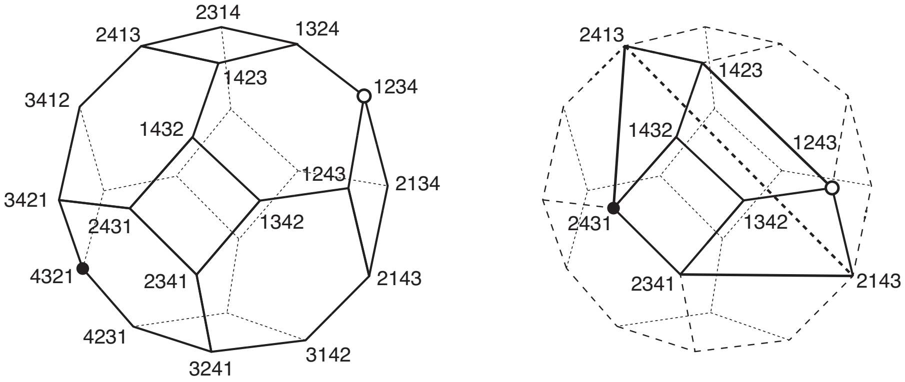
The Toda lattice and Bruhat interval polytopes

Lauren K. Williams, UC Berkeley



Plan of the talk

- Introduction to the Toda lattice
- The sorting property
- The positive flag variety, and generalized sorting.
- Bruhat interval polytopes and their faces
- The generalized lifting property and R-polynomials
- Combinatorics of Bruhat interval polytopes
- Bruhat interval polytopes for G/P

References

- (joint with Yuji Kodama) The full Kostant-Toda hierarchy on the positive flag variety, to appear in Comm. Math. Phys.
- (joint with Emmanuel Tsukerman) Bruhat Interval Polytopes, `arXiv:1406.5202`

The Toda lattice

The Toda lattice is defined by

$$\frac{dL}{dt} = [\pi(L), L],$$

where $L = L(t)$ is a tridiagonal symmetric matrix and $\pi(L)$ is its skew-symmetric projection:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}, \pi(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

- Model introduced by Toda in 1967 (Def above due to Flaschka 1974)
- Represents dynamics of n particles of unit mass, moving on a line under influence of exponential repulsive forces.
- Eigenvalues of $L(t)$ are independent of t.

The Toda lattice

The Toda lattice is defined by

$$\frac{dL}{dt} = [\pi(L), L],$$

where $L = L(t)$ is a tridiagonal symmetric matrix and $\pi(L)$ is its skew-symmetric projection:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}, \pi(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

- Model introduced by Toda in 1967 (Def above due to Flaschka 1974)
- Represents dynamics of n particles of unit mass, moving on a line under influence of exponential repulsive forces.
- Eigenvalues of $L(t)$ are independent of t.

The Toda lattice

The Toda lattice is defined by

$$\frac{dL}{dt} = [\pi(L), L],$$

where $L = L(t)$ is a tridiagonal symmetric matrix and $\pi(L)$ is its skew-symmetric projection:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}, \pi(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

- Model introduced by Toda in 1967 (Def above due to Flaschka 1974)
- Represents dynamics of n particles of unit mass, moving on a line under influence of exponential repulsive forces.
- Eigenvalues of $L(t)$ are independent of t.

The Toda lattice

The Toda lattice is defined by

$$\frac{dL}{dt} = [\pi(L), L],$$

where $L = L(t)$ is a tridiagonal symmetric matrix and $\pi(L)$ is its skew-symmetric projection:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}, \pi(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

- Model introduced by Toda in 1967 (Def above due to Flaschka 1974)
- Represents dynamics of n particles of unit mass, moving on a line under influence of exponential repulsive forces.
- Eigenvalues of $L(t)$ are independent of t.

