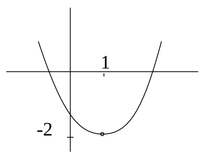
SOLUTIONS TO 18.01 EXERCISES

Unit 1. Differentiation

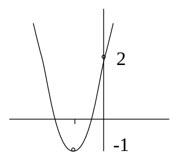
1A. Graphing

1A-1,2 a) $y = (x-1)^2 - 2$

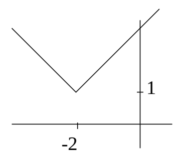
b) $y = 3(x^2 + 2x) + 2 = 3(x+1)^2 - 1$



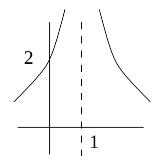
1a



1b



2a



2b

1A-3 a) $f(-x) = \dfrac{(-x)^3 - 3x}{1-(-x)^4} = \dfrac{-x^3-3x}{1-x^4} = -f(x)$, so it is odd.

b) $(\sin(-x))^2 = (\sin x)^2$, so it is even.

c) $\dfrac{\text{odd}}{\text{even}}$, so it is odd

d) $(1-x)^4 \neq \pm(1+x)^4$: neither.

e) $J_0((-x)^2) = J_0(x^2)$, so it is even.

1A-4 a) $p(x) = p_e(x) + p_o(x)$, where $p_e(x)$ is the sum of the even powers and $p_o(x)$ is the sum of the odd powers

b) $f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}$

$F(x) = \dfrac{f(x)+f(-x)}{2}$ is even and $G(x) = \dfrac{f(x)-f(-x)}{2}$ is odd because

$$F(-x) = \frac{f(-x)+f(-(-x))}{2} = F(x); \qquad G(-x) = \frac{f(x)-f(-x)}{2} = -G(-x).$$

c) Use part b:

$$\frac{1}{x+a} + \frac{1}{-x+a} = \frac{2a}{(x+a)(-x+a)} = \frac{2a}{a^2-x^2} \quad \text{even}$$

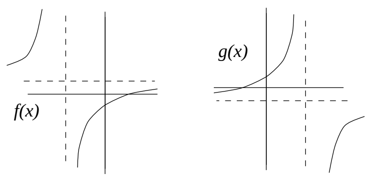
$$\frac{1}{x+a} - \frac{1}{-x+a} = \frac{-2x}{(x+a)(-x+a)} = \frac{-2x}{a^2-x^2} \quad \text{odd}$$

$$\implies \frac{1}{x+a} = \frac{a}{a^2-x^2} - \frac{x}{a^2-x^2}$$

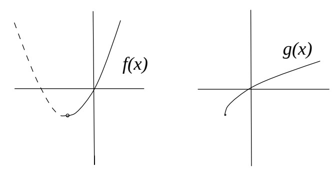
1A-5 a) $y = \dfrac{x-1}{2x+3}$. Crossmultiply and solve for x, getting $x = \dfrac{3y+1}{1-2y}$, so the inverse function is $\dfrac{3x+1}{1-2x}$.

b) $y = x^2 + 2x = (x+1)^2 - 1$

(Restrict domain to $x \le -1$, so when it's flipped about the diagonal $y = x$, you'll still get the graph of a function.) Solving for x, we get $x = \sqrt{y+1} - 1$, so the inverse function is $y = \sqrt{x+1} - 1$.



5a



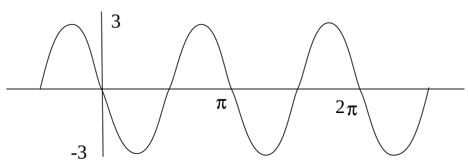
5b

1A-6 a) $A = \sqrt{1+3} = 2$, $\tan c = \dfrac{\sqrt{3}}{1}$, $c = \frac{\pi}{3}$. So $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$.

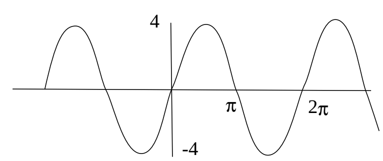
b) $\sqrt{2}\sin(x - \dfrac{\pi}{4})$

1A-7 a) $3\sin(2x - \pi) = 3\sin 2(x - \dfrac{\pi}{2})$, amplitude 3, period π, phase angle $\pi/2$.

b) $-4\cos(x + \dfrac{\pi}{2}) = 4\sin x$ amplitude 4, period 2π, phase angle 0.



7a



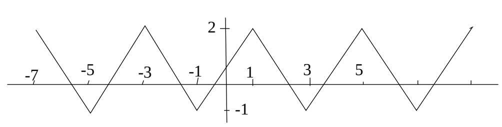
7b

1A-8

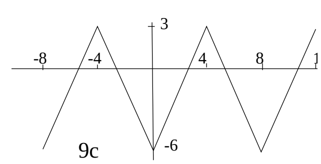
$f(x)$ odd $\Longrightarrow f(0) = -f(0) \Longrightarrow f(0) = 0$.

So $f(c) = f(2c) = \cdots = 0$, also (by periodicity, where c is the period).

