Diagrams of Affine Permutations and Their Labellings

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Oracle

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Contents

Based on the joint work with
Hwanchul Yoo (KIAS).

Section 1

Affine Balanced Labellings

Permutations

A **permutation** is a bijection $w : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
$\Sigma_n :=$ the **symmetric group**, group of (finite) permutations of size n.

Σ_n is generated by

1. the **simple reflections** $s_1, \ldots, s_{n-1}$
   (s_i is the permutation which interchanges the pair $(i, i+1)$)
2. and the following relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Affine Permutations

An **affine permutation** of period n is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i) + n \ \forall i \in \mathbb{Z}$ and $w(1) + w(2) + \cdots + w(n) = n(n+1)/2$.
$\widetilde{\Sigma}_n :=$ the **affine symmetric group**, group of affine permutations of period n.

$\widetilde{\Sigma}_n$ is generated by

1. the **simple reflections** $s_0, s_1, \ldots, s_{n-1}$
   (s_i interchanges the pairs $(i+rn, i+1+rn) \ \forall r \in \mathbb{Z}$.)
2. and the following relations

$$\begin{aligned}
s_i^2 &= 1 && \text{for all } i \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i \\
s_i s_j &= s_j s_i && \text{for } i, j \text{ not adjacent, } i \neq j.
\end{aligned}$$

where the indices are taken modulo n.

Affine Permutations

Example

$\widetilde{\Sigma}_3$ is generated by $\{s_0, s_1, s_2\}$.

	$\cdots$	-1	0	1	2	3	4	5	$\cdots$
id	$\cdots$	-1	0	1	2	3	4	5	$\cdots$
s_2	$\cdots$	0	-1	1	3	2	4	6	$\cdots$
$s_2 s_0$	$\cdots$	0	1	-1	3	4	2	6	$\cdots$
$s_2 s_0 s_1$	$\cdots$	-4	1	3	-1	4	6	2	$\cdots$
$s_2 s_0 s_1 s_0$	$\cdots$	-4	3	1	-1	6	4	2	$\cdots$

$$w = s_2 s_0 s_1 s_0 \in \widetilde{\Sigma}_3$$

$$w = [1, -1, 6] \, : \, \textbf{window notation}$$

Affine Permutations

Remark

A permutation $w = [w_1, \ldots, w_n]$ can be viewed as an affine permutation $[w_1, \ldots, w_n]$ with period n, written in window notation.

This corresponds to the natural embedding $\Sigma_n \hookrightarrow \widetilde{\Sigma}_n$, $s_i \longmapsto s_i$.

Reduced Words

A **reduced decomposition** of w is a decomposition $w = s_{i_1} \cdots s_{i_\ell}$ where ℓ is the minimal number for which such a decomposition exists.

That minimal $\ell = \ell(w)$ is the **length** of w.

The word $i_1 i_2 \cdots i_\ell$ is called a **reduced word** of w.

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Fact

The length of an affine permutation w is the number of **inversions** (i, j) *with $1 \leq i \leq n$.*

$$\ell(w) = |\{(i,j) \mid i < j,\ w(i) > w(j),\ 1 \leq i \leq n\}|.$$

Diagrams

The **affine permutation diagram** of $w \in \widetilde{\Sigma}_n$ is the set

$$D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

When w is a finite permutation, $D(w)$ consists of infinite number of identical copies of the Rothe diagram of w diagonally.