The Toda lattice

The Toda lattice is defined by

$$\frac{dL}{dt} = [\pi(L), L],$$

where $L = L(t)$ is a tridiagonal symmetric matrix and $\pi(L)$ is its skew-symmetric projection:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}, \pi(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a_{n-1} & 0 \end{pmatrix}.$$

- Model introduced by Toda in 1967 (Def above due to Flaschka 1974)
- Represents dynamics of n particles of unit mass, moving on a line under influence of exponential repulsive forces.
- Eigenvalues of $L(t)$ are independent of t.

Sorting property of the Toda lattice

Suppose the initial matrix $L(0)$ is generic: it has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $a_k(0) \neq 0$ for all k. Then the time evolution of the Toda lattice sorts the eigenvalues of L!

$$\lim_{t \to -\infty} L(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\lim_{t \to +\infty} L(t) = \begin{pmatrix} \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{pmatrix}$$

Sorting property of the Toda lattice

Suppose the initial matrix $L(0)$ is generic: it has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $a_k(0) \neq 0$ for all k. Then the time evolution of the Toda lattice sorts the eigenvalues of L!

$$\lim_{t \to -\infty} L(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\lim_{t \to +\infty} L(t) = \begin{pmatrix} \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{pmatrix}$$

Sorting property of the Toda lattice

Suppose the initial matrix $L(0)$ is generic: it has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $a_k(0) \neq 0$ for all k. Then the time evolution of the Toda lattice sorts the eigenvalues of L!

$$\lim_{t \to -\infty} L(t) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\lim_{t \to +\infty} L(t) = \begin{pmatrix} \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{pmatrix}$$

The full symmetric Toda lattice

The full symmetric Toda lattice is defined as before by

$$\frac{dL}{dt} = [\pi(L), L],$$

but now $L = L(t)$ is any symmetric matrix.

- Eigenvalues of $L(t)$ are independent of t.
- In generic case, the sorting property holds (Kodama-McLaughlin '96).
- In non-generic case, what can we say about $\lim_{t \to \pm\infty} L(t)$?

The full symmetric Toda lattice

The full symmetric Toda lattice is defined as before by

$$\frac{dL}{dt} = [\pi(L), L],$$

but now $L = L(t)$ is any symmetric matrix.

- Eigenvalues of $L(t)$ are independent of t.
- In generic case, the sorting property holds (Kodama-McLaughlin '96).
- In non-generic case, what can we say about $\lim_{t \to \pm\infty} L(t)$?

The full symmetric Toda lattice

The full symmetric Toda lattice is defined as before by

$$\frac{dL}{dt} = [\pi(L), L],$$

but now $L = L(t)$ is any symmetric matrix.

- Eigenvalues of $L(t)$ are independent of t.
- In generic case, the sorting property holds (Kodama-McLaughlin '96).
- In non-generic case, what can we say about $\lim_{t\to\pm\infty} L(t)$?

The full symmetric Toda lattice

The full symmetric Toda lattice is defined as before by

$$\frac{dL}{dt} = [\pi(L), L],$$

but now $L = L(t)$ is any symmetric matrix.

- Eigenvalues of $L(t)$ are independent of t.
- In generic case, the sorting property holds (Kodama-McLaughlin '96).
- In non-generic case, what can we say about $\lim_{t\to\pm\infty} L(t)$?

The full symmetric Toda lattice

The full symmetric Toda lattice is defined as before by

$$\frac{dL}{dt} = [\pi(L), L],$$

but now $L = L(t)$ is any symmetric matrix.

- Eigenvalues of $L(t)$ are independent of t.
- In generic case, the sorting property holds (Kodama-McLaughlin '96).
- In non-generic case, what can we say about $\lim_{t\to\pm\infty} L(t)$?