1A-9



9ab period = 4



9c

c) The graph is made up of segments joining $(0, -6)$ to $(4, 3)$ to $(8, -6)$. It repeats in a zigzag with period 8. * This can be derived using:

$$x/2 - 1 = -1 \implies x = 0 \text{ and } g(0) = 3f(-1) - 3 = -6$$
$$x/2 - 1 = 1 \implies x = 4 \text{ and } g(4) = 3f(1) - 3 = 3$$
$$x/2 - 1 = 3 \implies x = 8 \text{ and } g(8) = 3f(3) - 3 = -6$$

1B. Velocity and rates of change

1B-1 a) $h =$ height of tube $= 400 - 16t^2$.

$$\text{average speed } \frac{h(2) - h(0)}{2} = \frac{(400 - 16 \cdot 2^2) - 400}{2} = -32\text{ft/sec}$$

(The minus sign means the test tube is going down. You can also do this whole problem using the function $s(t) = 16t^2$, representing the distance down measured from the top. Then all the speeds are positive instead of negative.)

b) Solve $h(t) = 0$ (or $s(t) = 400$) to find landing time $t = 5$. Hence the average speed for the last two seconds is

$$\frac{h(5) - h(3)}{2} = \frac{0 - (400 - 16 \cdot 3^2)}{2} = -128\text{ft/sec}$$

c)

$$\begin{aligned} \frac{h(t) - h(5)}{t - 5} &= \frac{400 - 16t^2 - 0}{t - 5} = \frac{16(5 - t)(5 + t)}{t - 5} \\ &= -16(5 + t) \to -160\text{ft/sec as } t \to 5 \end{aligned}$$

1B-2 A tennis ball bounces so that its initial speed straight upwards is b feet per second. Its height s in feet at time t seconds is

$$s = bt - 16t^2$$

a)

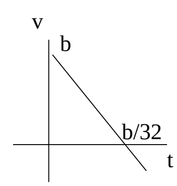
$$\begin{aligned} \frac{s(t + h) - s(t)}{h} &= \frac{b(t + h) - 16(t + h)^2 - (bt - 16t^2)}{h} \\ &= \frac{bt + bh - 16t^2 - 32th - 16h^2 - bt + 16t^2}{h} \\ &= \frac{bh - 32th - 16h^2}{h} \\ &= b - 32t - 16h \to b - 32t \text{ as } h \to 0 \end{aligned}$$

Therefore, $v = b - 32t$.

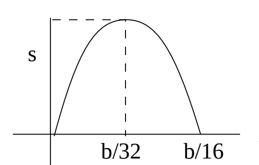
b) The ball reaches its maximum height exactly when the ball has finished going up. This is time at which $v(t) = 0$, namely, $t = b/32$.

c) The maximum height is $s(b/32) = b^2/64$.

d) The graph of v is a straight line with slope -32. The graph of s is a parabola with maximum at place where $v = 0$ at $t = b/32$ and landing time at $t = b/16$.



graph of velocity



graph of position

e) If the initial velocity on the first bounce was $b_1 = b$, and the velocity of the second bounce is b_2, then $b_2^2/64 = (1/2)b_1^2/64$. Therefore, $b_2 = b_1/\sqrt{2}$. The second bounce is at $b_1/16 + b_2/16$. *(continued →)*

f) If the ball continues to bounce then the landing times form a geometric series

$$\begin{aligned} b_1/16 + b_2/16 + b_3/16 + \cdots &= b/16 + b/16\sqrt{2} + b/16(\sqrt{2})^2 + \cdots \\ &= (b/16)(1 + (1/\sqrt{2}) + (1/\sqrt{2})^2 + \cdots) \\ &= \frac{b/16}{1 - (1/\sqrt{2})} \end{aligned}$$

Put another way, the ball stops bouncing after $1/(1 - (1/\sqrt{2})) \approx 3.4$ times the length of time the first bounce.

1C. Slope and derivative.

1C-1 a)

$$\begin{aligned} \frac{\pi(r+h)^2 - \pi r^2}{h} &= \frac{\pi(r^2 + 2rh + h^2) - \pi r^2}{h} = \frac{\pi(2rh + h^2)}{h} \\ &= \pi(2r + h) \\ &\to 2\pi r \text{ as } h \to 0 \end{aligned}$$

b)

$$\begin{aligned} \frac{(4\pi/3)(r+h)^3 - (4\pi/3)r^3}{h} &= \frac{(4\pi/3)(r^3 + 3r^2h + 3rh^2 + h^3) - (4\pi/3)r^3}{h} \\ &= \frac{(4\pi/3)(3r^2h + 3rh^2 + h^3)}{h} \\ &= (4\pi/3)(3r^2 + 3rh + h^2) \\ &\to 4\pi r^2 \text{ as } h \to 0 \end{aligned}$$

1C-2 $\dfrac{f(x) - f(a)}{x - a} = \dfrac{(x-a)g(x) - 0}{x - a} = g(x) \to g(a)$ as $x \to a$.

1C-3 a)

$$\begin{aligned} \frac{1}{h}\left[\frac{1}{2(x+h)+1} - \frac{1}{2x+1}\right] &= \frac{1}{h}\left[\frac{2x + 1 - (2(x+h)+1)}{(2(x+h)+1)(2x+1)}\right] \\ &= \frac{1}{h}\left[\frac{-2h}{(2(x+h)+1)(2x+1)}\right] \\ &= \frac{-2}{(2(x+h)+1)(2x+1)} \\ &\longrightarrow \frac{-2}{(2x+1)^2} \text{ as } h \to 0 \end{aligned}$$

b)

$$\frac{2(x+h)^2+5(x+h)+4-(2x^2+5x+4)}{h} = \frac{2x^2+4xh+2h^2+5x+5h-2x^2-5x}{h}$$
$$= \frac{4xh+2h^2+5h}{h} = 4x+2h+5$$
$$\longrightarrow 4x+5 \text{ as } h \to 0$$

c)

$$\frac{1}{h}\left[\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}\right] = \frac{1}{h}\left[\frac{(x^2+1)-((x+h)^2+1)}{((x+h)^2+1)(x^2+1)}\right]$$
$$= \frac{1}{h}\left[\frac{x^2+1-x^2-2xh-h^2-1}{((x+h)^2+1)(x^2+1)}\right]$$
$$= \frac{1}{h}\left[\frac{-2xh-h^2}{((x+h)^2+1)(x^2+1)}\right]$$
$$= \frac{-2x-h}{((x+h)^2+1)(x^2+1)}$$
$$\longrightarrow \frac{-2x}{(x^2+1)^2} \text{ as } h \to 0$$

d) Common denominator:

$$\frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] = \frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right]$$

