Unit 4. Applications of integration

4A. Areas between curves.

4A-1 Find the area between the following curves

a) $y = 2x^2$ and $y = 3x - 1$ b) $y = x^3$ and $y = ax$; assume $a > 0$

c) $y = x + 1/x$ and $y = 5/2$. d) $x = y^2 - y$ and the y axis.

4A-2 Find the area under the curve $y = 1 - x^2$ in two ways.

4A-3 Find the area between the curves $y = 4 - x^2$ and $y = 3x$ in two ways.

4A-4 Find the area between $y = \sin x$ and $y = \cos x$ from one crossing to the next.

4B. Volumes by slicing; volumes of revolution

4B-1 Find the volume of the solid of revolution generated by rotating the regions bounded by the curves given around the x-axis.

a) $y = 1 - x^2$, $y = 0$ b) $y = a^2 - x^2$, $y = 0$ c) $y = x$, $y = 0$, $x = 1$

d) $y = x$, $y = 0$, $x = a$ e) $y = 2x - x^2$, $y = 0$ f) $y = 2ax - x^2$, $y = 0$

g) $y = \sqrt{ax}$, $y = 0$, $x = a$ h) $x^2/a^2 + y^2/b^2 = 1$, $x = 0$

4B-2 Find the volume of the solid of revolution generated by rotating the regions in 4B-1 around the y-axis.

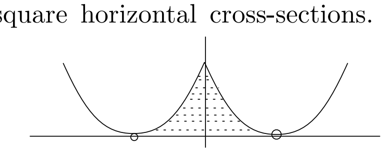
4B-3 Show that the volume of a pyramid with a rectangular base is $bh/3$, where b is the area of the base and h is the height. (Show in the process that the proportions of the rectangle do not matter.)

4B-4 Consider (x, y, z) such that $x^2 + y^2 < 1$, $x > 0$ and $0 \le z \le 5$. This describes one half of cylinder (a split log). Chop out a wedge out of the log along $z = 2x$. Find the volume of the wedge.

4B-5 Find the volume of the solid obtained by revolving an equilateral triangle of sidelength a around one of its sides.

4B-6 The base of a solid is the disk $x^2 + y^2 \le a^2$. Planes perpendicular to the xy-plane and perpendicular to the x-axis slice the solid in isoceles right triangles. The hypotenuse of these trianglesis the segment where the plane meets the disk. What is the volume of the solid?

4B-7 A tower is constructed with a square base and square horizontal cross-sections. Viewed from any direction perpendicular to a side, the tower has base $y = 0$ and profile lines $y = (x - 1)^2$ and $y = (x + 1)^2$. (See shaded region in picture.) Find the volume of the solid.



4C. Volumes by shells

4C-1 Assume that $0 < a < b$. Revolve the disk $(x-b)^2 + y^2 \le a^2$ around the y axis. This doughnut shape is known as a torus.

a) Set up the integral for volume using integration dx

b) Set up the integral for volume using integration dy

c) Evaluate (b).

d) (optional) Show that the (a) and (b) are the same using the substitution $z = x - b$.

4C-2 Find the volume of the region $0 \le y \le x^2$, $x \le 1$ revolved around the y-axis.

4C-3 Find the volume of the region $\sqrt{x} \le y \le 1$, $x \ge 0$ revolved around the y-axis by both the method of shells and the method of disks and washers.

4C-4 Set up the integrals for the volumes of the regions in 4B-1 by the method of shells. (Do not evaluate.)

4C-5 Set up the integrals for the volumes of the regions in 4B-2 by the method of shells. (Do not evaluate.)

4C-6 Let $0 < a < b$. Consider a ball of radius b and a cylinder of radius a whose axis passes through the center of the ball. Find the volume of the ball with the cylinder removed.

4D. Average value

4D-1 What is the average cross-sectional area of the solid obtained by revolving the region bounded by $x = 2$, the x-axis, and the curve $y = x^2$ about the x-axis? (Cross-sections are taken perpendicular to the x-axis.)

4D-2 Show that the average value of $1/x$ over the interval $[a, 2a]$ is of the form C/a, where C is a constant independent of a. (Assume $a > 0$.)

4D-3 A point is moving along the x-axis; the functions $x = s(t)$ and $v(t)$ give respectively its position and velocity at time t. Show that over a time interval $a \le t \le b$, the average value of the function $v(t)$ equals the average velocity of the point over this interval, as the uncalculused would calculate it.

4D-4 What is the average value of the square of the distance of a point P from a fixed point Q on the unit circle, where P is chosen at random on the circle? (Use coordinates; place Q on the x-axis.) Check your answer for reasonableness.