Diagrams

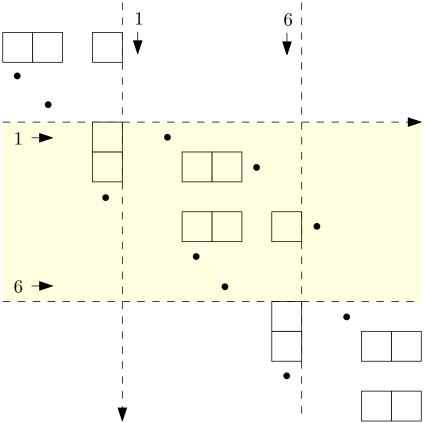

Figure: diagram of
$w = [2, 5, 0, 7, 3, 4] \in \widetilde{\Sigma}_6$

Diagrams

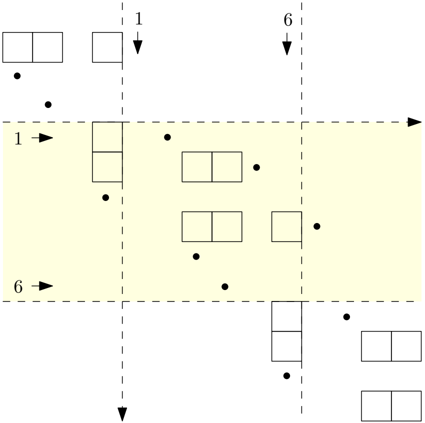

Figure: diagram of
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fundamental window
$[D(w)] := ([1,n] \times \mathbb{Z}) \cap D(w).$

cell
$= \{\underline{(i+rn, j+rn)} \mid r \in \mathbb{Z}\}$
$=: \overline{(i,j)}.$

$\ell(w) = \#(\text{cells}) = |[D(w)]|$

Balanced Hook

Labelling of a diagram = labelling of the *cells* with positive integers.

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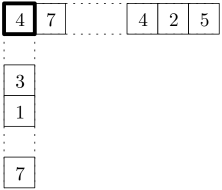

Hook $H_{ij} := (\{i\} \times \mathbb{Z}_{\geq j}) \cup (\mathbb{Z}_{\geq i} \times \{j\})$

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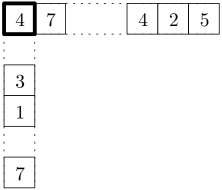

Hook $H_{ij} := (\{i\} \times \mathbb{Z}_{\geq j}) \cup (\mathbb{Z}_{\geq i} \times \{j\})$
Balanced hook: If one rearranges the labels in the hook so that they weakly increase from right to left and from top to bottom, then the corner label remains unchanged.

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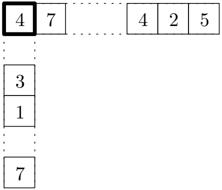

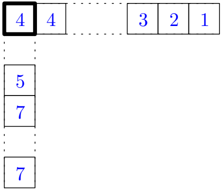

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Balanced Labellings

Definition (Fomin-Greene-Reiner-Shimozono 1997)

A labelling of a diagram D is called:

balanced if every *hook* is balanced

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A labelling of a diagram D is called:

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injective if each of the labels $\{1, 2, \ldots, \ell\}$ appears exactly once in $[D]$.

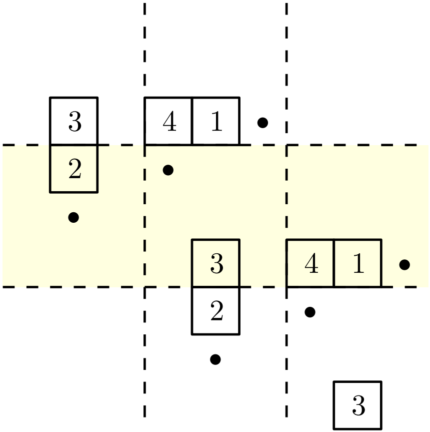

Figure: injective balanced labelling

Canonical Labellings

Let $a = a_1 a_2 \cdots a_\ell$ be a reduced word of $w \in \widetilde{\Sigma}_n$,
i.e. $w = s_{a_1} s_{a_2} \cdots s_{a_\ell}$.

We define the **canonical labelling** T_a of a.

e.g. $w = [1, -1, 6] = s_2 s_0 s_1 s_0$.