Now simplify the numerator by multiplying numerator and denominator by $\sqrt{x}+\sqrt{x+h}$, and using $(a-b)(a+b) = a^2-b^2$:

$$\frac{1}{h}\left[\frac{(\sqrt{x})^2-(\sqrt{x+h})^2}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right] = \frac{1}{h}\left[\frac{x-(x+h)}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]$$
$$= \frac{1}{h}\left[\frac{-h}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]$$
$$= \left[\frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]$$
$$\longrightarrow \frac{-1}{2(\sqrt{x})^3} = -\frac{1}{2}x^{-3/2} \text{ as } h \to 0$$

e) For part (a), $-2/(2x+1)^2 < 0$, so there are no points where the slope is 1 or 0. For slope -1,

$$-2/(2x+1)^2 = -1 \implies (2x+1)^2 = 2 \implies 2x+1 = \pm\sqrt{2} \implies x = -1/2 \pm \sqrt{2}/2$$

For part (b), the slope is 0 at $x = -5/4$, 1 at $x = -1$ and -1 at $x = -3/2$.

1C-4 Using Problem 3,

a) $f'(1) = -2/9$ and $f(1) = 1/3$, so $y = -(2/9)(x-1) + 1/3 = (-2x+5)/9$

b) $f(a) = 2a^2 + 5a + 4$ and $f'(a) = 4a + 5$, so

$$y = (4a+5)(x-a) + 2a^2 + 5a + 4 = (4a+5)x - 2a^2 + 4$$

c) $f(0) = 1$ and $f'(0) = 0$, so $y = 0(x-0) + 1$, or $y = 1$.

d) $f(a) = 1/\sqrt{a}$ and $f'(a) = -(1/2)a^{-3/2}$, so

$$y = -(1/2)a^{3/2}(x-a) + 1/\sqrt{a} = -a^{-3/2}x + (3/2)a^{-1/2}$$

1C-5 Method 1. $y'(x) = 2(x-1)$, so the tangent line through $(a, 1+(a-1)^2)$ is

$$y = 2(a-1)(x-a) + 1 + (a-1)^2$$

In order to see if the origin is on this line, plug in $x = 0$ and $y = 0$, to get the following equation for a.

$$0 = 2(a-1)(-a) + 1 + (a-1)^2 = -2a^2 + 2a + 1 + a^2 - 2a + 1 = -a^2 + 2$$

Therefore $a = \pm\sqrt{2}$ and the two tangent lines through the origin are

$$y = 2(\sqrt{2}-1)x \quad \text{and} \quad y = -2(\sqrt{2}+1)x$$

(Because these are lines throught the origin, the constant terms must cancel: this is a good check of your algebra!)

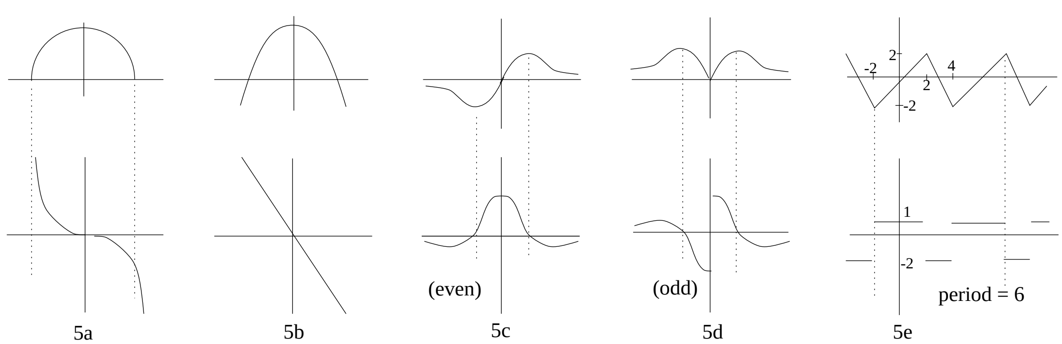
Method 2. Seek tangent lines of the form $y = mx$. Suppose that $y = mx$ meets $y = 1 + (x-1)^2$, at $x = a$, then $ma = 1 + (a-1)^2$. In addition we want the slope $y'(a) = 2(a-1)$ to be equal to m, so $m = 2(a-1)$. Substituting for m we find

$$2(a-1)a = 1 + (a-1)^2$$

This is the same equation as in method 1: $a^2 - 2 = 0$, so $a = \pm\sqrt{2}$ and $m = 2(\pm\sqrt{2} - 1)$, and the two tangent lines through the origin are as above,

$$y = 2(\sqrt{2}-1)x \quad \text{and} \quad y = -2(\sqrt{2}+1)x$$

1C-6



1D. Limits and continuity

1D-1 Calculate the following limits if they exist. If they do not exist, then indicate whether they are $+\infty$, $-\infty$ or undefined.

