formula until Unit 5. The average of the square of the distance is

$$\frac{1}{a}\int_{-a/2}^{a/2}(x^2+(a/2)^2)dx = \frac{2}{a}\int_0^{a/2}(x^2+(a/2)^2)dx = a^2/3$$

4D-9 $\frac{1}{\pi/a}\int_0^{\pi/a}\sin ax\,dx \; -\frac{1}{\pi}\cos(ax)\Big|_0^{\pi/a} = 2/\pi$

4D'. Work

4D'-1 According to Hooke's law, we have $F = kx$, where F is the force, x is the displacement (i.e., the added length), and k is the Hooke's law constant for the spring.

To find k, substitute into Hooke's law: $2{,}000 = k\cdot(1/2) \Rightarrow k = 4000$.

To find the work W, we have

$$W = \int_0^6 F\,dx = \int_0^6 4000x\,dx = 2000x^2\Big]_0^6 = 72{,}000 \text{ inch-pounds} = 6{,}000 \text{ foot-pounds.}$$

4D'-2 Let $W(h)$ = weight of pail and paint at height h.

$W(0) = 12$, $W(30) = 10 \Rightarrow W(h) = 12 - \frac{1}{15}h$, since the pulling and leakage both occur at a constant rate.

$$\text{work} = \int_0^{30} W(h)\,dh = \int_0^{30}(12 - \frac{h}{15})\,dh = 12h - \frac{h^2}{30}\Big]_0^{30} = 330 \text{ ft-lbs.}$$

4D'-3 Think of the hose as divided into many equal little infinitesimal pieces, of length dh, each of which must be hauled up to the top of the building.

The piece at distance h from the top end has weight $2\,dh$; to haul it up to the top requires $2h\,dh$ ft-lbs. Adding these up,

$$\text{total work} = \int_0^{50} 2h\,dh = h^2\Big]_0^{50} = 2500\text{ft-lbs.}$$

4D'-4 If they are x units apart, the gravitational force between them is $\frac{g\,m_1m_2}{x^2}$.

$$\text{work} = \int_d^{nd}\frac{g\,m_1m_2}{x^2}\,dx = -\frac{g\,m_1m_2}{x}\Big]_d^{nd} = -g\,m_1m_2\left(\frac{1}{nd}-\frac{1}{d}\right) = \frac{g\,m_1m_2}{d}\left(\frac{n-1}{n}\right).$$

The limit as $n \to \infty$ is $\dfrac{g\, m_1 m_2}{d}$.

4E. Parametric equations

4E-1 $y - x = t^2$, $y - 2x = -t$. Therefore,

$$y - x = (y - 2x)^2 \implies y^2 - 4xy + 4x^2 - y + x = 0 \quad \text{(parabola)}$$

4E-2 $x^2 = t^2 + 2 + 1/t^2$ and $y^2 = t^2 - 2 + 1/t^2$. Subtract, getting the hyperbola $x^2 - y^2 = 4$

4E-3 $(x-1)^2 + (y-4)^2 = \sin^2\theta + \cos^2 t = 1$ (circle)

4E-4 $1 + \tan^2 t = \sec^2 t \implies 1 + x^2 = y^2$ (hyperbola)

4E-5 $x = \sin 2t = 2\sin t \cos t = \pm 2\sqrt{1-y^2}\,y$. This gives $x^2 = 4y^2 - 4y^4$.

4E-6 $y' = 2x$, so $t = 2x$ and

$$x = t/2, \quad y = t^2/4$$

4E-7 Implicit differentiation gives $2x + 2yy' = 0$, so that $y' = -x/y$. So the parameter is $t = -x/y$. Substitute $x = -ty$ in $x^2 + y^2 = a^2$ to get

$$t^2y^2 + y^2 = a^2 \implies y^2 = a^2/(1+t^2)$$

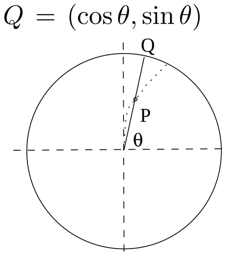
Thus

$$y = \frac{a}{\sqrt{1+t^2}}, \quad x = \frac{-at}{\sqrt{1+t^2}}$$

For $-\infty < t < \infty$, this parametrization traverses the upper semicircle $y > 0$ (going clockwise). One can also get the lower semicircle (also clockwise) by taking the negative square root when solving for y,

$$y = \frac{-a}{\sqrt{1+t^2}}, \quad x = \frac{at}{\sqrt{1+t^2}}$$

4E-8 The tip Q of the hour hand is given in terms of the angle θ by $Q = (\cos\theta, \sin\theta)$ (units are meters).