The complete flag variety

Let $G = \mathrm{SL}_n(\mathbb{R})$, $B^+ = B$ and B^- be the subgroups of upper and lower triangular matrices. Then G/B is the *complete flag variety*. Can identify elements with flags

$$\mathrm{Fl}_n = \{V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \mid \dim V_i = i\}.$$

Let $W = S_n$ the symmetric group. For $w \in W$, let $\dot{w}$ denote a representative in G. Have two opposite Schubert decompositions of G/B:

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B = \bigsqcup_{v \in W} B^-\dot{v}B/B.$$

Define the intersection of opposite Schubert cells (Richardson variety):

$$\mathcal{R}_{v,w} := (B\dot{w}B/B) \cap (B^-\dot{v}B/B)$$

The complete flag variety

Let $G = \mathrm{SL}_n(\mathbb{R})$, $B^+ = B$ and B^- be the subgroups of upper and lower triangular matrices. Then G/B is the *complete flag variety*. Can identify elements with flags

$$\mathrm{Fl}_n = \{V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \mid \dim V_i = i\}.$$

Let $W = S_n$ the symmetric group. For $w \in W$, let $\dot{w}$ denote a representative in G. Have two opposite Schubert decompositions of G/B:

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B = \bigsqcup_{v \in W} B^-\dot{v}B/B.$$

Define the intersection of opposite Schubert cells (Richardson variety):

$$\mathcal{R}_{v,w} := (B\dot{w}B/B) \cap (B^-\dot{v}B/B)$$

The complete flag variety

Let $G = \mathrm{SL}_n(\mathbb{R})$, $B^+ = B$ and B^- be the subgroups of upper and lower triangular matrices. Then G/B is the *complete flag variety*. Can identify elements with flags

$$\mathrm{Fl}_n = \{V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \mid \dim V_i = i\}.$$

Let $W = S_n$ the symmetric group. For $w \in W$, let $\dot{w}$ denote a representative in G. Have two opposite Schubert decompositions of G/B:

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B = \bigsqcup_{v \in W} B^-\dot{v}B/B.$$

Define the intersection of opposite Schubert cells (Richardson variety):

$$\mathcal{R}_{v,w} := (B\dot{w}B/B) \cap (B^-\dot{v}B/B)$$

The complete flag variety

Let $G = \mathrm{SL}_n(\mathbb{R})$, $B^+ = B$ and B^- be the subgroups of upper and lower triangular matrices. Then G/B is the *complete flag variety*. Can identify elements with flags

$$\mathrm{Fl}_n = \{V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n \mid \dim V_i = i\}.$$

Let $W = S_n$ the symmetric group. For $w \in W$, let $\dot{w}$ denote a representative in G. Have two opposite Schubert decompositions of G/B:

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B = \bigsqcup_{v \in W} B^-\dot{v}B/B.$$

Define the intersection of opposite Schubert cells (Richardson variety):

$$\mathcal{R}_{v,w} := (B\dot{w}B/B) \cap (B^-\dot{v}B/B)$$

The totally non-negative (tnn) flag variety (Lusztig)

Let U^+ and U^- be the subgroups of upper and lower triangular matrices in G with 1's on diagonal. Let $y_i(m) \in U^-$ be the element

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Let $U^-_{\geq 0}$ of U^- be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B in its real topology.

For comparison: $G/B = \overline{\{\, uB \mid u \in U^- \,\}}$.

The totally non-negative (tnn) flag variety (Lusztig)

Let U^+ and U^- be the subgroups of upper and lower triangular matrices in G with 1's on diagonal. Let $y_i(m) \in U^-$ be the element

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Let $U^-_{\geq 0}$ of U^- be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B in its real topology.

For comparison: $G/B = \overline{\{\, uB \mid u \in U^- \,\}}$.

The totally non-negative (tnn) flag variety (Lusztig)

Let U^+ and U^- be the subgroups of upper and lower triangular matrices in G with 1's on diagonal. Let $y_i(m) \in U^-$ be the element

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Let $U^-_{\geq 0}$ of U^- be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$.
The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B in its real topology.

For comparison: $G/B = \overline{\{\, uB \mid u \in U^- \,\}}$.