a) -4

b) $8/3$

c) undefined (both $\pm\infty$ are possible)

d) Note that $2-x$ is negative when $x>2$, so the limit is $-\infty$

e) Note that $2-x$ is positive when $x<2$, so the limit is $+\infty$ (can also be written ∞)

f) $\dfrac{4x^2}{x-2} = \dfrac{4x}{1-(2/x)} \to \dfrac{\infty}{1} = \infty$ as $x\to\infty$

g) $\dfrac{4x^2}{x-2} - 4x = \dfrac{4x^2-4x(x-2)}{x-2} = \dfrac{8x}{x-2} = \dfrac{8}{1-(2/x)} \to 8$ as $x\to\infty$

i) $\dfrac{x^2+2x+3}{3x^2-2x+4} = \dfrac{1+(2/x)+(3/x^2)}{3-(2/x)+4/x^2)} \to \dfrac{1}{3}$ as $x\to\infty$

j) $\dfrac{x-2}{x^2-4} = \dfrac{x-2}{(x-2)(x+2)} = \dfrac{1}{x+2} \to \dfrac{1}{4}$ as $x\to 2$

1D-2 a) $\displaystyle\lim_{x\to0+}\sqrt{x}=0$ b) $\displaystyle\lim_{x\to1+}\frac{1}{x-1}=\infty$ $\displaystyle\lim_{x\to1-}\frac{1}{x-1}=-\infty$

c) $\displaystyle\lim_{x\to1}(x-1)^{-4}=\infty$ (left and right hand limits are same)

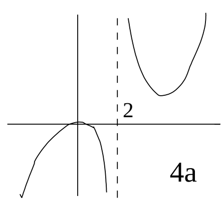
d) $\displaystyle\lim_{x\to0}|\sin x|=0$ (left and right hand limits are same)

e) $\displaystyle\lim_{x\to0+}\frac{|x|}{x}=1$ $\displaystyle\lim_{x\to0-}\frac{|x|}{x}=-1$

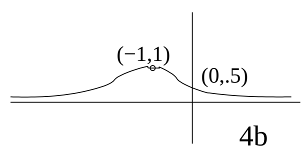
1D-3 a) $x=2$ removable $x=-2$ infinite b) $x=0,\pm\pi,\pm2\pi,\ldots$ infinite

c) $x=0$ removable d) $x=0$ removable e) $x=0$ jump f) $x=0$ removable

1D-4



4a



4b

1D-5 a) for continuity, want $ax+b=1$ when $x=1$. Ans.: all a,b such that $a+b=1$

b) $\dfrac{dy}{dx} = \dfrac{d(x^2)}{dx} = 2x = 2$ when $x=1$. We have also $\dfrac{d(ax+b)}{dx} = a$. Therefore, to make $f'(x)$ continuous, we want $a=2$.

Combining this with the condition $a+b$=1 from part (a), we get finally $b=-1$, $a=2$.

1D-6 a) $f(0) = 0^2 + 4 \cdot 0 + 1 = 1$. Match the function values:

$$f(0^-) = \lim_{x \to 0} ax + b = b, \qquad \text{so } b = 1 \text{ by continuity.}$$

Next match the slopes:

$$f'(0^+) = \lim_{x \to 0} 2x + 4 = 4$$

and $f'(0^-) = a$. Therefore, $a = 4$, since $f'(0)$ exists.

b)

$$f(1) = 1^2 + 4 \cdot 1 + 1 = 6 \text{ and } f(1^-) = \lim_{x \to 1} ax + b = a + b$$

Therefore continuity implies $a + b = 6$. The slope from the right is

$$f'(1^+) = \lim_{x \to 1} 2x + 4 = 6$$

Therefore, this must equal the slope from the left, which is a. Thus, $a = 6$ and $b = 0$.

1D-7

$$f(1) = c1^2 + 4 \cdot 1 + 1 = c + 5 \text{ and } f(1^-) = \lim_{x \to 1} ax + b = a + b$$

Therefore, by continuity, $c + 5 = a + b$. Next, match the slopes from left and right:

$$f'(1^+) = \lim_{x \to 1} 2cx + 4 = 2c + 4 \text{ and } f'(1^-) = \lim_{x \to 1} a = a$$

Therefore,

$$a = 2c + 4 \text{ and } b = -c + 1.$$

1D-8

a)

$$f(0) = \sin(2 \cdot 0) = 0 \text{ and } f(0^+) = \lim_{x \to 0} ax + b = b$$

Therefore, continuity implies $b = 0$. The slope from each side is

$$f'(0^-) = \lim_{x \to 0} 2\cos(2x) = 2 \text{ and } f'(0^+) = \lim_{x \to 0} a = a$$

Therefore, we need $a \neq 2$ in order that f not be differentiable.

b)

$$f(0) = \cos(2 \cdot 0) = 1 \text{ and } f(0^+) = \lim_{x \to 0} ax + b = b$$

Therefore, continuity implies $b = 1$. The slope from each side is

$$f'(0^-) = \lim_{x \to 0} -2\sin(2x) = 0 \text{ and } f'(0^+) = \lim_{x \to 0} a = a$$

Therefore, we need $a \neq 0$ in order that f not be differentiable.

1D-9 There cannot be any such values because every differentiable function is continuous.

1E: Differentiation formulas: polynomials, products, quotients

1E-1 Find the derivative of the following polynomials

a) $10x^9 + 15x^4 + 6x^2$

b) 0 ($e^2 + 1 \approx 8.4$ is a constant and the derivative of a constant is zero.)

c) 1/2

d) By the product rule: $(3x^2+1)(x^5+x^2)+(x^3+x)(5x^4+2x) = 8x^7+6x^5+5x^4+3x^2$. Alternatively, multiply out the polynomial first to get $x^8+x^6+x^5+x^3$ and then differentiate.