Next we express θ in terms of the time parameter t (hours). We have

$$\theta = \left\{\begin{matrix} \pi/2, t = 0 \\ \pi/3, t = 1 \end{matrix}\right\} \theta \text{ decreases linearly with t}$$

$$\implies \theta - \frac{\pi}{2} = \frac{\frac{\pi}{3} - \frac{\pi}{2} \cdot (t-0)}{1-0}$$. Thus we get $\theta = \frac{\pi}{2} - \frac{\pi}{6}t$.

Finally, for the snail's position P, we have

$P = (t\cos\theta, t\sin\theta)$, where t increases from 0 to 1. So,

$$x = t\cos(\frac{\pi}{2} - \frac{\pi}{6}t) = t\sin\frac{\pi}{6}t, \qquad y = t\sin(\tfrac{\pi}{2} - \tfrac{\pi}{6}t) = t\cos\tfrac{\pi}{6}t$$

4F. Arclength

4F-1 a) $ds = \sqrt{1+(y')^2}dx = \sqrt{26}dx$. Arclength $= \int_0^1 \sqrt{26}dx = \sqrt{26}$.

b) $ds = \sqrt{1+(y')^2}dx = \sqrt{1+(9/4)x}dx$.

$$\text{Arclength} = \int_0^1 \sqrt{1+(9/4)x}dx = (8/27)(1+9x/4)^{3/2}\Big|_0^1 = (8/27)((13/4)^{3/2}-1)$$

c) $y' = -x^{-1/3}(1-x^{2/3})^{1/2} = -\sqrt{x^{-2/3}-1}$. Therefore, $ds = x^{-1/3}dx$, and

$$\text{Arclength} = \int_0^1 x^{-1/3}dx = (3/2)x^{2/3}\Big|_0^1 = 3/2$$

d) $y' = x(2+x^2)^{1/2}$. Therefore, $ds = \sqrt{1+2x^2+x^4}dx = (1+x^2)dx$ and

$$\text{Arclength} = \int_1^2 (1+x^2)dx = x + x^3/3\Big|_1^2 = 10/3$$

4F-2 $y' = (e^x - e^{-x})/2$, so the hint says $1+(y')^2 = y^2$ and $ds = \sqrt{1+(y')^2}dx = ydx$. Thus,

$$\text{Arclength} = (1/2)\int_0^b (e^x + e^{-x})dx = (1/2)(e^x - e^{-x})\Big|_0^b = (e^b - e^{-b})/2$$

4F-3 $y' = 2x$, $\sqrt{1+(y')^2} = \sqrt{1+4x^2}$. Hence, arclength $= \int_0^b \sqrt{1+4x^2}dx$. **4F-4** $ds = \sqrt{(dx/dt)^2+(dy/dt)^2}dt = \sqrt{4t^2+9t^4}dt$. Therefore,

$$\text{Arclength} = \int_0^2 \sqrt{4t^2+9t^4}dt = \int_0^2 (4+9t^2)^{1/2}tdt$$
$$= (1/27)(4+9t^2)^{3/2}\Big|_0^2 = (40^{3/2}-8)/27$$

4F-5 $dx/dt = 1 - 1/t^2$, $dy/dt = 1 + 1/t^2$. Thus

$$ds = \sqrt{(dx/dt)^2+(dy/dt)^2}dt = \sqrt{2+2/t^4}dt \quad \text{and}$$

$$\text{Arclength} = \int_1^2 \sqrt{2+2/t^4}dt$$

4F-6 a) $dx/dt = 1-\cos t$, $dy/dt = \sin t$.

$$ds/dt = \sqrt{(dx/dt)^2+(dy/dt)^2} = \sqrt{2-2\cos t} \quad \text{(speed of the point)}$$