4D-5 If the average value of $f(t)$ between 0 and x is given by the function $g(x)$, express $f(x)$ in terms of $g(x)$.

4D-6 An amount of money A compounded continuously at interest rate r increases according to the law

$$A(t) = A_0 e^{rt} \qquad (t = \text{time in years})$$

a) What is the average amount of money in the bank over the course of T years?

b) Suppose r and T are small. Give an approximate answer to part (a) by using the quadratic approximation to your exact answer; check it for reasonableness.

4D-7 Find the average value of x^2 in $0 \leq x \leq b$.

4D-8 Find the average distance from a point on the perimeter of a square of sidelength a to the center. Find the average of the square of the distance.

4D-9 Find the average value of $\sin ax$ in its first hump.

4D'. Work

4D'-1 An extremely stiff spring is 12 inches long, and a force of 2,000 pounds extends it 1/2 inch. How many foot-pounds of work would be done in stretching it to 18 inches?

4D'-2 A heavy metal 2 pound pail initially is filled with 10 pounds of paint. Immediately after it is filled, it is pulled up at a steady rate to the top of a building 30 feet high. While being pulled, the paint leaks out through a hole in the pail at a steady rate so that by the time it reaches the top, 1/5 of the paint has leaked out. How many foot-pounds of work were done pulling the pail to the top of the building?

4D'-3 A heavy-duty rubber firehose hanging over the side of a building is 50 feet long and weighs 2 lb./foot. How much work is done winding it up on a windlass on the top of the building?

4D'-4 Two point-particles having respective masses m_1 and m_2 are at d units distance. How much work is required to move them n times as far apart (i.e., to distance nd)? What is the work to move them infinitely far apart?

4E. Parametric equations

4E-1 Find the rectangular equation for $x = t + t^2$, $y = t + 2t^2$.

4E-2 Find the rectangular equation for $x = t + 1/t$ and $y = t - 1/t$ (compute x^2 and y^2).

4E-3 Find the rectangular equation for $x = 1 + \sin t$, $y = 4 + \cos t$.

4E-4 Find the rectangular equation for $x = \tan t$, $y = \sec t$.

4E-5 Find the rectangular equation for $x = \sin 2t$, $y = \cos t$.

4E-6 Consider the parabola $y = x^2$. Find the parametrization using the slope of the curve at a point (x, y) as the parameter.

4E-7 Find the parametrization of the circle $x^2 + y^2 = a^2$ using the slope as the parameter. Which portion of the circle do you obtain in this way?

4E-8 At noon, a snail starts at the center of an open clock face. It creeps at a steady rate along the hour hand, reaching the end of the hand at 1:00 PM. The hour hand is 1 meter long. Write parametric equations for the position of the snail at time t, in some reasonable xy-coordinate system.

4E-9* a) What part of a train is moving backwards when the train moves forwards?

b) A circular disc has inner radius a and outer radius b. Its inner circle rolls along the positive x-axis without slipping . Find parametric equations for the motion of a point P

on its outer edge, assuming P starts at $(0, b)$. Use θ as parameter. (Your equations should reduce to those of the cycloid when $a = b$. Do they?)

c) Sketch the curve that P traces out.

d) Show from the parametric equations you found that P is moving backwards whenever it lies below the x-axis.

4F. Arclength

4F-1 Find the arclength of the following curves

a) $y = 5x + 2$, $0 \le x \le 1$. b) $y = x^{3/2}$, $0 \le x \le 1$.

c) $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$. d) $y = (1/3)(2 + x^2)^{3/2}$, $1 \le x \le 2$.

4F-2 Find the length of the curve $y = (e^x + e^{-x})/2$ for $0 \le x \le b$. Hint:

$$\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

4F-3 Express the length of the parabola $y = x^2$ for $0 \le x \le b$ as an integral. (Do not evaluate.)

4F-4 Find the length of the curve $x = t^2$, $y = t^3$ for $0 \le t \le 2$.

4F-5 Find an integral for the length of the curve given parametrically in Exercise 4E-2 for $1 \le t \le 2$. Simplify the integrand as much as possible but do not evaluate.

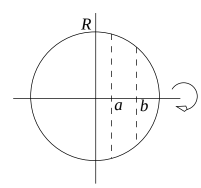
4F-6 a) The cycloid given parametrically by $x = t - \sin t$, $y = 1 - \cos t$ describes the path of a point on a rolling wheel. If t represents time, then the wheel is rotating at a constant speed. How fast is the point moving at each time t? When is the forward motion (dx/dt) largest and when is it smallest?

b) Find the length of the cycloid for one turn of the wheel. (Use a half angle formula.)