1E-2 Find the antiderivative of the following polynomials

a) $ax^2/2 + bx + c$, where a and b are the given constants and c is a third constant.

b) $x^7/7 + (5/6)x^6 + x^4 + c$

c) The only way to get at this is to multiply it out: $x^6 + 2x^3 + 1$. Now you can take the antiderivative of each separate term to get

$$\frac{x^7}{7} + \frac{x^4}{2} + x + c$$

Warning: The answer is not $(1/3)(x^3 + 1)^3$. (The derivative does not match if you apply the chain rule, the rule to be treated below in E4.)

1E-3 $y' = 3x^2 + 2x - 1 = 0 \implies (3x - 1)(x + 1) = 0$. Hence $x = 1/3$ or $x = -1$ and the points are $(1/3, 49/27)$ and $(-1, 3)$

1E-4 a) $f(0) = 4$, and $f(0^-) = \lim_{x\to 0} 5x^5 + 3x^4 + 7x^2 + 8x + 4 = 4$. Therefore the function is continuous for all values of the parameters.

$$f'(0^+) = \lim_{x\to 0} 2ax + b = b \text{ and } f'(0^-) = \lim_{x\to 0} 25x^4 + 12x^3 + 14x + 8 = 8$$

Therefore, $b = 8$ and a can have any value.

b) $f(1) = a + b + 4$ and $f(1^+) = 5 + 3 + 7 + 8 + 4 = 27$. So by continuity,

$$a + b = 23$$

$$f'(1^-) = \lim_{x\to 1} 2ax + b = 2a + b; \qquad f'(1^+) = \lim_{x\to 1} 25x^4 + 12x^3 + 14x + 8 = 59.$$

Therefore, differentiability implies

$$2a + b = 59$$

Subtracting the first equation, $a = 59 - 23 = 36$ and hence $b = -13$.

1E-5 a) $\dfrac{1}{(1+x)^2}$ b) $\dfrac{1-2ax-x^2}{(x^2+1)^2}$ c) $\dfrac{-x^2-4x-1}{(x^2-1)^2}$

d) $3x^2 - 1/x^2$

1F. Chain rule, implicit differentiation

1F-1 a) Let $u = (x^2 + 2)$

$$\frac{d}{dx}u^2 = \frac{du}{dx}\frac{d}{du}u^2 = (2x)(2u) = 4x(x^2+2) = 4x^3 + 8x$$

Alternatively,

$$\frac{d}{dx}(x^2+2)^2 = \frac{d}{dx}(x^4 + 4x^2 + 4) = 4x^3 + 8x$$

b) Let $u = (x^2 + 2)$; then $\frac{d}{dx}u^{100} = \frac{du}{dx}\frac{d}{du}u^{100} = (2x)(100u^{99}) = (200x)(x^2+2)^{99}$.

1F-2 Product rule and chain rule:

$$10x^9(x^2+1)^{10} + x^{10}[10(x^2+1)^9(2x)] = 10(3x^2+1)x^9(x^2+1)^9$$

1F-3 $y = x^{1/n} \implies y^n = x \implies ny^{n-1}y' = 1$. Therefore,

$$y' = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{1-n} = \frac{1}{n}x^{\frac{1}{n}-1}$$

1F-4 $(1/3)x^{-2/3} + (1/3)y^{-2/3}y' = 0$ implies

$$y' = -x^{-2/3}y^{2/3}$$

Put $u = 1 - x^{1/3}$. Then $y = u^3$, and the chain rule implies

$$\frac{dy}{dx} = 3u^2\frac{du}{dx} = 3(1-x^{1/3})^2(-(1/3)x^{-2/3}) = -x^{-2/3}(1-x^{1/3})^2$$

The chain rule answer is the same as the one using implicit differentiation because

$$y = (1-x^{1/3})^3 \implies y^{2/3} = (1-x^{1/3})^2$$

1F-5 Implicit differentiation gives $\cos x + y'\cos y = 0$. Horizontal slope means $y' = 0$, so that $\cos x = 0$. These are the points $x = \pi/2 + k\pi$ for every integer k. Recall that $\sin(\pi/2 + k\pi) = (-1)^k$, i.e., 1 if k is even and -1 if k is odd. Thus at $x = \pi/2 + k\pi$, $\pm 1 + \sin y = 1/2$, or $\sin y = \mp 1 + 1/2$. But $\sin y = 3/2$ has no solution, so the only solutions are when k is even and in that case $\sin y = -1 + 1/2$, so that $y = -\pi/6 + 2n\pi$ or $y = 7\pi/6 + 2n\pi$. In all there are two grids of points at the vertices of squares of side 2π, namely the points

$$(\pi/2 + 2k\pi, -\pi/6 + 2n\pi) \text{ and } (\pi/2 + 2k\pi, 7\pi/6 + 2n\pi); \qquad k, n \text{ any integers.}$$

1F-6 Following the hint, let $z = -x$. If f is even, then $f(x) = f(z)$ Differentiating and using the chain rule:

$$f'(x) = f'(z)(dz/dx) = -f'(z) \quad \text{because } dz/dx = -1$$

But this means that f' is odd. Similarly, if g is odd, then $g(x = -g(z)$. Differentiating and using the chain rule:

$$g'(x) = -g'(z)(dz/dx) = g'(z) \quad \text{because } dz/dx = -1$$

1F-7 a) $\dfrac{dD}{dx} = \dfrac{1}{2}((x-a)^2 + {y_0}^2)^{-1/2}(2(x-a)) = \dfrac{x-a}{\sqrt{(x-a)^2 + {y_0}^2}}$

b) $\dfrac{dm}{dv} = m_0 \cdot \dfrac{-1}{2}(1 - v^2/c^2)^{-3/2} \cdot \dfrac{-2v}{c^2} = \dfrac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}}$

c) $\dfrac{dF}{dr} = mg \cdot (-\dfrac{3}{2})(1 + r^2)^{-5/2} \cdot 2r = \dfrac{-3mgr}{(1 + r^2)^{5/2}}$

d) $\dfrac{dQ}{dt} = at \cdot \dfrac{-6bt}{(1 + bt^2)^4} + \dfrac{a}{(1 + bt^2)^3} = \dfrac{a(1 - 5bt^2)}{(1 + bt^2)^4}$