Forward motion (dx/dt) is largest for t an odd multiple of π ($\cos t = -1$). Forward motion is smallest for t an even multiple of π ($\cos t = 1$). *(continued →)*

Remark: The largest forward motion is when the point is at the top of the wheel and the smallest is when the point is at the bottom (since $y = 1 - \cos t$.)

b) $\int_0^{2\pi} \sqrt{2 - 2\cos t}dt = \int_0^{2\pi} 2\sin(t/2)dt = -4\cos(t/2)|_0^{2\pi} = 8$

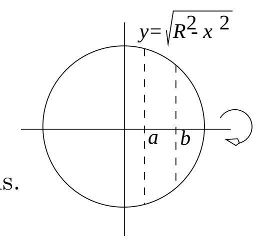
4F-7 $\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}dt$

4F-8 $dx/dt = e^t(\cos t - \sin t)$, $dy/dt = e^t(\cos t + \sin t)$.

$$ds = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}dt = e^t\sqrt{2\cos^2 t + 2\sin^2 t}dt = \sqrt{2}e^t dt$$

Therefore, the arclength is

$$\int_0^{10} \sqrt{2}e^t dt = \sqrt{2}(e^{10} - 1)$$



4G. Surface Area

4G-1 The curve $y = \sqrt{R^2 - x^2}$ for $a \leq x \leq b$ is revolved around the x-axis.

Since we have $y' = -x/\sqrt{R^2 - x^2}$, we get

$$ds = \sqrt{1 + (y')^2}dx = \sqrt{1 + x^2/(R^2 - x^2)}dx = \sqrt{R^2/(R^2 - x^2)}dx = (R/y)dx$$

Therefore, the area element is

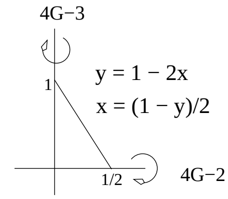
$$dA = 2\pi y ds = 2\pi R dx$$

and the area is

$$\int_a^b 2\pi R dx = 2\pi R(b - a)$$

4G-2 Limits are $0 \leq x \leq 1/2$. $ds = \sqrt{5}dx$, so

$$dA = 2\pi y ds = 2\pi(1-2x)\sqrt{5}dx \implies A = 2\pi\sqrt{5}\int_0^{1/2}(1-2x)dx = \sqrt{5}\pi/2$$



4G-3 Limits are $0 \le y \le 1$. $x = (1-y)/2$, $dx/dy = -1/2$. Thus

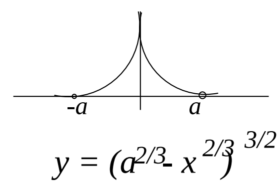
$$ds = \sqrt{1 + (dx/dy)^2}dy = \sqrt{5/4}dy;$$

$$dA = 2\pi y ds = \pi(1-y)(\sqrt{5}/2)dx \implies A = (\sqrt{5}\pi/2)\int_0^1 (1-y)dy = \sqrt{5}\pi/4$$

4G-4 $A = \int 2\pi y ds = \int_0^4 2\pi x^2 \sqrt{1+4x^2}dx$

4G-5 $x = \sqrt{y}$, $dx/dy = -1/2\sqrt{y}$, and $ds = \sqrt{1 + 1/4y}dy$

$$\begin{aligned} A &= \int 2\pi x ds = \int_0^2 2\pi\sqrt{y}\sqrt{1+1/4y}dy \\ &= \int_0^2 2\pi\sqrt{y+1/4}dy \\ &= (4\pi/3)(y+1/4)^{3/2}\Big|_0^2 = (4\pi/3)((9/4)^{3/2} - (1/4)^{3/2}) \\ &= 13\pi/3 \end{aligned}$$

