G. GRAPHING FUNCTIONS

To get a quick insight int o how the graph of a function looks, it is very helpful to know how certain simple operations on the graph are related to the way the function expression looks. We consider these here.

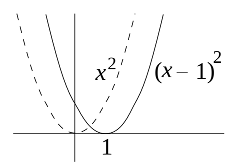
1. Right-left translation.

Let $c > 0$. Start with the graph of some function $f(x)$. Keep the x-axis and y-axis fixed, but move the graph c units to the right, or c units to the left. (See the pictures below.) You get the graphs of two new functions:

$$(1) \qquad \textit{Moving the } f(x) \textit{ graph } c \textit{ units to the } \begin{cases} \textit{right} \\ \textit{left} \end{cases} \quad \textit{gives the graph of} \quad \begin{cases} f(x-c) \\ f(x+c) \end{cases} .$$

If $f(x)$ is given by a formula in x, then $f(x-c)$ is the function obtained by replacing x by $x-c$ wherever it occurs in the formula. For instance,

$$f(x) = x^2 + x \quad \Rightarrow \quad f(x-1) = (x-1)^2 + (x-1) \quad = \quad x^2 - x, \text{ by algebra.}$$

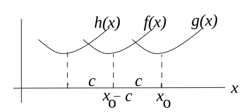


Example 1. Sketch the graph of $f(x) = x^2 - 2x + 1$.

Solution. By algebra, $f(x) = (x-1)^2$. Therefore by (1), its graph is the one obtained by moving the graph of x^2 one unit to the right, as shown. The result is a parabola touching the x-axis at $x = 1$.

To see the reason for the rule (1), suppose the graph of $f(x)$ is moved c units to the right: it becomes then the graph of a new function $g(x)$, whose relation to $f(x)$ is described by (see the picture):

$$\text{value of } g(x) \text{ at } x_0 \quad = \quad \text{value of } f(x) \text{ at } x_0 - c \quad = \quad f(x_0 - c) \; .$$

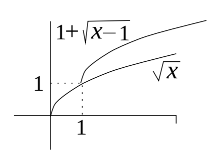


This shows that $g(x) = f(x-c)$. The reasoning is similar if the graph is translated c units to the left. Try giving the argument yourself while referring to the picture.

The effect of **up-down translation** of the graph is much simpler to see. If $c > 0$,

$$(2) \qquad \textit{Moving the } f(x) \textit{ graph } c \textit{ units } \begin{cases} \textit{up} \\ \textit{down} \end{cases} \quad \textit{gives the graph of} \quad \begin{cases} f(x)+c \\ f(x)-c \end{cases} .$$

since for example moving the graph up by c units has the effect of adding c units to each function value, and therefore gives us the graph of the function $f(x) + c$



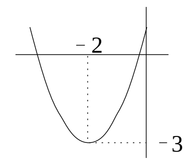
Example 2. Sketch the graph of $1 + \sqrt{x-1}$.

Solution Combine rules (1) and (2). First sketch $\sqrt{x}$, then move its graph 1 unit to the right to get the graph of $\sqrt{x-1}$, then 1 unit up to get the graph of $1 + \sqrt{x-1}$, as shown.

Example 3. Sketch the curve $y = x^2 + 4x + 1$.

Solution We "complete the square":

$$x^2 + 4x + 1 \;=\; (x^2 + 4x + 4) - 3 \;=\; (x + 2)^2 - 3 \;,$$

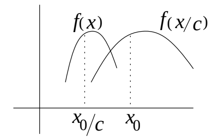


so we move the graph of x^2 to the left 2 units, then 3 units down, getting the graph shown.

2. Changing scale: stretching and shrinking.

Let $c > 1$. To stretch the x-axis by the factor c means to move the point 1 to the position formerly occupied by c, and in general, the point x_0 to the position formerly occupied by cx_0. Similarly, to shrink the x-axis by the factor c means to move x_0 to the position previously taken by x_0/c. What happens to the graph of $f(x)$ when we stretch or shrink the x-axis?

$$(3) \quad \left\{ \begin{array}{l} \textit{Stretching} \\ \textit{Shrinking} \end{array} \right. \quad \textit{the } x\textit{-axis by } c \textit{ changes the graph of } f(x) \textit{ into that of } \left\{ \begin{array}{l} f(x/c) \\ f(cx) \end{array} \right. .$$

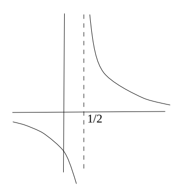


The picture explains this rule; it illustrates stretching by the factor $c > 1$. The new function has the same value at x_0 that $f(x)$ has at x_0/c, so that it is given by the rule $x_0 \to f(x_0/c)$, which means that it is the function $f(x/c)$.