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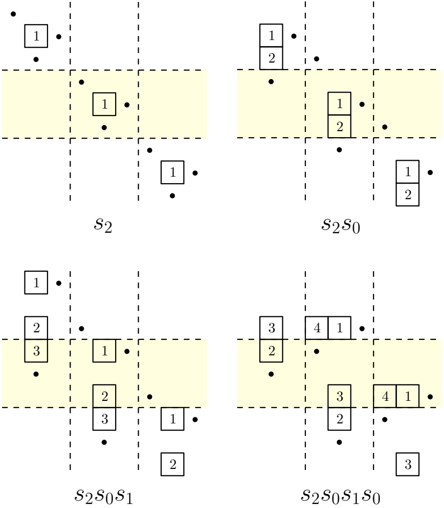


Canonical Labellings

Theorem (Yoo-Y. 2012, FGRS 1997 for finite case)

For all $w \in \widetilde{\Sigma}_n$, the map

$$a \longmapsto T_a \text{ (the canonical labelling of } a)$$

is a bijection

$$\{\text{reduced words of } w\} \longrightarrow \{\text{injective balanced labellings of } D(w)\}.$$

Proof.

Find the inverse map. □

Section 2

Symmetric Functions

Stanley Symmetric Functions

$v \in \Sigma_n$ is called **decreasing** if it has a decreasing reduced word.
e.g. $v = s_4 s_2 s_1 \in \Sigma_5$.
$w = v_1 v_2 \cdots v_r$ is a **decreasing factorization** of w if each $v_i \in \Sigma_n$ is decreasing and $\ell(w) = \sum_{i=1}^r \ell(v_i)$.
$(\ell(v_1), \ell(v_2), \ldots, \ell(v_r))$ is the **type** of the factorization.

Definition (Stanley 1984)

Let $w \in \Sigma_n$ be a permutation. The **Stanley symmetric function** $F_w(x)$ of w is defined by

$$F_w(x) := F_w(x_1, x_2, \cdots) = \sum_{w = v_1 v_2 \cdots v_r} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)},$$

where the sum is over all decreasing factorization of w.

Affine Stanley Symmetric Functions

A word $a_1 a_2 \cdots a_\ell$ with letters in $\mathbb{Z}/n\mathbb{Z}$ is **cyclically decreasing** if
(1) each letter appears at most once, and
(2) whenever i and $i+1$ both appears in the word, $i+1$ precedes i.

$v \in \widetilde{\Sigma}_n$ is cyclically decreasing if it has a cyclically decreasing reduced word. e.g. $v = s_2 s_0 s_4 \in \widetilde{\Sigma}_5$.

$w = v_1 v_2 \cdots v_r$ is a **cyclically decreasing factorization** of w if each $v_i \in \widetilde{\Sigma}_n$ is cyclically decreasing and $\ell(w) = \sum_{i=1}^{r} \ell(v_i)$.

Definition (Lam 2006)

$w \in \widetilde{\Sigma}_n$. The **affine Stanley symmetric function** $\widetilde{F}_w(x)$ of w is

$$\widetilde{F}_w(x) := \widetilde{F}_w(x_1, x_2, \cdots) = \sum_{w = v_1 v_2 \cdots v_r} x_1^{\ell(v_1)} x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)},$$

where the sum is over all cyclically decreasing factorization of w.

Column-Strict Balanced Labellings

Theorem (Yoo-Y. 2012, FGRS 1997 for finite case)

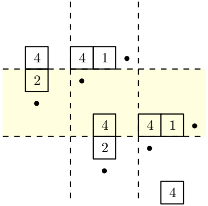
$\mathcal{CB}(D) = \{$*column-strict balanced labellings of a diagram* $D\}$.
("column strict" := no column contains two boxes with equal labels.)

For all $w \in \widetilde{\Sigma}_n$,

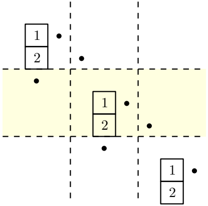
$$\widetilde{F}_w(x) = \sum_{T \in \mathcal{CB}(D(w))} x^T$$

where x^T *denotes the monomial* $\prod_{(i,j) \in [D(w)]} x_{T(i,j)}$.