4F-7 Express the length of the ellipse $x^2/a^2 + y^2/b^2 = 1$ using the parametrization $x = a\cos t$ and $y = b\sin t$. (Do not evaluate.)

4F-8 Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ for $0 \le t \le 10$.

4G. Surface Area



4G-1 Consider the sphere of radius R formed by revolving the circle $x^2 + y^2 = R^2$ around the x-axis. Show that for $-R \le a < b \le R$, the portion of the sphere $a \le x \le b$ has surface area $2\pi R(b - a)$. For example, the hemisphere, $a = 0$, $b = R$ has area $2\pi R^2$.

4G-2 Find the area of the segment of $y = 1 - 2x$ in the first quadrant revolved around the x-axis.

4G-3 Find the area of the segment of $y = 1 - 2x$ in the first quadrant revolved around the y-axis.

4G-4 Find an integral formula for the area of $y = x^2$, $0 \le x \le 4$ revolved around the x-axis. (Do not evaluate.)

4G-5 Find the area of $y = x^2$, $0 \le x \le 4$ revolved around the y-axis.

4G-6 Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ revolved around the x-axis.

4G-7 Conside the torus of Problem 4C-1.

a) Set up the integral for surface area using integration dx

b) Set up the integral for surface area using integration dy

c) Evaluate (b) using the substitution $y = a\sin\theta$.

4H. Polar coordinate graphs

4H-1 For each of the following points given in rectangular coordinates, give its polar coordinates. (For points below the x-axis, give two expressions for its polar coordinates, using respectively positive and negative values for θ.)

a) $(0, 3)$ b) $(-2, 0)$ c) $(1, \sqrt{3})$ d) $(-2, 2)$
e) $(1, -1)$ f) $(0, -2)$ g) $(\sqrt{3}, -1)$ h) $(-2, -2)$

4H-2

a) Find using two different methods the equation in polar coordinates for the circle of radius a with center at $(a, 0)$ on the x-axis, as follows:

(i) write its equation in rectangular coordinates, and then change it to polar coordinates (substitute $x = r\cos\theta$ and $y = r\sin\theta$, and then simplify).

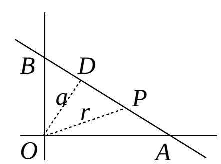
(ii) treat it as a locus problem: let OQ be the diameter lying along the x-axis, and $P : (r, \theta)$ a point on the circle; use $\triangle OPQ$ and trigonometry to find the relation connecting r and θ.

b) Carry out the analogue of 4H-2a for the circle of radius a with center at $(0, a)$ on the y-axis; OQ is now the diameter lying along the y-axis.

c) (i) Find the polar equation for the line intersecting the positive x- and y-axes respectively at A and B, and having perpendicular distance a from the origin.

(Let $\alpha = \angle DOA$; use the right triangle DOP to get the equation connecting r, θ, α and a.

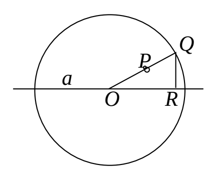
(ii) Convert your polar equation to the usual rectangular equation involving A and B, by using trigonometry.



d) In the accompanying figure, the point Q moves around the circle of radius a centered at the origin; QR is a perpendicular to the x-axis. P is a point on ray OQ such that $|QP| = |QR|$: P is the point inside the circle in the first two quadrants, but outside the circle in the last two quadrants.

(i) Sketch the locus of P; the locus is called a *cardioid* (cf. 4H-3c).

(ii) find the polar equation of this locus.



e) The point P moves in a locus so that the product of its distances from the two points $Q : (-a, 0)$ and $R : (a, 0)$ is constant. Assuming the locus of P goes through the origin, determine the value of the constant, and derive the polar equation of the locus of P.

(Work with the squares of the distances, rather than the distances themselves, and use the law of cosines; the identities $(A+B)(A-B) = A^2 - B^2$ and $\cos 2\theta = 2\cos^2\theta - 1$ simplify the algebra and produce a simple answer at the end. The resulting curve is a *lemniscate*, cf. 4H-3g.)