If the y-axis is stretched by the factor $c > 1$, each y-value is multiplied by c, so the new graph is that of the function $cf(x)$:

$$(4) \quad \left\{ \begin{array}{l} \textit{Stretching} \\ \textit{Shrinking} \end{array} \right. \quad \textit{the } y\textit{-axis by } c \textit{ changes the graph of } f(x) \textit{ into that of } \left\{ \begin{array}{l} c\,f(x) \\ f(x)/c \end{array} \right. .$$

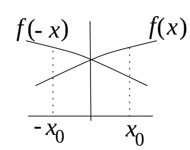
Example 4. Sketch the graph of $\dfrac{1}{2x-1}$.



Solution. Start with the graph of $1/x$, move it 1 unit to the right to get the graph of $1/(x-1)$, then shrink the x-axis by the factor 2 to get the graph of the given function. See the picture.

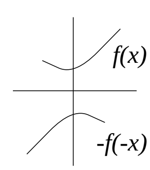
3. Reflecting in the x- and y-axes: even and odd functions.

To reflect the graph of $f(x)$ in the y-axis, just flip the plane over around the y-axis. This carries the point (x, y) into the point $(-x, y)$, and the graph of $f(x)$ into the graph of $f(-x)$. Namely, the new function has the same y-value at x_0 as $f(x)$ has at $-x_0$, so it is given by the rule $x_0 \to f(-x_0)$ and is the function $f(-x)$.



Similarly, reflecting the xy-plane in the x-axis carries (x, y) to the point $(x, -y)$ and the graph of $f(x)$ gets carried into that of $-f(x)$.

Finally, relecting first in the y-axis and then in the x-axis carries the point (x, y) into the point $(-x, -y)$. This is called a *reflection through the origin.* The graph of $f(x)$ gets carried into the graph of $-f(-x)$, by combining the above two results. Summarizing:



$$(5) \qquad \textit{Reflecting in the} \begin{cases} y\textit{-axis} \\ x\textit{-axis} \\ \textit{origin} \end{cases} \textit{moves the graph of } f(x) \textit{ into that of} \begin{cases} f(-x) \\ -f(x) \\ -f(-x). \end{cases}$$

Of importance are those functions $f(x)$ whose graphs are symmetric with respect to the y-axis — that is, reflection in the y-axis doesn't change the graph; such functions are called **even**. Functions whose graphs are symmetric with respect to the origin are called **odd**. In terms of their expression in x,

(6) $f(-x) = f(x)$ *definition of even function*

(7) $f(-x) = -f(x)$ *definition of odd function*

Example 5. Show that a polynomial with only even powers, like $x^4 - 2x^2 + 7$, is an even function, and a polynomial with only odd powers, like $3x^5 - x^3 + 2x$, is an odd function — this, by the way, explains the terminology "even" and "odd" used for functions.

Solution. We have to show (6) and (7) hold for polynomials with respectively only even or odd powers, but this follows immediately from the fact that for any non-negative integer n, we have

$$(-x)^n = (-1)^n x^n = \begin{cases} x^n, & \text{if } n \text{ is even,} \\ -x^n, & \text{if } n \text{ is odd.} \end{cases}$$ □

The following easily proved rules predict the odd- or even-ness of the product or quotient of two odd or even functions:

(8) *even · even = even* *odd · odd = even* *odd · even = odd*

(9) *even/even = even* *odd/odd = even* *odd/even = odd*

Example 6. $\dfrac{x^3}{1-x^2}$ is of the form *odd/even*, therefore it is odd;

$(3 + x^4)^{1/2}(x - x^3)$ has the form *even · odd*, so it is odd.

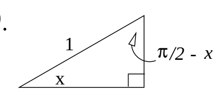
4. The trigonometric functions.

The trigonometric functions offer further illustrations of the ideas about translation, change of scale, and symmetry that we have been discussing. Your book reviews the standard facts about them in section 9.1, which you should refer to as needed.