The totally non-negative (tnn) flag variety (Lusztig)

Let U^+ and U^- be the subgroups of upper and lower triangular matrices in G with 1's on diagonal. Let $y_i(m) \in U^-$ be the element

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Let $U^-_{\geq 0}$ of U^- be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B in its real topology.

For comparison: $G/B = \overline{\{\, uB \mid u \in U^- \,\}}$.

The totally non-negative (tnn) flag variety (Lusztig)

Let U^+ and U^- be the subgroups of upper and lower triangular matrices in G with 1's on diagonal. Let $y_i(m) \in U^-$ be the element

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Let $U^-_{\geq 0}$ of U^- be the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}},$$

where the closure is taken inside G/B in its real topology.

For comparison: $G/B = \overline{\{\, uB \mid u \in U^- \,\}}$.

The cell decomposition of the tnn flag variety

Recall: $U^-_{\geq 0}$ of U^- is the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{\, uB \mid u \in U^-_{\geq 0} \,\}}.$$

Rietsch's theorem

For $v, w \in W$ with $v \leq w$ in Bruhat order, let

$$\mathcal{R}^{>0}_{v,w} := \mathcal{R}_{v,w} \cap (G/B)_{\geq 0}.$$

This is a topological cell of dimension $\ell(w) - \ell(v)$.
So the tnn flag variety $(G/B)_{\geq 0}$ has a cell decomposition,

$$(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w} \right). \tag{1}$$

The cell decomposition of the tnn flag variety

Recall: $U^-_{\geq 0}$ of U^- is the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{ uB \mid u \in U^-_{\geq 0} \}}.$$

Rietsch's theorem

For $v, w \in W$ with $v \leq w$ in Bruhat order, let

$$\mathcal{R}^{>0}_{v,w} := \mathcal{R}_{v,w} \cap (G/B)_{\geq 0}.$$

This is a topological cell of dimension $\ell(w) - \ell(v)$.
So the tnn flag variety $(G/B)_{\geq 0}$ has a cell decomposition,

$$(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w} \right). \tag{1}$$

The cell decomposition of the tnn flag variety

Recall: $U^-_{\geq 0}$ of U^- is the semigroup in U^- generated by the $y_i(p)$ for $p \in \mathbb{R}_{>0}$. The *tnn flag variety* $(G/B)_{\geq 0}$ is

$$(G/B)_{\geq 0} := \overline{\{ uB \mid u \in U^-_{\geq 0} \}}.$$

Rietsch's theorem

For $v, w \in W$ with $v \leq w$ in Bruhat order, let

$$\mathcal{R}^{>0}_{v,w} := \mathcal{R}_{v,w} \cap (G/B)_{\geq 0}.$$

This is a topological cell of dimension $\ell(w) - \ell(v)$.
So the tnn flag variety $(G/B)_{\geq 0}$ has a cell decomposition,

$$(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w} \right). \tag{1}$$

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Let $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in SO_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Let $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in SO_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.

Let $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in SO_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Let $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Let $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

From $(G/B)_{\geq 0}$ to the full symmetric Toda lattice

Fix real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let $\mathcal{F}_\Lambda$ be the set of symmetric matrices with fixed eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

Let $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

To each $gB \in (G/B)_{\geq 0}$, we associate an initial matrix $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ as follows:

- Use the QR-decomposition to write $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$. This uniquely defines q_0.
- Set $\mathcal{L}^0 = q_0^T \Lambda q_0 \in \mathcal{F}_\Lambda$.
- We can now consider the solution $\mathcal{L}(t)$ to the full symmetric Toda lattice, with initial data $\mathcal{L}(0) := \mathcal{L}^0$.

Theorem (Kodama-W.): generalized sorting property

Recall that $(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}_{v,w}^{>0} \right)$.

Let $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ be the initial matrix associated to $gB \in \mathcal{R}_{v,w}^{>0}$, defined by:

- Factoring $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$
- Setting $\mathcal{L}^0 = q_0^T \Lambda q_0$.

