



**3B-4** Calculate the difference between the upper and lower Riemann sums for the following integrals with  $n$  intervals

$$\text{a) } \int_0^b x^2 dx \qquad \text{b) } \int_0^b x^3 dx$$

Does the difference tend to zero as  $n$  tends to infinity?

**3B-5** Evaluate the limit, by relating it to a Riemann sum.

$$\lim_{n \rightarrow \infty} \frac{\sin(b/n) + \sin(2b/n) + \cdots + \sin((n-1)b/n) + \sin(nb/n)}{n}$$

**3B-6\*** Calculate  $\int_0^1 e^x dx$  by using upper Riemann sums.

Hints: The sum is a geometric progression. You will need the limit  $\lim_{n \rightarrow \infty} n(e^{1/n} - 1)$ . This can be evaluated putting  $h = 1/n$  and relating the limit to the derivative of  $e^x$  at  $x = 0$ .

**3B-7\*** Evaluate the limit

$$\lim_{n \rightarrow \infty} \frac{2^{b/n} + 2^{2b/n} + \cdots + 2^{(n-1)b/n} + 2^{nb/n}}{n}$$

(See **3B-6**.)

### 3C. Fundamental theorem of calculus

**3C-1** Find the area under the graph of  $y = 1/\sqrt{x-2}$  for  $3 \leq x \leq 6$

**3C-2** Calculate

$$\text{a) } \int_0^2 \sqrt{3x+5} dx \qquad \text{b) } \int_0^2 (3x+5)^n dx \qquad \text{c) } \int_{3\pi/4}^{\pi} \frac{\sin x dx}{\cos^3 x}$$

**3C-3** Calculate

$$\text{a) } \int_1^2 \frac{x dx}{x^2+1} \qquad \text{b) } \int_b^{2b} \frac{x dx}{x^2+b^2}$$

**3C-4** Calculate  $\lim_{b \rightarrow \infty} \int_1^b x^{-10} dx$ . What area does this integral describe?

**3C-5** Find the area

$$\text{a) under one arch of } \sin x. \qquad \text{b) under one arch of } \sin ax \text{ for a positive constant } a.$$

**3C-6** Find the area between the  $x$ -axis and

$$\text{a) the curve } y = x^2 - 4 \qquad \text{b) the curve } y = x^2 - a \text{ for } a > 0.$$

**3D. Second fundamental theorem**

**3D-1** a) Prove that  $\int_0^x \frac{dt}{\sqrt{t^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) - \ln a, a > 0, x > 0.$

b) For what  $c$  is  $\int_c^x \frac{dt}{\sqrt{t^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$  ?

**3D-2\*** Show that the function  $y = \int_0^x \sqrt{1 - t^2} dt$  satisfies the differential equation with side conditions:

$$y' y'' = -x; \quad y(0) = 0, y'(0) = 1.$$

**3D-3** Discuss the function  $F(x) = \int_0^x \frac{1 - t^2}{1 + t^2} dt$ , including a sketch; describe

a) domain

b) relative maxima and minima, where increasing or decreasing, points of inflection

c) behavior as  $x \rightarrow \infty$  (Hint: Evaluate the integrand as  $t \rightarrow \pm\infty$ .)

d) symmetry about y-axis or origin

**3D-4** Find a function whose derivative is  $\sin(x^3)$  and whose value at 0 is

a) 0

b) 2

c) find one whose value at 1 is  $-1$

**3D-5** Evaluate  $\lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_1^{1+\Delta x} \frac{t}{\sqrt{1+t^4}} dt$  two ways:

a) by interpreting the integral as the area under a curve

b) by relating the limit to  $F'(1)$ , where  $F(x) = \int_0^x \frac{t}{\sqrt{1+t^4}} dt$

**3D-6** For different values of  $a$ , the functions  $F(x) = \int_a^x dt$  differ from each other by constants. Show this two ways:

a) directly

b) using the corollary to the mean-value theorem quoted ((8), p.FT.5)

**3D-7** Evaluate  $F'(x)$  if  $F(x) =$

a)  $\int_0^{x^2} \sqrt{u} \sin u du$       b)  $\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$

c)  $\int_x^{x^2} \tan u du$

**3D-8** Let  $f(x)$  be continuous. Find  $f(\pi/2)$  if :

a)  $\int_0^x f(t) dt = 2x(\sin x + 1)$

b)  $\int_0^{x/2} f(t) dt = 2x(\sin x + 1)$

**3E. Change of variables; Estimating integrals**

**3E-1** Prove directly from the definition  $L(x) = \int_1^x \frac{dt}{t}$  that  $L(\frac{1}{a}) = -L(a)$ , by making a change of variables in the definite integral.

