Unit 5. Integration techniques

5A. Inverse trigonometric functions; Hyperbolic functions

5A-1 Evaluate

a) $\tan^{-1}\sqrt{3}$ b) $\sin^{-1}(\sqrt{3}/2)$

c) If $\theta = \tan^{-1} 5$, then evaluate $\sin\theta$, $\cos\theta$, $\cot\theta$, $\csc\theta$, and $\sec\theta$.

d) $\sin^{-1}\cos(\pi/6)$ e) $\tan^{-1}\tan(\pi/3)$

f) $\tan^{-1}\tan(2\pi/3)$ g) $\lim_{x\to-\infty} \tan^{-1} x$.

5A-2 Calculate

a) $\displaystyle\int_1^2 \frac{dx}{x^2+1}$ b) $\displaystyle\int_b^{2b} \frac{dx}{x^2+b^2}$ c) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$.

5A-3 Calculate the derivative with respect to x of the following

a) $\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$ b) $\tanh x$

c) $\ln(x+\sqrt{x^2+1})$ d) y such that $\cos y = x$, $0 \le x \le 1$ and $0 \le y \le \pi/2$.

e) $\sin^{-1}(x/a)$ f) $\sin^{-1}(a/x)$

g) $\tan^{-1}(x/\sqrt{1-x^2})$ h) $\sin^{-1}\sqrt{1-x}$

5A-4 a) If the tangent line to $y = \cosh x$ at $x = a$ goes through the origin, what equation must a satisfy?

b) Solve for a using Newton's method.

5A-5 a) Sketch the graph of $y = \sinh x$, by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$, and sketch its graph, indicating the domain of definition. (The inverse hyperbolic sine.)

c) Find $\dfrac{d}{dx}\sinh^{-1} x$.

d) Use your work to evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$

5A-6 a) Find the average value of y with respect to arclength on the semicircle $x^2+y^2=1$, $y>0$, using polar coordinates.

b) A weighted average of a function is

$$\int_a^b f(x)w(x)dx \Big/ \int_a^b w(x)dx$$

Do part (a) over again expressing arclength as $ds = w(x)dx$. The change of variables needed to evaluate the numerator and denominator will bring back part (a).

c) Find the average height of $\sqrt{1-x^2}$ on $-1 < x < 1$ with respect to dx. Notice that this differs from part (b) in both numerator and denominator.

5B. Integration by direct substitution

Evaluate the following integrals

5B-1. $\int x\sqrt{x^2-1}dx$ 5B-2. $\int e^{8x}dx$ 5B-3. $\int \frac{\ln x dx}{x}$

5B-4. $\int \frac{\cos x dx}{2+3\sin x}$ 5B-5. $\int \sin^2 x\cos x dx$ 5B-6. $\int \sin 7x dx$

5B-7. $\int \frac{6x dx}{\sqrt{x^2+4}}$ 5B-8. $\int \tan 4x dx$ 5B-9. $\int e^x(1+e^x)^{-1/3}dx$

5B-10. $\int \sec 9x dx$ 5B-11. $\int \sec^2 9x dx$ 5B-12. $\int xe^{-x^2}dx$

5B-13. $\int \frac{x^2dx}{1+x^6}$. Hint: Try $u = x^3$.

Evaluate the following integrals by substitution and changing the limits of integration.

5B-14. $\int_0^{\pi/3} \sin^3 x\cos x dx$ 5B-15. $\int_1^e \frac{(\ln x)^{3/2}dx}{x}$ 5B-16. $\int_{-1}^1 \frac{\tan^{-1} x dx}{1+x^2}$

5C. Trigonometric integrals

Evaluate the following

5C-1. $\int \sin^2 x dx$ 5C-2. $\int \sin^3(x/2)dx$ 5C-3. $\int \sin^4 x dx$

5C-4. $\int \cos^3(3x)dx$ 5C-5. $\int \sin^3 x\cos^2 x dx$ 5C-6. $\int \sec^4 x dx$

5C-7. $\int \sin^2(4x)\cos^2(4x)dx$ 5C-8. $\int \tan^2(ax)\cos(ax)dx$ 5C-9. $\int \sin^3 x\sec^2 x dx$

5C-10. $\int (\tan x+\cot x)^2dx$ 5C-11. $\int \sin x\cos(2x)dx$ (Use double angle formula.)

5C-12. $\int_0^{\pi} \sin x\cos(2x)dx$ (See 27.)

5C-13. Find the length of the curve $y = \ln\sin x$ for $\pi/4 \le x \le \pi/2$.

5C-14. Find the volume of one hump of $y = \sin ax$ revolved around the x-axis.

5D. Integration by inverse substitution

Evaluate the following integrals

5D-1. $\int \frac{dx}{(a^2-x^2)^{3/2}}$ 5D-2. $\int \frac{x^3dx}{\sqrt{a^2-x^2}}$ 5D-3. $\int \frac{(x+1)dx}{4+x^2}$

5D-4. $\int \sqrt{a^2+x^2}dx$ 5D-5. $\int \frac{\sqrt{a^2-x^2}dx}{x^2}$ 5D-6. $\int x^2\sqrt{a^2+x^2}dx$

(For 5D-4,6 use $x = a\sinh y$, and $\cosh^2 y = (\cosh(2y)+1)/2$, $\sinh 2y = 2\sinh y\cosh y$.)