The graphs of $\sin x$ and $\cos x$ are crudely sketched below. (In calculus, the variable x is always to be in radians; review radian measure in section 9.1 if you have forgotten it. Briefly, there are 2π radians in a 360^o angle, so that for example a right angle is $\pi/2$ radians.)

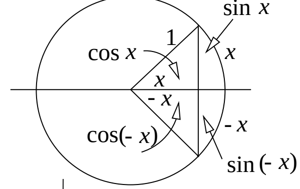
As the graphs suggest and the unit circle picture shows,

(10) $\cos(-x) = \cos x$ *(even function)* $\sin(-x) = -\sin x$ *(odd function).*



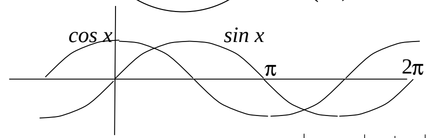
From the standard triangle at the right, one sees that

$$\cos(\pi/2 - x) = \sin x ,$$



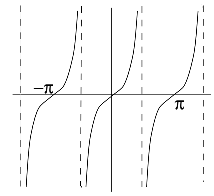
and since $\cos x$ is an even function, this shows that

(11) $$\cos(x - \pi/2) \;=\; \sin x\ .$$



From (11), we see that moving the graph of $\cos x$ to the right by $\pi/2$ units turns it into the graph of $\sin x$. (See picture.)

The trigonometric function



(12) $$\tan x \;=\; \frac{\sin x}{\cos x}$$

is also important; its graph is sketched at the right. It is an odd function, by (9) and (10), since it has the form odd/even.

Periodicity

An important property of the trigonometric functions is that they repeat their values:

(13) $$\sin(x + 2\pi) = \sin x, \qquad\qquad \cos(x + 2\pi) = \cos x\ .$$

This is so because $x + 2\pi$ and x represent in radians the same angle.

From the graphical point of view, equations (13) say that if we move the graph of $\sin x$ or $\cos x$ to the left by 2π units, it will coincide with itself.

From the function viewpoint, equations (13) say that $\sin x$ and $\cos x$ are *periodic* functions, with period 2π. In general, let $c > 0$; we say that $f(x)$ is **periodic**, with **period** c, if

(14) $$f(x + c) = f(x) \quad \text{for all } x, \text{ and}$$

(14') c is the smallest positive number for which (14) is true.

By rule (1), the graph of a periodic function having period c coincides with itself when it is translated c units to the left. If we replace x by $x - c$ in (14), we see that the graph will also coincide with itself if it is moved to the right by c units. But beware: if a function is made by combining other periodic functions, you cannot always predict the period. For example, although it is true that

$$\tan(x + 2\pi) = \tan x \qquad \text{and} \qquad \cos^2(x + 2\pi) = \cos^2 x\ ,$$

the period of both $\tan x$ and $cos^2 x$ is actually π, as the above figure suggests for $\tan x$.

The general sinusoidal wave.

The graph of $\sin x$ is referred to as a "pure wave" or a "sinusoidal oscillation". We now consider to what extent we can change how it looks by applying the geometric operations of translation and scale change discussed earlier.

a) Start with $\sin x$, which has period 2π and oscillates between ± 1.

b) Stretch the y axis by the factor $A > 0$; by (4) this gives $A \sin x$, which has period 2π and oscillates between $\pm A$.

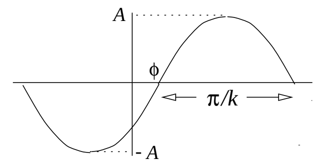
c) Shrink the x-axis by the factor $k > 0$; by (3), this gives $A \sin kx$, which has period $2\pi/k$, since

$$A \sin k(x + \frac{2\pi}{k}) \;=\; A \sin(kx + 2\pi) = A \sin kx.$$

d) Move the graph ϕ units to the right; by (1), this gives

(15) $A \sin k(x - \phi)$, $A, k > 0, \phi \geq 0,$ *general sinusoidal wave*

which has



period $2\pi/k$	(the wave repeats itself every $2\pi/k$ units);
angular frequency k	(has k complete cycles as x goes from 0 to 2π);
amplitude A	(the wave oscillates between A and $-A$);
phase angle ϕ	(the midpoint of the wave is at $x = \phi$).

Notice that the function (15) depends on three constants: k, A, and ϕ. We call such constants *parameters*; their value determines the shape and position of the wave.