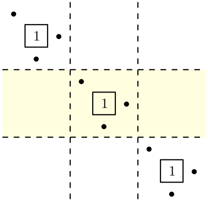
Column-Strict Balanced Labellings



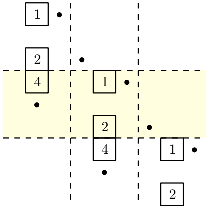
$()()()(s_0)$



$()()(\mathrm{id})(s_1 s_0)$



$(s_2)(s_0)(\mathrm{id})(s_1 s_0)$



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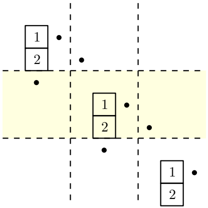

$()(s_0)(\mathrm{id})(s_1 s_0)$

$$\Rightarrow \quad w = v_1 v_2 v_3 v_4$$

$$v_1 = s_2, \; v_2 = s_0,$$
$$v_3 = \mathrm{id}, \; v_4 = s_1 s_0.$$

Section 3

Properties of Permutation Diagrams

Affine Diagram

A collection D of unit square boxes on $\mathbb{Z} \times \mathbb{Z}$ is called an **affine diagram** (of period n) if there are finite number of boxes on each row and column, and

$$(i, j) \in D \Leftrightarrow (i + n, j + n) \in D.$$

Clearly, any affine permutation diagram of $w \in \widetilde{\Sigma}_n$ is an affine diagram of period n.

Question

When is an (affine) diagram an (affine) permutation diagram?

Theorem (Yoo-Y. 2012, FGRS 1997 for finite case)

D affine permutation diagram of w,

$$\text{Reduced word of } w \iff \text{Injective balanced labelling of } D.$$

Proof.

Find the inverse map. $\square$

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$\Longrightarrow$ Try to find an injective balanced labelling of D.

$\Longrightarrow$ Using the "inverse map", find the reduced word $a_1 a_2 \cdots a_\ell$ corresponding to the labelling.

$\Longrightarrow$ Check if $w = s_{a_1} s_{a_2} \cdots s_{a_\ell}$ gives you the diagram D.

Classification Theorem

Theorem (Yoo-Y. 2012)

D an affine diagram. D is an affine permutation diagram if and only if it is **North-West** *and admits a* **content map**.

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Theorem (Yoo-Y. 2012)

D an affine diagram. D is an affine permutation diagram if and only if it is **North-West** *and admits a* **content map***.*

An affine diagram is **North-West** if whenever there is a box at (i, j) and at (k, ℓ) with $i < k$ and $j > \ell$, there must be a box at (i, ℓ).

Content Map

Definition (Yoo-Y. 2012)

D an affine diagram with period n. $\mathcal{C} : D \to \mathbb{Z}$ is a **content map** if it satisfies the following four conditions.

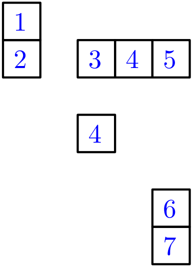
(C1) If boxes b_1 and b_2 are in the same row (respectively, column), b_2 being to the east (resp., south) to b_1, and there are no boxes between b_1 and b_2, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = 1$.

(C2) If b_2 is strictly to the southeast of b_1, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) \geq 2$.

(C3) For each row (resp., column), the content of the leftmost (resp., topmost) box is equal to the row (resp., column) index.

(C4) If $b_1 = (i, j)$ and $b_2 = (i + n, j + n)$ coordinate-wise, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = n$.