5D-7. $\int \frac{\sqrt{x^2-a^2}dx}{x^2}$ 5D-8. $\int x\sqrt{x^2-9}dx$

5D-9. Find the arclength of $y = \ln x$ for $1 \le x \le b$.

Completing the square

Calculate the following integrals

5D-10. $\displaystyle\int \frac{dx}{(x^2+4x+13)^{3/2}}$ 5D-11. $\displaystyle\int x\sqrt{-8+6x-x^2}dx$ 5D-12. $\displaystyle\int \sqrt{-8+6x-x^2}dx$

5D-13. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$ 5D-14. $\displaystyle\int \frac{xdx}{\sqrt{x^2+4x+13}}$ 5D-15. $\displaystyle\int \frac{\sqrt{4x^2-4x+17dx}}{2x-1}$

5E. Integration by partial fractions

5E-1. $\displaystyle\int \frac{dx}{(x-2)(x+3)}dx$ 5E-2. $\displaystyle\int \frac{xdx}{(x-2)(x+3)}dx$ 5E-3. $\displaystyle\int \frac{xdx}{(x^2-4)(x+3)}dx$

5E-4. $\displaystyle\int \frac{3x^2+4x-11}{(x^2-1)(x-2)}dx$ 5E-5. $\displaystyle\int \frac{3x+2}{x(x+1)^2}dx$ 5E-6. $\displaystyle\int \frac{2x-9}{(x^2+9)(x+2)}dx$

5E-7 The equality (1) of Notes F is valid for $x \neq 1, -2$. Therefore, the equality (4) is also valid only when $x \neq 1, -2$, since it arises from (1) by multiplication. Why then is it legitimate to substitute $x = 1$ into (4)?

5E-8 Express the following as a sum of a polynomial and a proper rational function

a) $\dfrac{x^2}{x^2-1}$ b) $\dfrac{x^3}{x^2-1}$ c) $\dfrac{x^2}{3x-1}$

d) $\dfrac{x+2}{3x-1}$ e) $\dfrac{x^8}{(x+2)^2(x-2)^2}$ (just give the form of the solution)

5E-9 Integrate the functions in Problem **5E-8**.

5E-10 Evaluate the following integrals

a) $\displaystyle\int \frac{dx}{x^3-x}$ b) $\displaystyle\int \frac{(x+1)dx}{(x-2)(x-3)}$ c) $\displaystyle\int \frac{(x^2+x+1)dx}{x^2+8x}$

d) $\displaystyle\int \frac{(x^2+x+1)dx}{x^2+8x}$ e) $\displaystyle\int \frac{dx}{x^3+x^2}$ f) $\displaystyle\int \frac{(x^2+1)dx}{x^3+2x^2+x}$

g) $\displaystyle\int \frac{x^3dx}{(x+1)^2(x-1)}$ h) $\displaystyle\int \frac{(x^2+1)dx}{x^2+2x+2}$

5E-11 Solve the differential equation $dy/dx = y(1-y)$.

5E-12 This problem shows how to integrate any rational function of $\sin\theta$ and $\cos\theta$ using the substitution $z = \tan(\theta/2)$. The integrand is transformed into a rational function of z, which can be integrated using the method of partial fractions.

a) Show that

$$\cos\theta = \frac{1-z^2}{1+z^2}, \quad \sin\theta = \frac{2z}{1+z^2}, \quad d\theta = \frac{2dz}{1+z^2}.$$

Calculate the following integrals using the substitution $z = \tan(\theta/2)$ of part (a).

b) $\displaystyle\int_0^\pi \frac{d\theta}{1+\sin\theta}$ c) $\displaystyle\int_0^\pi \frac{d\theta}{(1+\sin\theta)^2}$ d) $\displaystyle\int_0^\pi \sin\theta d\theta$ (Not the easiest way!)

5E-13 a) Use the polar coordinate formula for area to compute the area of the region $0 < r < 1/(1+\cos\theta)$, $0 \le \theta \le \pi/2$. Hint: Problem 12 shows how the substitution $z = \tan(\theta/2)$ allows you to integrate any rational function of a trigonometric function.

b) Compute this same area using rectangular coordinates and compare your answers.

5F. Integration by parts. Reduction formulas

Evaluate the following integrals

5F-1 a) $\displaystyle\int x^a \ln x dx$ $(a \neq -1)$ b) Evaluate the case $a = -1$ by substitution.

5F-2 a) $\displaystyle\int xe^x dx$ b) $\displaystyle\int x^2 e^x dx$ c) $\displaystyle\int x^3 e^x dx$

d) Derive the reduction formula expressing $\displaystyle\int x^n e^{ax} dx$ in terms of $\displaystyle\int x^{n-1} e^{ax} dx$.

5F-3 Evaluate $\displaystyle\int \sin^{-1}(4x) dx$

5F-4 Evaluate $\displaystyle\int e^x \cos x dx$. (Integrate by parts twice.)

5F-5 Evaluate $\displaystyle\int \cos(\ln x) dx$. (Integrate by parts twice.)

5F-6 Show the substitution $t = e^x$ transforms the integral $\displaystyle\int x^n e^x dx$, into $\displaystyle\int (\ln t)^n dt$. Use a reduction procedure to evaluate this integral.