4H-3 For each of the following,

(i) give the corresponding equation in rectangular coordinates;

(ii) draw the graph; indicate the direction of increasing θ.

a) $r = \sec\theta$

b) $r = 2a\cos\theta$

c) $r = (a + b\cos\theta)$ (This figure is a cardioid for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

d) $r = a/(b + c\cos\theta)$ (Assume the constants a and b are positive. This figure is an ellipse for $b > |c| > 0$, a circle for $c = 0$, a parabola for $b = |c|$, and a hyperbola for $b < |c|$.)

e) $r = a\sin(2\theta)$ (4-leaf rose)

f) $r = a\cos(2\theta)$ (4-leaf rose)

g) $r^2 = a^2\sin(2\theta)$ (lemniscate)

h) $r^2 = a^2\cos(2\theta)$ (lemniscate)

i) $r = e^{a\theta}$ (logarithmic spiral)

4I. Area and arclength in polar coordinates

4I-1 Find the arclength element $ds = w(\theta)d\theta$ for the curves of 4H-3.

4I-2 Find the area of one leaf of a three-leaf rose $r = a\cos(3\theta)$.

4I-3 Find the area of the region $0 \le r \le e^{3\theta}$ for $0 \le \theta \le \pi$

4I-4 Find the area of one loop of the lemniscate $r^2 = a^2\sin(2\theta)$

4I-5 What is the average distance of a point on a circle of radius a from a fixed point Q on the circle? (Place the circle so Q is at the origin and use polar coordinates.)

4I-6 What is the average distance from the x-axis of a point chosen at random on the cardioid $r = a(1-\cos\theta)$, if the point is chosen

a) by letting a ray $\theta = c$ sweep around at uniform velocity, stopping at random and taking the point where it intersects the cardioid;

b) by letting a point P travel around the cardioid at uniform velocity, stopping at random; (the answers to (a) and (b) are different...)

4I-7 Calculate the area and arclength of a circle, parameterized by $x = a\cos\theta, y = a\sin\theta$.

4J. Other Applications

4J-1 Suppose it takes k units of energy to lift a cubic meter of water one meter. About how much energy E will it take to pump dry a circular hole one meter in diameter and 100 meters deep that is filled with water? (Give reasoning.)

4J-2 The amount x (in grams) of a radioactive material declines exponentially over time (in minutes), according to the law $x = x_0e^{-kt}$, where x_0 is the amount initially present at time $t = 0$. If one gram of the material produces r units of radiation/minute, about how much radiation R is produced over one hour by x_0 grams of the material? (Give reasoning.)

4J-3 A very shallow circular reflecting pool has uniform depth D, and radius R (meters). A disinfecting chemical is released at its center, and after a few hours of symmetrical diffusion outwards, the concentration of chemical at a point r meters from the center is $\dfrac{k}{1+r^2}$ g/m^3.

What amount A of the chemical was released into the pool? (Give reasoning.)

4J-4 Assume a heated outdoor pool requires k units of heat/hour for each degree F it is maintained above the external air temperature.

If the external temperature T varies between 50^o and 70^o over a 24 hour period starting at midnight, according to $T = 10\big(6 - \cos(\pi t/12)\big)$, how many heat units will be required to maintain the pool at a steady 75^o temperature? (Give reasoning.)

4J-5 A manufacturers cost for storing one unit of inventory is c dollars/day for space and insurance. Over the course of 30 days, production P rises from 10 to 40 units/day according to $P = 10+t$. Assuming no units are sold, what is the inventory cost for this period? (Give reasoning.)

4J-6 A water tank for a town has the shape of a sphere of radius r feet, and its center is at a height h above the ground. If the weight of a cubic foot of water is w lbs., how much work is required to fill the tank when empty by pumping water from the ground? (Give reasoning using infinitesimals.)

4J-6 Divide the water in the tank into thin horizontal slices of width dy.
If the slice is at height y above the center of the tank, its radius is $\sqrt{r^2 - y^2}$.
volume of water in the slice $= \pi(r^2 - y^2)\,dy$
weight of water in the slice $= \pi w(r^2 - y^2)\,dy$
work to lift this slice from the ground $= \pi w(r^2 - y^2)\,dy\,(h + y)$.

$$\text{Total work} \;=\; \int_{-r}^{r} \pi w(r^2-y^2)(h+y)\,dy \;=\; \pi w \int_{-r}^{r} (r^2h+r^2y-hy^2-y^3) \;=\; \pi w\left[r^2hy+\frac{r^2y^2}{2}-h\frac{y^3}{3}-\frac{y^4}{4}\right]_{-r}^{r}.$$

The even powers of y have the same value at $-r$ and r, so contribute 0 to the value; we get

$$=\; \pi wh\left[r^2y - \frac{y^3}{3}\right]_{-r}^{r} = 2\pi wh\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi whr^3.$$