1F-8 a) $V = \dfrac{1}{3}\pi r^2 h \implies 0 = \dfrac{1}{3}\pi(2rr'h + r^2) \implies r' = \dfrac{-r^2}{2rh} = \dfrac{-r}{2h}$

b) $PV^c = nRT \implies P'V^c + P \cdot cV^{c-1} = 0 \implies P' = -\dfrac{cPV^{c-1}}{V^c} = -\dfrac{cP}{V}$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta$ implies

$$0 = 2aa' + 2b - 2(\cos\theta(a'b + a)) \implies a' = \frac{-2b + 2\cos\theta \cdot a}{2a - 2\cos\theta \cdot b} = \frac{a\cos\theta - b}{a - b\cos\theta}$$

1G. Higher derivatives

1G-1 a) $6 - x^{-3/2}$ b) $\dfrac{-10}{(x+5)^3}$ c) $\dfrac{-10}{(x+5)^3}$ d) 0

1G-2 If $y''' = 0$, then $y'' = c_0$, a constant. Hence $y' = c_0 x + c_1$, where c_1 is some other constant. Next, $y = c_0 x^2/2 + c_1 x + c_2$, where c_2 is yet another constant. Thus, y must be a quadratic polynomial, and any quadratic polynomial will have the property that its third derivative is identically zero.

1G-3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \implies y' = -(b^2/a^2)(x/y)$$

Thus,

$$\begin{aligned} y'' &= -\left(\frac{b^2}{a^2}\right)\left(\frac{y - xy'}{y^2}\right) = -\left(\frac{b^2}{a^2}\right)\left(\frac{y + x(b^2/a^2)(x/y)}{y^2}\right) \\ &= -\left(\frac{b^4}{y^3 a^2}\right)(y^2/b^2 + x^2/a^2) = -\frac{b^4}{a^2 y^3} \end{aligned}$$

1G-4 $y = (x+1)^{-1}$, so $y^{(1)} = -(x+1)^{-2}$, $y^{(2)} = (-1)(-2)(x+1)^{-3}$, and

$$y^{(3)} = (-1)(-2)(-3)(x+1)^{-4}.$$

The pattern is

$$y^{(n)} = (-1)^n (n!)(x+1)^{-n-1}$$

1G-5 a) $y' = u'v + uv' \quad \Longrightarrow \quad y'' = u''v + 2u'v' + uv''$

b) Formulas above do coincide with Leibniz's formula for $n = 1$ and $n = 2$. To calculate $y^{(p+q)}$ where $y = x^p(1+x)^q$, use $u = x^p$ and $v = (1+x)^q$. The only term in the Leibniz formula that is not 0 is $\binom{n}{k} u^{(p)} v^{(q)}$, since in all other terms either one factor or the other is 0. If $u = x^p$, $u^{(p)} = p!$, so

$$y^{(p+q)} = \binom{n}{p} p!q! = \frac{n!}{p!q!} \cdot p!q! = n!$$

1H. Exponentials and Logarithms: Algebra

1H-1 a) To see when $y = y_0/2$, we must solve the equation $\dfrac{y_0}{2} = y_0 e^{-kt}$, or $\frac{1}{2} = e^{-kt}$.

Take ln of both sides: $-\ln 2 = -kt$, from which $t = \dfrac{\ln 2}{k}$.

b) $y_1 = y_0 e^{kt_1}$ by assumption, $\lambda = \dfrac{-\ln 2}{k} y_0 e^{k(t_1+\lambda)} = y_0 e^{kt_1} \cdot e^{k\lambda} = y_1 \cdot e^{-\ln 2} = y_1 \cdot \dfrac{1}{2}$

1H-2 $pH = -\log_{10}[H^+]$; by assumption, $[H^+]_{dil} = \frac{1}{2}[H^+]_{orig}$. Take $-\log_{10}$ of both sides (note that $\log 2 \approx .3$):

$-\log [H^+]_{dil} = \log 2 - \log [H^+]_{orig} \implies pH_{dil} = pH_{orig} + \log_2$.

1H-3 a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$; exponentiating both sides and solving for y:

$(y+1)\cdot(y-1) = e^{2x} \cdot x \implies y^2 - 1 = xe^{2x} \implies y = \sqrt{xe^{2x}+1}$, since $y > 0$.

b) $\log(y+1) - \log(y-1) = -x^2$; exponentiating, $\dfrac{y+1}{y-1} = 10^{-x^2}$. Solve for y; to simplify the algebra, let $A = 10^{-x^2}$. Crossmultiplying, $y + 1 = Ay - A \implies y = \dfrac{A+1}{A-1} = \dfrac{10^{-x^2}+1}{10^{-x^2}-1}$

c) $2\ln y - \ln(y+1) = x$; exponentiating both sides and solving for y:

$\dfrac{y^2}{y+1} = e^x \implies y^2 - e^x y - e^x = 0 \implies y = \dfrac{e^x\sqrt{e^{2x}+4e^x}}{2}$, since $y - 1 > 0$.