**3E-2** The function defined by  $E(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du$  is used in probability and statistics and has the same importance as sine and cosine functions have to trigonometry.

a) Express  $E(x)$  in terms of the function of example 5 of Notes FT,  $F(x) = \int_0^x e^{-t^2} dt$ , by making a change of variable. It is known that  $\lim_{x \rightarrow \infty} F(x) = \sqrt{\pi}/2$ . What is  $\lim_{x \rightarrow \infty} E(x)$ ?

b) Evaluate  $\lim_{N \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^N e^{-u^2/2} du$  and  $\lim_{x \rightarrow -\infty} E(x)$ .

c) Express  $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} du$  in terms of the function  $E$  and the constants  $a$  and  $b$ , where  $a < b$ .

**3E-3** Evaluate by making a substitution and changing both the variable and the limits of integration.

a)  $\int_1^e \frac{\sqrt{\ln x}}{x} dx$  ( $u = \ln x$ )

b)  $\int_0^\pi \frac{\sin x}{(2 + \cos x)^3} dx$

c)  $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$  ( $x = \sin u$ )

d)  $\int_{18}^{19} \frac{dx}{x^2 - 34x + 289}$  ( $x = z + 17$ )

**3E-4** From the definite integral  $\int_{-1}^1 \sqrt{1-x^2} dx = \pi/2$  deduce the value of  $\int_{-a}^a \sqrt{a^2-x^2} dx$  by making a suitable change of variable of the form  $x = ct$  ( $c$  constant).

**3E-5** Let  $F(x) = \int_0^x f(t) dt$ .

a) Prove that if  $f(t)$  is even, then  $F(x)$  is odd.

b) Prove that if  $f(t)$  is odd, then  $F(x)$  is even.

Hint: Make the change of variable  $u = -t$  in the definite integral. (Compare with **4-6**.)

**3E-6** By comparing the given integral with an integral that is easier to evaluate, establish each of the following estimations:

a)  $\int_0^1 \frac{dx}{1+x^3} > 0.65$

b)  $\int_0^\pi \sin^2 x dx < 2$

c)  $\int_{10}^{20} \sqrt{x^2+1} dx > 150$

**3E-7** Show  $\left| \int_1^N \frac{\sin x}{x^2} dx \right| < 1$

### 3F. Differential equations: separation of variables

**3F-1** Solve the following differential equations

a)  $dy/dx = (2x + 5)^4$

b)  $dy/dx = (y + 1)^{-1}$

c)  $dy/dx = 3/\sqrt{y}$

d)  $dy/dx = xy^2$

**3F-2** Solve each differential equation with the given initial condition, and evaluate the solution at the given value of  $x$ :

a)  $dy/dx = 4xy$ ,  $y(1) = 3$ . Find  $y(3)$ .

b)  $dy/dx = \sqrt{y+1}$ ,  $y(0) = 1$ . Find  $y(3)$ .

c)  $dy/dx = x^2y^{-1}$ ,  $y(0) = 10$ . Find  $y(5)$ .

d)  $dy/dx = 3y + 2$ ,  $y(0) = 0$ . Find  $y(8)$ .

e)  $dy/dx = e^y$ ,  $y(3) = 0$ . Find  $y(0)$ . For which values of  $x$  is the solution  $y$  defined?

**3F-3** a) Solve  $dy/dx = y^2$  with  $y = 1$  at  $x = 0$ . Evaluate  $y$  at  $x = 1/2$ , at  $x = -1$ , and at  $x = 1$ .

b) Graph the solution and use the graph to discuss the range of validity of the formula for  $y$ . In particular, explain why the apparent value at  $x = 3/2$  is suspect.