Content Map



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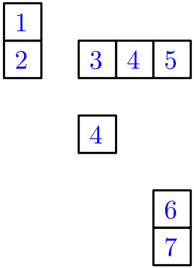

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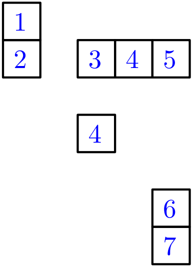

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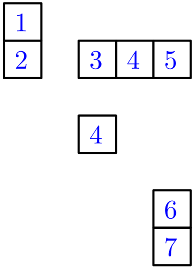

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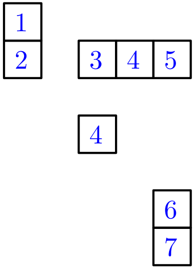

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(C4) If $b_1 = (i, j)$ and $b_2 = (i + n, j + n)$ coordinate-wise, then $\mathcal{C}(b_2) - \mathcal{C}(b_1) = n$.

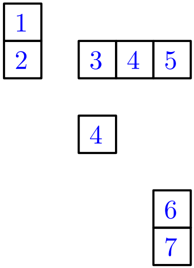
Theorem (Yoo-Y. 2012)

D an (affine) diagram. D is an (affine) permutation diagram if and only if it is North-West and admits a content map.

Proof.

We re-construct the affine permutation from a North-West diagram with a content map. $\square$

Proof of Classification Theorem (Wiring Diagrams)



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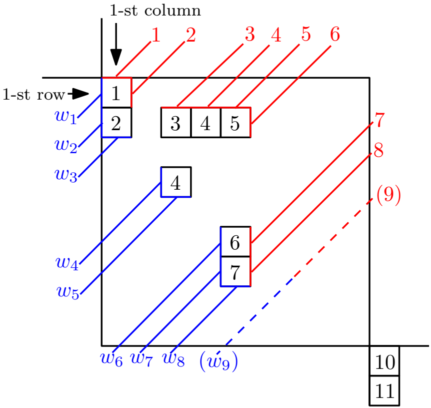

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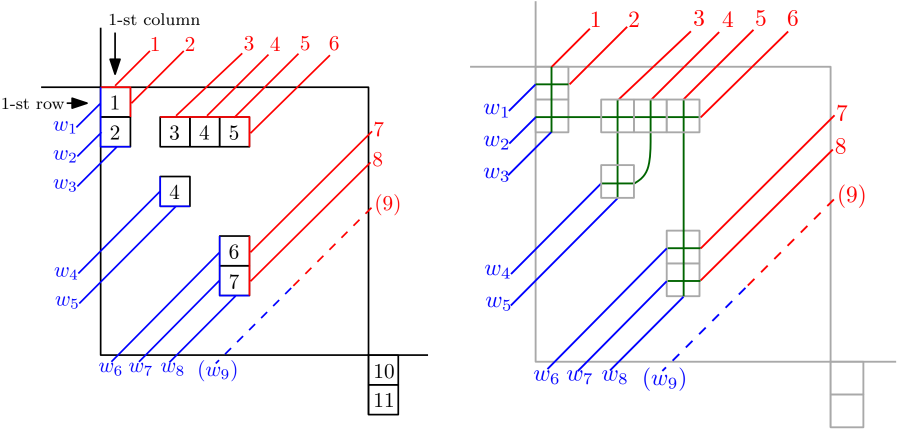
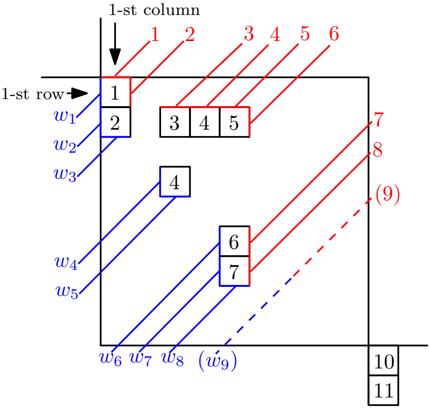
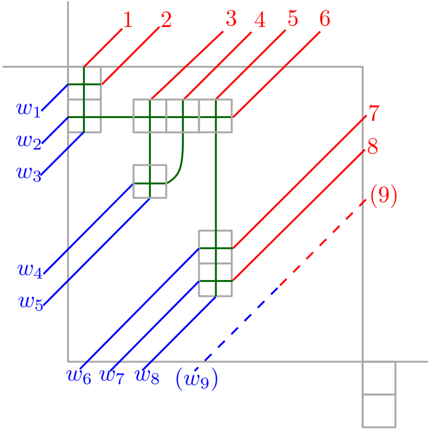