By using trigonometric identities, it is possible to write (15) in another form, which also has three parameters:

(16) $$a \sin kx + b \cos kx$$

The relation between the parameters in the two forms is:

(17) $$a = A \cos k\phi, \quad b = -A \sin k\phi; \qquad A = \sqrt{a^2 + b^2}, \quad \tan k\phi = -\frac{b}{a} \ .$$

Proof of the equivalence of (15) and (16).

(15) $\Rightarrow$ (16): from the identity $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, we get

$$A \sin(k(x - \phi)) = A \sin(kx - k\phi) = A \cos k\phi \sin kx - A \sin k\phi \cos kx$$

which has the form of (16), with the values for a and b given in (17).

(16) $\Rightarrow$ (15): square the two equations on the left of (17) and add them; this gives

$$a^2 + b^2 = A^2(\cos^2 k\phi + \sin^2 k\phi) = A^2, \text{ showing that } A = \sqrt{a^2 + b^2} \ .$$

If instead we take the ratio of the two equations on the left of (17), we get $-b/a = \tan k\phi$, as promised. □

Example 7. Find the period, frequency, amplitude, and phase angle of the wave represented by the functions

a) $2 \sin(3x - \pi/6)$ b) $-2 \cos(2x - \pi/2)$

Solution.

a) Writing the function in the form (15), we get $2 \sin 3(x - \pi/18)$, which shows it has period $2\pi/3$, frequency 3, amplitude 2, and phase angle $\pi/18$ (or 10^o).

b) We get rid of the $-$ sign by using $-\cos x = \cos(x - \pi)$ — translating the cosine curve π units to the right is the same as reflecting it in the x-axis (this is the best way to remember such relations). We get then

$$\begin{aligned} -2\cos(2x - \pi/2) &= 2\cos(2x - \pi/2 - \pi) \\ &= 2\sin(2x - \pi), \qquad \text{by (11);} \\ &= 2 \sin 2(x - \pi/2). \end{aligned}$$

Thus the period is π, the frequency 2, the amplitude 2, and the phase angle $\pi/2$. (Note that the first three could have been read off immediately without making the above transformation.)

Example 8. Sketch the curve $\sin 2x + \cos 2x$.

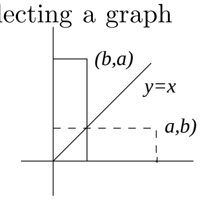
Solution Transforming it into the form (15), we can get A and ϕ by using (19):

$$A = \sqrt{2}; \qquad \tan 2\phi = -1 \quad \Rightarrow \quad 2\phi = 135^o = 3\pi/4, \quad \Rightarrow \quad \phi = 3\pi/8 \ .$$

So the function is also representable as $\sqrt{2}\sin 2(x - 3\pi/8)$; it is a wave of amplitude $\sqrt{2}$, period π, frequncy 2, and phase angle $3\pi/8$, and can be sketched using this data.

5. Reflection in the diagonal line; inverse functions.

As our final geometric operation on graphs, we consider the effect of reflecting a graph in the diagonal line $y = x$.



This reflection can be carried out by flipping the plane over about the diagonal line. Each point of the diagonal stays fixed; the x-and y-axes are interchanged. The points (a, b) and (b, a) are interchanged, as the picture shows, because the two rectangles are interchanged.

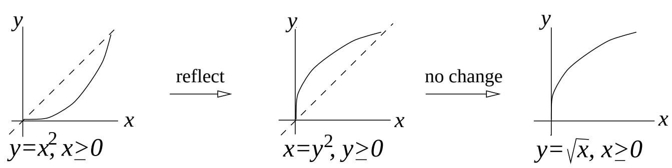
To see the effect of this on the function, let's consider first a simple example.

Example 9. If the graph of $f(x) = x^2, \quad x \geq 0$ is reflected in the diagonal, what function corresponds to the reflected graph?

Solution. The original curve is the graph of the equation: $y = x^2, \quad x \geq 0$.

Reflection corresponds to interchanging the two axes; thus the reflected curve is the graph of the equation: $x = y^2, \quad y \geq 0$.

To find the corresponding function, we have to express y explicitly in terms of x, which we do by solving the equation for y: $y = \sqrt{x}, \quad x \geq 0$; the restriction on x follows because if $x = y^2$ and $y \geq 0$, then $x \geq 0$ also.



Remarks.

