Unit 6. Additional Topics

6A. Indeterminate forms; L'Hospital's rule

6A-1 Find the following limits

a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$ b) $\displaystyle\lim_{x\to 0} \frac{\cos(x/2)-1}{x^2}$ c) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x}$

d) $\displaystyle\lim_{x\to 0} \frac{x^2-3x-4}{x+1}$ e) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x}{5x}$ f) $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$

g) $\displaystyle\lim_{x\to 1} \frac{x^a-1}{x^b-1}$ h) $\displaystyle\lim_{x\to 1} \frac{\tan(x)}{\sin(3x)}$ i) $\displaystyle\lim_{x\to \pi} \frac{\ln\sin(x/2)}{x-\pi}$

j) $\displaystyle\lim_{x\to \pi} \frac{\ln\sin(x/2)}{(x-\pi)^2}$

6A-2 Evaluate the following limits.

a) $\displaystyle\lim_{x\to 0^+} x^x$ b) $\displaystyle\lim_{x\to 0^+} x^{1/x}$ c) $\displaystyle\lim_{x\to 0^+} (1/x)^{\ln x}$

d) $\displaystyle\lim_{x\to 0^+} (\cos x)^{1/x}$ e) $\displaystyle\lim_{x\to \infty} x^{1/x}$ f) $\displaystyle\lim_{x\to 0^+} (1+x^2)^{1/x}$

g) $\displaystyle\lim_{x\to 0^+} (1+3x)^{10/x}$ h) $\displaystyle\lim_{x\to \infty} \frac{x+\cos x}{x}$ i) $\displaystyle\lim_{x\to \infty} x\sin\frac{1}{x}$

j) $\displaystyle\lim_{x\to 0^+} \left(\frac{x}{\sin x}\right)^{1/x^2}$ k) $\displaystyle\lim_{x\to \infty} x^a(\ln x)^b$. Consider all values of a and b.

6A-3 The power x^{-1} is the exceptional case among the integrals of the powers of x. It would be nice if

$$\lim_{a\to -1} \int x^a\,dx = \int x^{-1}\,dx$$

It seems hopeless for this to be true[1] since

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + c \quad \text{for} \quad a \neq -1$$

involves only powers, yet the integral of x^{-1} is a logarithm. But it can be rescued using the definite integral. Show using L'Hospital's rule that

$$\lim_{a\to -1} \int_1^x t^a\,dt = \int_1^x t^{-1}\,dt \quad (= \ln x)$$

6A-4 Show that as a tends to -1 of a well-chosen solution to E30/1(a) tends to the answer in part (b). Hint: Follow the method of the preceding problem.

6A-5 By repeated use of L'Hospital's rule,

$$\lim_{x\to 0} \frac{3x^2-4x}{2x-x^2} = \lim_{x\to 0} \frac{6x-4}{2-2x} = \lim_{x\to 0} \frac{6}{-2} = -3,$$

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[1]It seems hopeless because for almost all choices of c the indefinite integral has an infinite limit as $a \to -1$. The definite integral leads to the correct choice of c, namely, $c = -1/(a+1)$. The constant c is a constant with respect to x, but there is no reason why it can't vary with a. And the right choice of c makes the limit as $a \to -1$ finite.

yet when $x \simeq 0$, $\dfrac{3x^2 - 4x}{2x - x^2} \simeq \dfrac{-4x}{2x} = -2$. Resolve the contradiction.

6A-6 Graph the following functions. (L'Hospital's rule will help with some of the limiting values at the ends.)

a) $y = xe^{-x}$ b) $y = x \ln x$ c) $y = x/\ln x$

6B. Improper integrals

Test the following improper integrals for convergence by using comparison with a simpler integral.

6B-1. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+5}}$ **6B-2.** $\displaystyle\int_0^\infty \frac{x^2dx}{x^3+2}$ **6B-3.** $\displaystyle\int_0^1 \frac{dx}{x^3+x^2}$

6B-4. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ **6B-5.** $\displaystyle\int_0^\infty \frac{e^{-x}dx}{x}$ **6B-6.** $\displaystyle\int_1^\infty \frac{\ln x dx}{x^2}$

6B-7 Decide whether the following integrals are convergent or divergent and evaluate if convergent.

a) $\displaystyle\int_0^\infty e^{-8x}dx$ b) $\displaystyle\int_1^\infty x^{-n}dx,\ n > 1$ c) $\displaystyle\int_1^\infty x^{-n}dx,\ 0 < n \le 1$

d) $\displaystyle\int_0^2 \frac{xdx}{\sqrt{4-x^2}}$ e) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2-x}}$ f) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$

g) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ h) $\displaystyle\int_0^1 \frac{dx}{x^3}$ i) $\displaystyle\int_{-1}^1 \frac{dx}{x}$

j) $\displaystyle\int_0^1 \ln x dx$ k) $\displaystyle\int_0^\infty e^{-2x}\cos x dx$ l) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)}$. (Use (f).)

m) $\displaystyle\int_0^\infty \frac{dx}{(x+2)^3}$ n) $\displaystyle\int_0^\infty \frac{dx}{(x-2)^3}$ o) $\displaystyle\int_0^{10} \frac{(\ln x)^2}{x}dx$

p) $\displaystyle\int_0^\pi \sec x dx$

6B-8 Find the following limits. (Use the fundamental theorem of calculus.)

a) $\displaystyle\lim_{x\to\infty} e^{-x^2}\int_0^x e^{t^2}dt$ b) $\displaystyle\lim_{x\to\infty} xe^{-x^2}\int_0^x e^{t^2}dt$ c) $\displaystyle\lim_{x\to\infty} e^{x^2}\int_0^x e^{-t^2}dt$

d) $\displaystyle\lim_{a\to 0^+} \sqrt{a}\int_a^1 \frac{dx}{\sqrt{x}}$ e) $\displaystyle\lim_{a\to 0^+} \sqrt{a}\int_a^1 \frac{dx}{x^{3/2}}$ f) $\displaystyle\lim_{b\to(\pi/2)^+} (b-\pi/2)\int_0^b \frac{dx}{1-\sin x}$

6C. Infinite Series

6C-1 Find the sum of the following geometric series:

a) $1 + 1/5 + 1/25 + \cdots$ b) $8 + 2 + 1/2 + \cdots$ c) $1/4 + 1/5 + \cdots$

Write the two following infinite decimals as the quotient of two integers:

d) $0.4444\ldots$ e) $0.0602602602602\ldots$

6C-2 Decide whether the following series are convergent or divergent; indicate reasoning. (Do not evaluate the sum.)

a) $1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$; use comparison with an integral.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$; consider the cases $p > 1$ and $p \leq 1$.

c) $1/2 + 1/4 + 1/6 + 1/8 + \cdots$

d) $1 + 1/3 + 1/5 + 1/7 + \cdots$

e) $1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots$ Hint: Combine pairs of consecutive terms to take advantage of the cancellation. Then use comparison.

f) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n!}$. g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{5}-1}{2}\right)^n$. h) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{5}+1}{2}\right)^n 5^{-n/2}$.

i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. j) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. k) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^4-5}$.

l) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^{1/3}}{(n^4+5)^{1/3}}$. m) $\displaystyle\sum_{n=1}^{\infty} \ln(\cos\frac{1}{n})$ n) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

o) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-\sqrt{n}}$

6C-3 a) Use the upper and lower Riemann sums of

$$\ln n = \int_1^n \frac{dx}{x}$$

to show that

$$\ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

b) Suppose that it takes 10^{-10} seconds for a computer to add one term in the series $\sum 1/n$. About how long would it take for the partial sum to reach 1000?