1H-4 $\dfrac{\ln a}{\ln b} = c \Rightarrow \ln a = c \ln b \Rightarrow a = e^{c\ln b} = e^{\ln b^c} = b^c$. Similarly, $\dfrac{\log a}{\log b} = c \Rightarrow a = b^c$.

1H-5 a) Put $u = e^x$ (multiply top and bottom by e^x first): $\dfrac{u^2+1}{u^2-1} = y$; this gives $u^2 = \dfrac{y+1}{y-1} = e^{2x}$; taking ln: $2x = \ln(\dfrac{y+1}{y-1})$, $x = \dfrac{1}{2}\ln(\dfrac{y+1}{y-1})$

b) $e^x + e^{-x} = y$; putting $u = e^x$ gives $u + \dfrac{1}{u} = y$; solving for u gives $u^2 - yu + 1 = 0$ so that $u = \dfrac{y \pm \sqrt{y^2-4}}{2} = e^x$; taking ln: $x = \ln(\dfrac{y \pm \sqrt{y^2-4}}{2})$

1H-6 $A = \log e \cdot \ln 10 = \ln(10^{\log e}) = \ln(e) = 1$; similarly, $\log_b a \cdot \log_a b = 1$

1H-7 a) If I_1 is the intensity of the jet and I_2 is the intensity of the conversation, then

$$\log_{10}(I_1/I_2) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}(I_1/I_0) - \log_{10}(I_2/I_0) = 13 - 6 = 7$$

Therefore, $I_1/I_2 = 10^7$.

b) $I = C/r^2$ and $I = I_1$ when $r = 50$ implies

$$I_1 = C/50^2 \implies C = I_1 50^2 \implies I = I_1 50^2/r^2$$

This shows that when $r = 100$, we have $I = I_1 50^2/100^2 = I_1/4$. It follows that

$$10\log_{10}(I/I_0) = 10\log_{10}(I_1/4I_0) = 10\log_{10}(I_1/I_0) - 10\log_{10} 4 \approx 130 - 6.0 \approx 124$$

The sound at 100 meters is 124 decibels.

The sound at 1 km has 1/100 the intensity of the sound at 100 meters, because $100m/1km = 1/10$.

$$10\log_{10}(1/100) = 10(-2) = -20$$

so the decibel level is $124 - 20 = 104$.

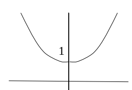
1I. Exponentials and Logarithms: Calculus

1I-1 a) $(x+1)e^x$ b) $4xe^{2x}$ c) $(-2x)e^{-x^2}$ d) $\ln x$ e) $2/x$ f) $2(\ln x)/x$ g) $4xe^{2x^2}$

h) $(x^x)' = \left(e^{x\ln x}\right)' = (x\ln x)'e^{x\ln x} = (\ln x + 1)e^{x\ln x} = (1 + \ln x)x^x$

i) $(e^x - e^{-x})/2$ j) $(e^x + e^{-x})/2$ k) $-1/x$ l) $-1/x(\ln x)^2$ m) $-2e^x/(1+e^x)^2$

1I-2



1I-3 a) As $n \to \infty$, $h = 1/n \to 0$.

$$n\ln(1+\frac{1}{n}) = \frac{\ln(1+h)}{h} = \frac{\ln(1+h) - \ln(1)}{h} \underset{h\to 0}{\longrightarrow} \frac{d}{dx}\ln(1+x)\Big|_{x=0} = 1$$

Therefore,

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = 1$$

b) Take the logarithm of both sides. We need to show

$$\lim_{n\to\infty} \ln(1+\frac{1}{n})^n = \ln e = 1$$

But

$$\ln(1+\frac{1}{n})^n = n\ln(1+\frac{1}{n})$$

so the limit is the same as the one in part (a).

1I-4 a)

$$\left(1+\frac{1}{n}\right)^{3n} = \left(\left(1+\frac{1}{n}\right)^{n}\right)^{3} \longrightarrow e^3 \text{ as } n \to \infty,$$

b) Put $m = n/2$. Then

$$\left(1+\frac{2}{n}\right)^{5n} = \left(1+\frac{1}{m}\right)^{10m} = \left(\left(1+\frac{1}{m}\right)^{m}\right)^{10} \longrightarrow e^{10} \text{ as } m \to \infty$$

c) Put $m = 2n$. Then

$$\left(1+\frac{1}{2n}\right)^{5n} = \left(1+\frac{1}{m}\right)^{5m/2} = \left(\left(1+\frac{1}{m}\right)^{m}\right)^{5/2} \longrightarrow e^{5/2} \text{ as } m \to \infty$$

1J. Trigonometric functions

1J-1 a) $10x\cos(5x^2)$ b) $6\sin(3x)\cos(3x)$ c) $-2\sin(2x)/\cos(2x) = -2\tan(2x)$

d) $-2\sin x/(2\cos x) = -\tan x$. (Why did the factor 2 disappear? Because $\ln(2\cos x) = \ln 2 + \ln(\cos x)$, and the derivative of the constant $\ln 2$ is zero.)