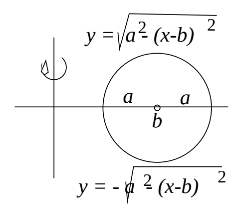


4G-6 $y = (a^{2/3} - x^{2/3})^{3/2} \implies y' = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$. Hence

$$ds = \sqrt{1 + x^{-2/3}(a^{2/3} - x^{2/3})}dx = a^{1/3}x^{-1/3}dx$$

Therefore, (using symmetry on the interval $-a \le x \le a$)

$$\begin{aligned} A &= \int 2\pi y ds = 2\int_0^a 2\pi(a^{2/3} - x^{2/3})^{3/2}a^{1/3}x^{-1/3}dx \\ &= (4\pi)(2/5)(-3/2)a^{1/3}(a^{2/3} - x^{2/3})^{5/2}\Big|_0^a \\ &= (12\pi/5)a^2 \end{aligned}$$



upper and lower surfaces are symmetrical and equal

4G-7 a) Top half: $y = \sqrt{a^2 - (x-b)^2}$, $y' = (b-x)/y$. Hence,

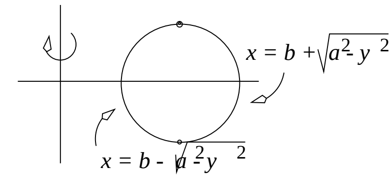
$$ds = \sqrt{1 + (b-x)^2/y^2}dx = \sqrt{(y^2 + (b-x)^2)/y^2}dx = (a/y)dx$$

Since we are only covering the top half we double the integral for area:

$$A = \int 2\pi x ds = 4\pi a \int_{b-a}^{b+a} \frac{xdx}{\sqrt{a^2 - (x-b)^2}}$$

b) We need to rotate two curves $x_2 = b + \sqrt{a^2 - y^2}$ and $x_1 = b - \sqrt{a^2 - y^2}$ around the y-axis. The value

$$dx_2/dy = -(dx_1/dy) = -y/\sqrt{a^2 - y^2}$$



inner and outer surfaces are not symmetrical and not equal

So in both cases,

$$ds = \sqrt{1 + y^2/(a^2 - y^2)}dy = (a/\sqrt{a^2 - y^2})dy$$

The integral is

$$A = \int 2\pi x_2 ds + \int 2\pi x_1 ds = \int_{-a}^{a} 2\pi(x_1 + x_2)\frac{ady}{\sqrt{a^2 - y^2}}$$

But $x_1 + x_2 = 2b$, so

$$A = 4\pi ab \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}}$$

c) Substitute $y = a\sin\theta$, $dy = a\cos\theta d\theta$ to get

$$A = 4\pi ab \int_{-\pi/2}^{\pi/2} \frac{a\cos\theta d\theta}{a\cos\theta} = 4\pi ab \int_{-\pi/2}^{\pi/2} d\theta = 4\pi^2 ab$$

4H. Polar coordinate graphs

4H-1 We give the polar coordinates in the form (r, θ):

a) $(3, \pi/2)$ b) $(2, \pi)$ c) $(2, \pi/3)$ d) $(2\sqrt{2}, 3\pi/4)$

e) $(\sqrt{2}, -\pi/4$ or $7\pi/4)$ f) $(2, -\pi/2$ or $3\pi/2)$

g) $(2, -\pi/6$ or $11\pi/6)$ h) $(2\sqrt{2}, -3\pi/4$ or $5\pi/4)$

4H-2 a) (i) $(x-a)^2 + y^2 = a^2 \Rightarrow x^2 - 2ax + y^2 = 0 \Rightarrow r^2 - 2ar\cos\theta = 0 \Rightarrow r = 2a\cos\theta$.

(ii) $\angle OPQ = 90^o$, since it is an angle inscribed in a semicircle.
In the right triangle OPQ, $|OP| = |OQ|\cos\theta$, i.e., $r = 2a\cos\theta$.

b) (i) Analogous to 4H-2a(i); ans: $r = 2a\sin\theta$.
(ii) analogous to 4H-2a(ii); note that $\angle OQP = \theta$, since both angles are complements of $\angle POQ$.

c) (i) OQP is a right triangle, $|OP| = r$, and $\angle POQ = \alpha - \theta$.
The polar equation is $r\cos(\alpha - \theta) = a$, or in expanded form,
$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = a$, or finally,

$$\frac{x}{A} + \frac{y}{B} = 1,$$

since from the right triangles OAQ and OBQ, we have $\cos\alpha = \dfrac{a}{A}$, $\sin\alpha = \cos BOQ = \dfrac{a}{B}$.