Proof of Classification Theorem (Wiring Diagrams)

$\Longrightarrow w = [2, 6, 1, 4, 3, 7, 8, 5, 9].$

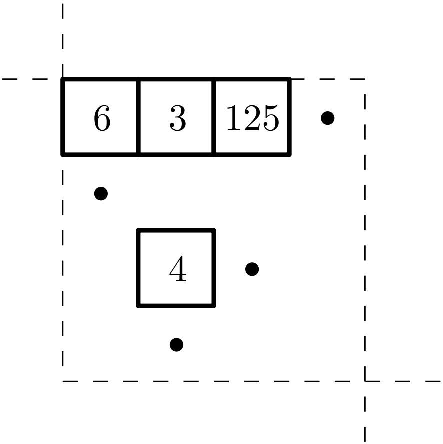




Section 4

Set-Valued Balanced Labellings

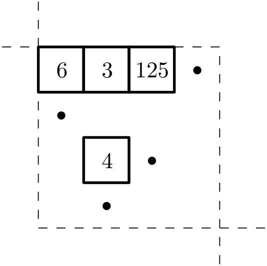
Set-Valued Labellings



$$u_3 u_3 u_2 u_3 u_2 u_1$$

Related Objects

labelling balanced labelling	set-valued (s-v) labelling s-v balanced labelling
reduced words Stanley symm. func. Schubert poly. $\mathfrak{S}_w$	*nilHecke words* *stable Grothendieck poly.* (Lascoux-Schützenberger) *Grothendieck poly.* $\mathfrak{G}_w$ (Lascoux-Schützenberger)
affine symm. group affine Stanley symm. func. $\widetilde{F}_w$	*affine nilHecke algebra* $\widetilde{\mathcal{U}}_n$ (Lam) *affine stable Grothendieck poly.* $\widetilde{G}_w$ (Lam)



$u_3u_3u_2u_3u_2u_1$

Theorem (Yoo-Y. 2013)

$w \in \widetilde{\Sigma}_n$, $\widetilde{\mathcal{U}}_n$ **affine nilHecke algebra**. *There is a bijection from*

$$\{\textit{nilHecke words } a \textit{ in } \widetilde{\mathcal{U}}_n \textit{ with } S(a) = w\}$$

to

$$\{\textit{s-v injective balanced labellings of } D(w)\}.$$

Theorem (Yoo-Y. 2013)

w an affine permutation, $\widetilde{G}_w$ *affine stable Grothendieck polynomial.*

$$\widetilde{G}_w(x) = \sum_T (-1)^{|T|-\ell(w)} x^T,$$

over all s-v column-strict balanced labellings T *of* $D(w)$*,*
$x^T := \prod_{b \in [D(w)]} \prod_{k \in T(b)} x_k.$

Theorem (Buch 2002)

λ/μ *a skew Young diagram (equiv., a diagram of* 321*-avoiding finite permutation).*

$$G_{\lambda/\mu}(x) = \sum_T (-1)^{|T|-|\lambda/\mu|} x^T,$$

over all set-valued tableaux T *of shape* λ/μ*,* $x^T := \prod_{b \in \lambda/\mu} \prod_{k \in T(b)} x_k.$

Theorem (Yoo-Y. 2013)

w a finite permutation. $\mathfrak{G}_w$ Grothendieck polynomial.

$$\mathfrak{G}_w(x) = \sum_T (-1)^{|T|-\ell(w)} x^T,$$

over all column-strict s-v balanced labellings T of $D(w)$ with flag conditions: $\forall t \in T(i,j)$, $t \leq i$.

Yoo-Yun, *Diagrams of affine permutations, balanced labellings, and symmetric functions*, arXiv:1305.0129.

(math.tedyun.com, arXiv:1305.0129)

Happy Birthday, Richard!