Then

$$\lim_{t \to -\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{v(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{v(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{v(n)} \end{pmatrix}$$

$$\lim_{t \to +\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{w(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{w(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{w(n)} \end{pmatrix}$$

Theorem (Kodama-W.): generalized sorting property

Recall that $(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}_{v,w}^{>0} \right)$.

Let $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ be the initial matrix associated to $gB \in \mathcal{R}_{v,w}^{>0}$, defined by:

- Factoring $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$
- Setting $\mathcal{L}^0 = q_0^T \Lambda q_0$.

Then

$$\lim_{t \to -\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{v(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{v(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{v(n)} \end{pmatrix}$$

$$\lim_{t \to +\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{w(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{w(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{w(n)} \end{pmatrix}$$

Theorem (Kodama-W.): generalized sorting property

Recall that $(G/B)_{\geq 0} = \bigsqcup_{w \in S_n} \left(\bigsqcup_{v \leq w} \mathcal{R}_{v,w}^{>0} \right).$

Let $\mathcal{L}^0 \in \mathcal{F}_\Lambda$ be the initial matrix associated to $gB \in \mathcal{R}_{v,w}^{>0}$, defined by:
- Factoring $g = q_0 b_0$ where $q_0 \in \mathrm{SO}_n(\mathbb{R})$ and $b_0 \in B$
- Setting $\mathcal{L}^0 = q_0^T \Lambda q_0$.

Then

$$\lim_{t \to -\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{v(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{v(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{v(n)} \end{pmatrix}$$

$$\lim_{t \to +\infty} \mathcal{L}(t) = \begin{pmatrix} \lambda_{w(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{w(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{w(n)} \end{pmatrix}$$

Other extensions

The full symmetric Toda hierarchy

The full symmetric Toda hierarchy has $n-1$ parameters $t_1, \dots, t_{n-1}$, and is given by

$$\frac{\partial L}{\partial t_k} = [\pi(L^k), L], \qquad \text{for} \quad k = 1, 2, \dots, n-1.$$

Theorem (Kodama - W.)

Suppose that $gB \in \mathcal{R}_{v,w}^{>0}$ and consider the corresponding solution to the full symmetric Toda hierarchy. Then for each permutation z such that $v \le z \le w$, there exists a direction $\mathbf{t}(s)$ such that $\mathcal{L}(\mathbf{t}(s))$ tends to

$$\begin{pmatrix} \lambda_{z(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{z(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{z(n)} \end{pmatrix}$$

Other extensions

The full symmetric Toda hierarchy

The full symmetric Toda hierarchy has $n-1$ parameters $t_1, \dots, t_{n-1}$, and is given by

$$\frac{\partial L}{\partial t_k} = [\pi(L^k), L], \qquad \text{for} \quad k = 1, 2, \dots, n-1.$$

Theorem (Kodama - W.)

Suppose that $gB \in \mathcal{R}_{v,w}^{>0}$ and consider the corresponding solution to the full symmetric Toda hierarchy. Then for each permutation z such that $v \le z \le w$, there exists a direction $\mathbf{t}(s)$ such that $\mathcal{L}(\mathbf{t}(s))$ tends to

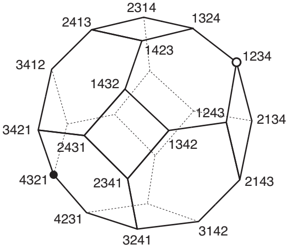
$$\begin{pmatrix} \lambda_{z(1)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_{z(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{z(n)} \end{pmatrix}$$

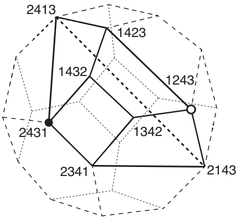
Bruhat interval polytopes

After analyzing the moment map images of flows in the full symmetric Toda hierarchy, we were led to study the following polytopes:

Definition (Kodama -W.)