d) Since $|OQ| = \sin\theta$, we have:
if P is above the x-axis, $\sin\theta > 0$, $OP| = |OQ| - |QR|$, or $r = a - a\sin\theta$;
if P is below the x-axis, $\sin\theta < 0$, $OP| = |OQ| + |QR|$, or $r = a + a|\sin\theta| = a - a\sin\theta$.
Thus the equation is $r = a(1 - \sin\theta)$.

e) Briefly, when $P = (0, 0)$, $|PQ||PR| = a \cdot a = a^2$, the constant.
Using the law of cosines,

$|PR|^2 = r^2 + a^2 - 2ar\cos\theta$;

$|PQ|^2 = r^2 + a^2 - 2ar\cos(\pi - \theta) = r^2 + a^2 + 2ar\cos\theta$

Therefore

$|PQ|^2|PR|^2 = (r^2 + a^2)^2 - (2ar\cos\theta)^2 = (a^2)^2$

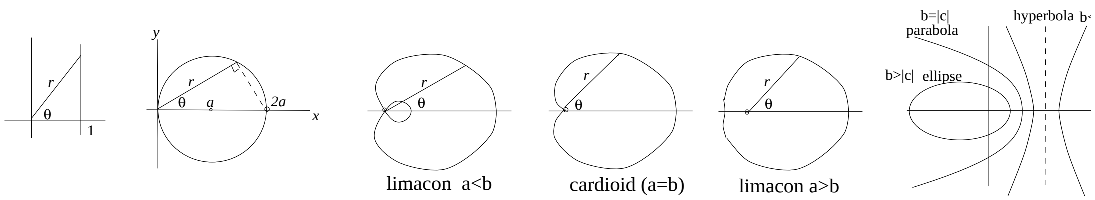
which simplifies to

$$r^2 = 2a^2\cos 2\theta.$$

4H-3 a) $r = \sec\theta \implies r\cos\theta = 1 \implies x = 1$ b) $r = 2a\cos\theta \implies r^2 = r \cdot 2a\cos\theta = 2ax \implies x^2 + y^2 = 2ax$

c) $r = (a + b\cos\theta)$ (This figure is a cardiod for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

$r^2 = ar + br \cdot \cos\theta = ar + bx \implies x^2 + y^2 = a\sqrt{x^2 + y^2} + bx$



8a 8b 8c 8d

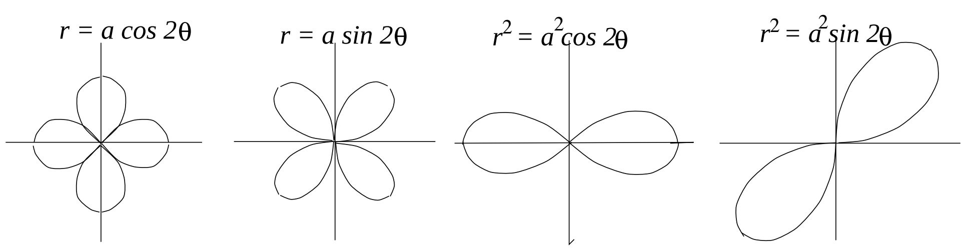
(d) $r = a/(b + c\cos\theta) \implies r(b + c\cos\theta) = a \implies rb + cx = a$

$\implies rb = a - cx \implies r^2b^2 = a^2 - 2acx + c^2x^2$

$\implies a^2 - 2acx + (c^2 - b^2)x^2 - b^2y^2 = 0$

(e) $r = a\sin(2\theta) \implies r = 2a\sin\theta\cos\theta = 2axy/r^2$

$\implies r^3 = 2axy \implies (x^2 + y^2)^{3/2} = 2axy$



f) $r = a\cos(2\theta) = a(2\cos^2\theta - 1) = a(\dfrac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^{3/2} = a(x^2 - y^2)$

g) $r^2 = a^2\sin(2\theta) = 2a^2\sin\theta\cos\theta = 2a^2\dfrac{xy}{r^2} \implies r^4 = 2a^2xy \implies (x^2 + y^2)^2 = 2axy$

h) $r^2 = a^2\cos(2\theta) = a^2(\dfrac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^2 = a^2(x^2 - y^2)$

i) $r = e^{a\theta} \implies \ln r = a\theta \implies \ln\sqrt{x^2 + y^2} = a\tan^{-1}\dfrac{y}{x}$