e) $\dfrac{x\cos x - \sin x}{x^2}$ f) $-(1+y')\sin(x+y)$ g) $-\sin(x+y)$ h) $2\sin x\cos x e^{\sin^2 x}$

i) $\dfrac{(x^2\sin x)'}{x^2\sin x} = \dfrac{2x\sin x + x^2\cos x}{x^2\sin x} = \dfrac{2}{x} + \cot x$. Alternatively,

$$\ln(x^2\sin x) = \ln(x^2) + \ln(\sin x) = 2\ln x + \ln\sin x$$

Differentiating gives $\dfrac{2}{x} + \dfrac{\cos x}{\sin x} = \dfrac{2}{x} + \cot x$

j) $2e^{2x}\sin(10x) + 10e^{2x}\cos(10x)$ k) $6\tan(3x)\sec^2(3x) = 6\sin x/\cos^3 x$

l) $-x(1-x^2)^{-1/2}\sec(\sqrt{1-x^2})\tan(\sqrt{1-x^2})$

m) Using the chain rule repeatedly and the trigonometric double angle formulas,

$$\begin{aligned}(\cos^2 x - \sin^2 x)' &= -2\cos x\sin x - 2\sin x\cos x = -4\cos x\sin x;\\ (2\cos^2 x)' &= -4\cos x\sin x;\\ (\cos(2x))' &= -2\sin(2x) = -2(2\sin x\cos x).\end{aligned}$$

The three functions have the same derivative, so they differ by constants. And indeed,

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1, \qquad (\text{using } \sin^2 x = 1 - \cos^2 x).$$

n)

$$5(\sec(5x)\tan(5x))\tan(5x) + 5(\sec(5x)(\sec^2(5x)) = 5\sec(5x)(\sec^2(5x) + \tan^2(5x))$$

Other forms: $5\sec(5x)(2\sec^2(5x) - 1)$; $10\sec^3(5x) - 5\sec(5x)$

o) 0 because $\sec^2(3x) - \tan^2(3x) = 1$, a constant — or carry it out for practice.

p) Successive use of the chain rule:

$$\begin{aligned}(\sin(\sqrt{x^2+1}))' &= \cos(\sqrt{x^2+1}) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}} \cos(\sqrt{x^2+1})\end{aligned}$$

q) Chain rule several times in succession:

$$\begin{aligned}(\cos^2\sqrt{1-x^2})' &= 2\cos\sqrt{1-x^2} \cdot (-\sin\sqrt{1-x^2}) \cdot \frac{-x}{\sqrt{1-x^2}} \\ &= \frac{x}{\sqrt{1-x^2}} \sin(2\sqrt{1-x^2})\end{aligned}$$

r) Chain rule again:

$$\begin{aligned}\left(\tan^2(\frac{x}{x+1})\right) &= 2\tan(\frac{x}{x+1}) \cdot \sec^2(\frac{x}{x+1}) \cdot \frac{x+1-x}{(x+1)^2} \\ &= \frac{2}{(x+1)^2} \tan(\frac{x}{x+1}) \sec^2(\frac{x}{x+1})\end{aligned}$$

1J-2 Because $\cos(\pi/2) = 0$,

$$\lim_{x\to\pi/2} \frac{\cos x}{x-\pi/2} = \lim_{x\to\pi/2} \frac{\cos x - \cos(\pi/2)}{x-\pi/2} = \frac{d}{dx}\cos x|_{x=\pi/2} = -\sin x|_{x=\pi/2} = -1$$

1J-3 a) $(\sin(kx))' = k\cos(kx)$. Hence

$$(\sin(kx))'' = (k\cos(kx))' = -k^2\sin(kx).$$

Similarly, differentiating cosine twice switches from sine and then back to cosine with only one sign change, so

$$(\cos(kx)'' = -k^2\cos(kx)$$

Therefore,

$$\sin(kx)'' + k^2\sin(kx) = 0 \ \text{ and } \ \cos(kx)'' + k^2\cos(kx) = 0$$

Since we are assuming $k > 0$, $k = \sqrt{a}$.

b) This follows from the linearity of the operation of differentiation. With $k^2 = a$,

$$\begin{aligned}&(c_1\sin(kx) + c_2\cos(kx))'' + k^2(c_1\sin(kx) + c_2\cos(kx)) \\ &\quad = c_1(\sin(kx))'' + c_2(\cos(kx))'' + k^2c_1\sin(kx) + k^2c_2\cos(kx) \\ &\quad = c_1[(\sin(kx))'' + k^2\sin(kx)] + c_2[(\cos(kx))'' + k^2\cos(kx)] \\ &\quad = c_1\cdot 0 + c_2\cdot 0 = 0\end{aligned}$$

c) Since ϕ is a constant, $d(kx+\phi)/dx = k$, and $(\sin(kx+\phi)' = k\cos(kx+\phi)$,

$$(\sin(kx+\phi)'' = (k\cos(kx+\phi))' = -k^2\sin(kx+\phi)$$

Therefore, if $a = k^2$,

$$(\sin(kx+\phi)'' + a\sin(kx+\phi) = 0$$

d) The sum formula for the sine function says

$$\sin(kx+\phi) = \sin(kx)\cos(\phi) + \cos(kx)\sin(\phi)$$

In other words

$$\sin(kx+\phi) = c_1\sin(kx) + c_2\cos(kx)$$

with $c_1 = \cos(\phi)$ and $c_2 = \sin(\phi)$.

1J-4 a) The Pythagorean theorem implies that

$$c^2 = \sin^2\theta + (1-\cos\theta)^2 = \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta = 2 - 2\cos\theta$$

Thus,

$$c = \sqrt{2-2\cos\theta} = 2\sqrt{\frac{1-\cos\theta}{2}} = 2\sin(\theta/2)$$

b) Each angle is $\theta = 2\pi/n$, so the perimeter of the n-gon is

$$n\sin(2\pi/n)$$

As $n\to\infty$, $h = 2\pi/n$ tends to 0, so

$$n\sin(2\pi/n) = \frac{2\pi}{h}\sin h = 2\pi\frac{\sin h - \sin 0}{h} \to 2\pi\frac{d}{dx}\sin x|_{x=0} = 2\pi\cos x|_{x=0} = 2\pi$$