Let $u \leq v$ in the Bruhat order on S_n. The Bruhat interval polytope $\mathsf{Q}_{u,v}$ is
$$\mathsf{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}.$$





Prop. (K.W.): $\mathsf{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

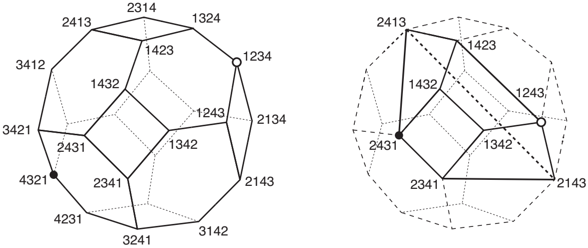
Bruhat interval polytopes

After analyzing the moment map images of flows in the full symmetric Toda hierarchy, we were led to study the following polytopes:

Definition (Kodama -W.)

Let $u \leq v$ in the Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is

$$\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}.$$



Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

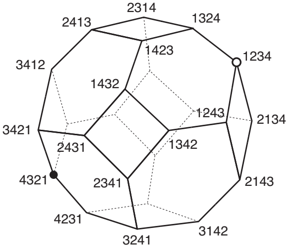
Bruhat interval polytopes

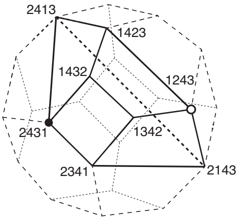
After analyzing the moment map images of flows in the full symmetric Toda hierarchy, we were led to study the following polytopes:

Definition (Kodama -W.)

Let $u \leq v$ in the Bruhat order on S_n. The Bruhat interval polytope $\mathrm{Q}_{u,v}$ is

$$\mathrm{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid u \leq z \leq v\}.$$





Prop. (K.W.): $\mathrm{Q}_{u,v}$ is the Minkowski sum of $n-1$ matroid (positroid) polytopes. It is a generalized permutohedron (in sense of Postnikov).

Combinatorics of Bruhat interval polytopes

Remark: Recall that the edges of the permutohedron correspond to cover relations in the weak Bruhat order. Faces of permutohedra are isomorphic to products of smaller permutohedra.

Theorem (Tsukerman-W.)

The face of every Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$. In particular, each edge of $Q_{u,v}$ comes from a cover relation in the (strong) Bruhat order.

Our proof uses:

- the classical Bjorner-Wachs theorem that the order complex of every interval in Bruhat order is homeomorphic to a sphere;
- a generalization of the *lifting property.*

Combinatorics of Bruhat interval polytopes

Remark: Recall that the edges of the permutohedron correspond to cover relations in the weak Bruhat order. Faces of permutohedra are isomorphic to products of smaller permutohedra.

Theorem (Tsukerman-W.)

The face of every Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$. In particular, each edge of $Q_{u,v}$ comes from a cover relation in the (strong) Bruhat order.

Our proof uses:

- the classical Bjorner-Wachs theorem that the order complex of every interval in Bruhat order is homeomorphic to a sphere;
- a generalization of the *lifting property*.

Combinatorics of Bruhat interval polytopes

Remark: Recall that the edges of the permutohedron correspond to cover relations in the weak Bruhat order. Faces of permutohedra are isomorphic to products of smaller permutohedra.

Theorem (Tsukerman-W.)

The face of every Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \le x \le y \le v$. In particular, each edge of $Q_{u,v}$ comes from a cover relation in the (strong) Bruhat order.

Our proof uses:

- the classical Bjorner-Wachs theorem that the order complex of every interval in Bruhat order is homeomorphic to a sphere;
- a generalization of the *lifting property*.

Combinatorics of Bruhat interval polytopes

Remark: Recall that the edges of the permutohedron correspond to cover relations in the weak Bruhat order. Faces of permutohedra are isomorphic to products of smaller permutohedra.