4I. Area and arclength in polar coordinates

4I-1 $\sqrt{(dr/d\theta)^2 + r^2}\,d\theta$

a) $\sec^2\theta d\theta$

b) $2a d\theta$

c) $\sqrt{a^2 + b^2 + 2ab\cos\theta}\,d\theta$

d) $\dfrac{a\sqrt{b^2 + c^2 + 2bc\cos\theta}}{(b + c\cos\theta)^2}d\theta$

e) $a\sqrt{4\cos^2(2\theta) + \sin^2(2\theta)}\,d\theta$

f) $a\sqrt{4\sin^2(2\theta) + \cos^2(2\theta)}\,d\theta$

g) Use implicit differentiation:

$$2rr' = 2a^2\cos(2\theta) \implies r' = a^2\cos(2\theta)/r \implies (r')^2 = a^2\cos^2(2\theta)/\sin(2\theta)$$

Hence, using a common denominator and $\cos^2 + \sin^2 = 1$,

$$ds = \sqrt{a^2\cos^2(2\theta)/\sin(2\theta) + a^2\sin(2\theta)}\,d\theta = \frac{a}{\sqrt{\sin(2\theta)}}d\theta$$

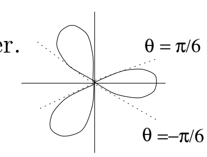
h) This is similar to (g):

$$ds = \frac{a}{\sqrt{\cos(2\theta)}}d\theta$$

i) $\sqrt{1 + a^2}e^{a\theta}d\theta$

4I-2 $dA = (r^2/2)d\theta$. The main difficulty is to decide on the endpoints of integration. Endpoints are successive times when $r = 0$.

$$\cos(3\theta) = 0 \implies 3\theta = \pi/2 + k\pi \implies \theta = \pi/6 + k\pi/3, \quad k \text{ an integer.}$$

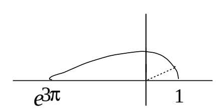


three-leaf rose
three empty sectors

$$\text{Thus,}\quad A = \int_{-\pi/6}^{\pi/6}(a^2\cos^2(3\theta)/2)d\theta = a^2\int_0^{\pi/6}\cos^2(3\theta)d\theta.$$

(Stop here in Unit 4. Evaluated in Unit 5.)

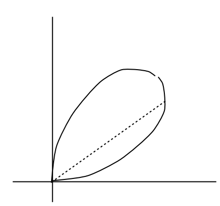
4I-3 $A = \int (r^2/2)d\theta = \int_0^{\pi}(e^{6\theta}/2)d\theta = (1/12)e^{6\theta}\Big|_0^{\pi} = (e^{6\pi} - 1)/12$

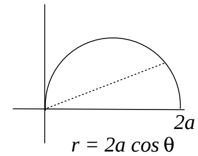


4I-4 Endpoints are successive time when $r = 0$.

$$\sin(2\theta) = 0 \implies 2\theta = k\pi, \quad k \text{ an integer.}$$

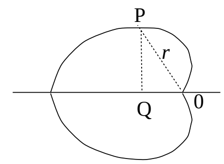
Thus, $A = \int (r^2/2)d\theta = \int_0^{\pi/2}(a^2/2)\sin(2\theta)d\theta = -(a^2/4)\cos(2\theta)\Big|_0^{\pi/2} = a^2/2.$





4I-5 $r = 2a\cos\theta$, $ds = 2ad\theta$, $-\pi/2 < \theta < \pi/2$. (The range was chosen carefully so that $r > 0$.) Total length of the circle is $2\pi a$. Since the upper and lower semicircles are symmetric, it suffices to calculate the average over the upper semicircle:

$$\frac{1}{\pi a}\int_0^{\pi/2} 2a\cos\theta(2a)d\theta = \frac{4a}{\pi}\sin\theta\Big|_0^{\pi/2} = \frac{4a}{\pi}$$