**3F-4** Newton's law of cooling says that the rate of change of temperature is proportional to the temperature difference. In symbols, if a body is at a temperature  $T$  at time  $t$  and the surrounding region is at a constant temperature  $T_e$  ( $e$  for external), then the rate of change of  $T$  is given by

$$dT/dt = k(T_e - T).$$

The constant  $k > 0$  is a constant of proportionality that depends properties of the body like specific heat and surface area.

a) Why is  $k > 0$  the only physically realistic choice?

b) Find the formula for  $T$  if the initial temperature at time  $t = 0$  is  $T_0$ .

c) Show that  $T \rightarrow T_e$  as  $t \rightarrow \infty$ .

d) Suppose that an ingot leaves the forge at a temperature of  $680^\circ$  Celsius in a room at  $40^\circ$  Celsius. It cools to  $200^\circ$  in eight hours. How many hours does it take to cool from  $680^\circ$  to  $50^\circ$ ? (It is simplest to keep track of the temperature difference  $T - T_e$ , rather than  $T$ . The temperature difference undergoes exponential decay.)

e) Suppose that an ingot at  $1000^\circ$  cools to  $800^\circ$  in one hour and to  $700^\circ$  in two hours. Find the temperature of the surrounding air.

f) Show that  $y(t) = T(t - t_0)$  also satisfies Newton's law of cooling for any constant  $t_0$ . Write out the formula for  $T(t - t_0)$  and show that it is the same as the formula in E10/17 for  $y(t)$  by identifying the constants  $k$ ,  $T_e$  and  $T_0$  with their corresponding values in the displayed formula in E10/17.

**3F-5\*** Air pressure satisfies the differential equation  $dp/dh = -(0.13)p$ , where  $h$  is the altitude from sea level measured in kilometers.

a) At sea level the pressure is<sup>1</sup> 1 kg/cm<sup>2</sup>. Solve the equation and find the pressure at the top of Mt. Everest (10 km).

b) Find the difference in pressure between the top and bottom of the Green Building. (Pretend it's 100 meters tall starting at sea level.) Compute the numerical value using a calculator. Then use instead the linear approximation to  $e^x$  near  $x = 0$  to estimate the percentage drop in pressure from the bottom to the top of the Green Building.

c) Use the linear approximation  $\Delta p \approx p'(0)\Delta h$  and compute  $p'(0)$  directly from the differential equation to find the drop in pressure from the bottom to top of the Green Building. Notice that this gives an answer without even knowing the solution to the differential equation. Compare with the approximation in part (b). What does the linear approximation  $p'(0)\Delta h$  give for the pressure at the top of Mt. Everest?

d) What is the differential equation for  $p$  if altitude is measured in meters instead of kilometers?

**3F-6** Let  $y = \cos^3 u - 3 \cos u$ ,  $x = \sin^4 u$ . Find  $dy$ ,  $dx$ , and  $dy/dx$ . Simplify.

**3F-7** Solve:

a)  $y' = -xy$ ,  $y(0) = 1$       b)  $\cos x \sin y dy = \sin x dx$ ,  $y(0) = 0$ .

**3F-8** a) Find all plane curves such that the tangent line at P intersects the  $x$ -axis 1 unit to the left of the projection of P on the the  $x$ -axis.

b) Find all plane curves in the first quadrant such that for every point P on the curve, P bisects the part of the tangent line at P that lies in the first quadrant.

### 3G. Numerical Integration

**3G-1** Find approximations to the following integrals using four intervals using Riemann sums with left endpoints, using the trapezoidal rule, and using Simpson's rule. Also give numerical approximations to the exact values of the integrals given to see how good these approximation methods are.

a)  $\int_0^1 \sqrt{x} dx$  (= 2/3.)      b)  $\int_0^\pi \sin x dx$  (= 2.)  
 c)  $\int_0^1 \frac{dx}{1+x^2}$  (=  $\pi/4$ ; cf. unit 5)      d)  $\int_1^2 \frac{dx}{x}$  (=  $\ln 2$ )

**3G-2** Show that the value given by Simpson's rule for two intervals for the integral

$$\int_0^b f(x) dx$$

gives the exact answer when  $f(x) = x^3$ . (Since a cubic polynomial is a sum of a quadratic polynomial and a polynomial  $ax^3$ , and Simpson's rule is exact for any quadratic polynomial, the result of this exercise implies by linearity (cf. Notes PI) that Simpson's rule will also be exact for any cubic polynomial.)

<sup>1</sup>using the correspondence between weight and mass on Earth of  $F = ma$  with  $a = 10m/sec^2$

3. INTEGRATION

**3G-3** Use the trapezoidal rule to estimate  $\sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{10,000}$ . Is your estimate too high or too low?

**3G-4** Use the trapezoidal rule to estimate the sum of the reciprocals of the first  $n$  integers. Is your estimate too high or too low?

**3G-5** If the trapezoidal rule is used to estimate the value of  $\int_a^b f(x)dx$  under what hypotheses on  $f(x)$  will the estimate be too low? too high?