1. When we flip the curve about the diagonal line, we do not interchange the labels on the x- and y-axes. The coordinate axes remain the same — it is only the curve that is moved (imagine it drawn on an overhead-projector transparency, and the transparency flipped over). This is analogous to our discussion in section 1 of translation, where the curve was moved to the right, but the coordinate axes themselves remained unchanged.

2. It was necessary in the previous example to restrict the domain of x in the original function x^2, so that after being flipped, its graph was still the graph of a function. If we

hadn't, the flipped curve would have been a parabola lying on its side; this is not the graph of a function, since it has two y-values over each x-value.

The function having the reflected graph, $y = \sqrt{x}, \quad x \geq 0$ is called the *inverse function* to the original function $y = x^2, \quad x \geq 0$. The general procedure may be represented schematically by:

$y = f(x)$	$\longrightarrow$	$x = f(y)$	$\longrightarrow$	$y = g(x)$
original graph	switch x and y	reflected graph	solve for y	reflected graph

In this scheme, the equations $x = f(y)$ and $y = g(x)$ *have the same graph;* all that has been done is to transform the equation algebraically, so that y appears as an explicit function of x. This function $g(x)$ is called the **inverse function to** $f(x)$ over the given interval; in general it will be necessary to restrict the domain of $f(x)$ to an interval, so that the reflected graph will be the graph of a function.

To summarize: $f(x)$ and $g(x)$ are inverse functions if

(i) *geometrically*, the graphs of $f(x)$ and $g(x)$ are reflections of each other in the diagonal line $y = x$;

(ii) *analytically*, $x = f(y)$ and $y = g(x)$ are equivalent equations, either arising from the other by solving explicitly for the relevant variable.

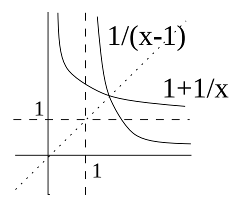
Example 10. Find the inverse function to $\dfrac{1}{x-1}, \quad x > 1$.

Solution. We introduce a dependent variable y, then interchange x and y, getting

$$x = \frac{1}{y-1}, \qquad y > 1 .$$

We solve this algebraically for y, getting

(20) $$y = 1 + \frac{1}{x}, \qquad x > 0 .$$



(The domain is restricted because if $y > 1$, then equation (20) implies that $x > 0$.) The right side of (20) is the desired inverse function. The graphs are sketched.

It often happens that in determining the inverse to $f(x)$, the equation

(21) $$x = f(y)$$

cannot be solved explicitly in terms of previously known functions. In that case, the corresponding equation

(22) $$y = g(x)$$

is viewed as *defining* the inverse function to $f(x)$, when taken with (21). Once again, care must be taken to restrict the domain of $f(x)$ as necessary to ensure that the relected will indeed define a function $g(x)$, i.e., will not be multiple-valued. A typical example is the following.

Example 11. Find the inverse function to $\sin x$.

Solution. Considering its graph, we see that for the reflected graph to define a function, we have to restrict the domain. The most natural choice is to consider the restricted function

$$y = \sin x, \qquad -\pi/2 \le x \le \pi/2. \tag{23}$$

The inverse function is then denoted $\sin^{-1} x$, or sometimes Arcsin x; it is defined by the pair of equivalent equations

$$x = \sin y, \quad -\pi/2 \le x \le \pi/2 \quad \Longleftrightarrow \quad y = \sin^{-1} x, \quad -1 \le x \le 1. \tag{24}$$

The domain $[-1, 1]$ of $\sin^{-1} x$ is evident from the picture — it is the same as the range of $\sin x$ over $[-\pi/2, \pi/2]$.

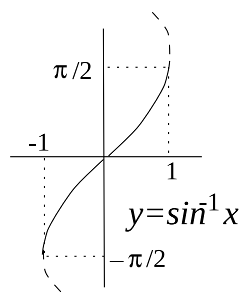
As examples of its values, $\sin^{-1} = \pi/2$, since $\sin \pi/2 = 1$; similarly, $\sin^{-1} 1/2 = \pi/6$.

Care is needed in handling this function. For example, substituting the left equation in (24) into the right equation says that

$$\sin^{-1}(\sin y) \ = \ y, \qquad -\pi/2 \le y \le \pi/2 \ . \tag{25}$$

It is common to see the restriction on y carelessly omitted, since the equation by itself seems "obvious". But without the restriction, it is not even true; for example if $y = \pi$,

$$\sin^{-1}(\sin \pi) = 0.$$



Exercises: Section 1A