4I-6 a) Since the upper and lower halves of the cardiod are symmetric, it suffices to calculate the average distance to the x-axis just for a point on the upper half. We have $r = a(1-\cos\theta)$, and the distance to the x-axis is $r\sin\theta$, so

$$\frac{1}{\pi}\int_0^{\pi} r\sin\theta d\theta = \frac{1}{\pi}\int_0^{\pi} a(1-\cos\theta)\sin\theta d\theta = \frac{a}{2\pi}(1-\cos\theta)^2\Big|_0^{\pi} = \frac{2a}{\pi}$$

(b) $$ds = \sqrt{(dr/d\theta)^2 + r^2}d\theta = a\sqrt{(1-\cos\theta)^2 + \sin^2\theta d\theta}$$
$$= a\sqrt{2-2\cos\theta}d\theta = 2a\sin(\theta/2)d\theta, \qquad \text{using the half angle formula.}$$

$$\text{arclength} = \int_0^{2\pi} 2a\sin(2\theta)d\theta = -4a\cos(\theta/2)|_0^{2\pi} = 8a$$

For the average, don't use the half-angle version of the formula for ds, and use the interval $-\pi < \theta < \pi$, where $\sin\theta$ is odd:

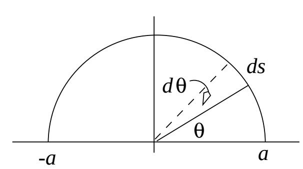
$$\text{Average} = \frac{1}{8a}\int_{-\pi}^{\pi} |r\sin\theta| a\sqrt{2-2\cos\theta}d\theta = \frac{1}{8a}\int_{-\pi}^{\pi} |\sin\theta|\sqrt{2}a^2(1-\cos\theta)^{3/2}d\theta$$
$$= \frac{\sqrt{2}a}{4}\int_0^{\pi}(1-\cos\theta)^{3/2}\sin\theta d\theta = \frac{\sqrt{2}a}{10}(1-\cos\theta)^{5/2}\Big|_0^{\pi} = \frac{4}{5}a$$

4I-7 $dx = -a\sin\theta d\theta$. So the semicircle $y > 0$ has area

$$\int_{-a}^{a} ydx = \int_{\pi}^{0} a\sin\theta(-a\sin\theta)d\theta = a^2\int_0^{\pi}\sin^2\theta d\theta$$

But
$$\int_0^{\pi}\sin^2\theta d\theta = \frac{1}{2}\int_0^{\pi}(1-\cos(2\theta)d\theta = \pi/2$$

So the area is $\pi a^2/2$ as it should be for a semicircle.



Arclength: $ds^2 = dx^2 + dy^2$

$\Longrightarrow (ds)^2 = (-a\sin\theta d\theta)^2 + (a\cos\theta d\theta)^2 = a^2(\sin^2 d\theta + \cos^2 d\theta)(d\theta)^2$

$\Longrightarrow ds = ad\theta$ (obvious from picture).

$$\int ds = \int_0^{2\pi} ad\theta = 2\pi a$$

4J. Other applications

4J-1 Divide the water in the hole into n equal circular discs of thickness Δy.

Volume of each disc: $\pi\left(\frac{1}{2}\right)^2 \Delta y$

Energy to raise the disc of water at depth y_i to surface: $\frac{\pi}{4} k y_i \Delta y$.

Adding up the energies for the different discs, and passing to the limit,

$$E = \lim_{n\to\infty} \sum_1^n \frac{\pi}{4} k y_i \Delta y = \int_0^{100} \frac{\pi}{4} k y \, dy = \frac{\pi k}{4} \frac{y^2}{2}\Big]_0^{100} = \frac{\pi k 10^4}{8}.$$

4J-2 Divide the hour into n equal small time intervals Δt.

At time t_i, $i = 1, \ldots, n$, there are $x_0 e^{-kt_i}$ grams of material, producing approximately $r x_0 e^{-kt_i} \Delta t$ radiation units over the time interval $[t_i, t_i + \Delta t]$.

Adding and passing to the limit,

$$R = \lim_{n\to\infty} \sum_1^n r\, x_0 e^{-kt_i} \Delta t = \int_0^{60} r\, x_0 e^{-kt} \, dt = r\, x_0 \frac{e^{-kt}}{-k}\Big]_0^{60} = \frac{r\, x_0}{k}\left(1 - e^{-60k}\right).$$

4J-3 Divide up the pool into n thin concentric cylindrical shells, of radius r_i, $i = 1, \ldots, n$, and thickness Δr.

The volume of the i-th shell is approximately $2\pi\, r_i D \,\Delta r$.

The amount of chemical in the i-th shell is approximately $\frac{k}{1 + r_i^2} 2\pi\, r_i D \,\Delta r$.

Adding, and passing to the limit,

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_1^n \frac{k}{1 + r_i^2} 2\pi\, r_i D \,\Delta r = \int_0^R 2\pi k D \frac{r}{1 + r^2} \, dr \\ &= \pi k D \ln(1 + r^2)\Big]_0^R = \pi k D \ln(1 + R^2) \text{ gms.} \end{aligned}$$

4J-4 Divide the time interval into n equal small intervals of length Δt by the points t_i, $i = 1, \ldots, n$.

The approximate number of heating units required to maintain the temperature at 75^o over the time interval $[t_i, t_i + \Delta t]$: is

$$\left[75 - 10\left(6 - \cos\frac{\pi t_i}{12}\right)\right] \cdot k \,\Delta t.$$

Adding over the time intervals and passing to the limit:

$$\begin{aligned} \text{total heat} &= \lim_{n\to\infty} \sum_1^n \left[75 - 10\left(6 - \cos\frac{\pi t_i}{12}\right)\right] \cdot k \,\Delta t \\ &= \int_0^{24} k\left[75 - 10\left(6 - \cos\frac{\pi t}{12}\right)\right] dt \\ &= \int_0^{24} k\left(15 + 10\cos\frac{\pi t}{12}\right) dt = k\left[15t + \frac{120}{\pi}\sin\frac{\pi t}{12}\right]_0^{24} = 360k. \end{aligned}$$

4J-5 Divide the month into n equal intervals of length Δt by the points t_i, $i = 1, \dots, n$.
Over the time interval $[t_i . t_i + \Delta t]$, the number of units produced is about $(10 + t_i)\,\Delta t$.
The cost of holding these in inventory until the end of the month is $c(30 - t_i)(10 + t_i)\,\Delta t$.
Adding and passing to the limit,

$$\begin{aligned} \text{total cost} &= \lim_{n\to\infty} \sum_1^n c(30 - t_i)(10 + t_i)\,\Delta t \\ &= \int_0^{30} c(30 - t)(10 + t)\,dt \;=\; c\left[300t + 10t^2 - \frac{t^3}{3}\right]_0^{30} \;=\; 9000c. \end{aligned}$$

4J-6 Divide the water in the tank into thin horizontal slices of width dy.
If the slice is at height y above the center of the tank, its radius is $\sqrt{r^2 - y^2}$.
This formula for the radius of the slice is correct even if $y < 0$ – i.e., the slice is below the center of the tank – as long as $-r < y < r$, so that there really is a slice at that height.
Volume of water in the slice $= \pi(r^2 - y^2)\,dy$
Weight of water in the slice $= \pi w(r^2 - y^2)\,dy$
Work to lift this slice from the ground to the height $h + y = \pi w(r^2 - y^2)\,dy\,(h + y)$.

$$\begin{aligned} \text{Total work} &= \int_{-r}^{r} \pi w(r^2 - y^2)(h + y)\,dy \\ &= \pi w \int_{-r}^{r} (r^2h + r^2y - hy^2 - y^3) \\ &= \pi w \left[r^2hy + \frac{r^2y^2}{2} - \frac{hy^3}{3} - \frac{y^4}{4}\right]_{-r}^{r}. \end{aligned}$$

In this last line, the even powers of y have the same value at $-r$ and r, so contribute 0 when it is evaluated; we get therefore

$$= \pi wh\left[r^2y - \frac{y^3}{3}\right]_{-r}^{r} = 2\pi wh\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi whr^3.$$