Theorem (Tsukerman-W.)

The face of every Bruhat interval polytope $Q_{u,v}$ has the form $Q_{x,y}$ where $u \leq x \leq y \leq v$. In particular, each edge of $Q_{u,v}$ comes from a cover relation in the (strong) Bruhat order.

Our proof uses:

- the classical Bjorner-Wachs theorem that the order complex of every interval in Bruhat order is homeomorphic to a sphere;
- a generalization of the *lifting property*.

Combinatorics of Bruhat interval polytopes

Remark: Recall that the edges of the permutohedron correspond to cover relations in the weak Bruhat order. Faces of permutohedra are isomorphic to products of smaller permutohedra.

Theorem (Tsukerman-W.)

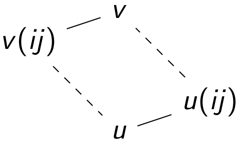
The face of every Bruhat interval polytope $\mathrm{Q}_{u,v}$ has the form $\mathrm{Q}_{x,y}$ where $u \leq x \leq y \leq v$. In particular, each edge of $\mathrm{Q}_{u,v}$ comes from a cover relation in the (strong) Bruhat order.

Our proof uses:

- the classical Bjorner-Wachs theorem that the order complex of every interval in Bruhat order is homeomorphic to a sphere;
- a generalization of the *lifting property*.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



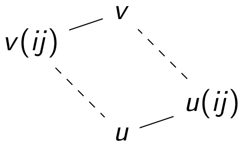
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



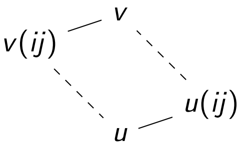
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



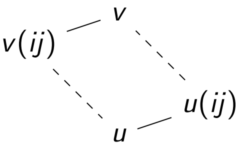
Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

The lifting property and a generalization

Lifting property: Suppose $u < v$ in Bruhat order and s is a simple reflection such that $vs \lessdot v$ and $us \gtrdot u$. Then $u \leq vs \lessdot v$ and $u \lessdot us \leq v$.
Caution: *such an s may not exist.*



Def: Say a transposition (ij) is *inversion-minimal* on (u, v) if $[i, j]$ is minimal (with respect to inclusion) such that $v(ij) < v$ and $u(ij) > u$.

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in Bruhat order on S_n. Choose a transposition (ij) which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

Example of the Generalized lifting property

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in the Bruhat order on S_n. Choose a transposition $t = (ij)$ which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

Example of the Generalized lifting property

Theorem (T.W.) - Generalized lifting property

Suppose $u < v$ in the Bruhat order on S_n. Choose a transposition $t = (ij)$ which is inversion-minimal on (u, v); one always exists. Then $u \leq v(ij) \lessdot v$ and $u \lessdot u(ij) \leq v$.

$$\begin{tikzcd}
 & v = 3241 \arrow[dl, dash, "t = (24)"'] \arrow[dr, dash, "(12)"] & \\
3142 \arrow[dr, dash, "(14)"'] & & 2341 \arrow[dl, dash, "t = (24)"] \\
 & u = 2143 &
\end{tikzcd}$$

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad \text{if } u \not\leq v.$
2. $R_{u,v}(q) = 1, \quad \text{if } u = v.$
3. If $vs < v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us < u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us > u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0$, if $u \not\leq v$.
2. $R_{u,v}(q) = 1$, if $u = v$.
3. If $vs \lessdot v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0, \quad$ if $u \not\leq v$.
2. $R_{u,v}(q) = 1, \quad$ if $u = v$.
3. If $vs \lessdot v$ (s a simple reflection) then

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

Generalization of the recurrence for R-polynomials $R_{u,v}(q)$

Kazhdan and Lusztig introduced R-polynomials as a tool for computing Kazhdan-Lusztig polynomials. Geometric interpretation: $R_{u,v}(q) = \#\mathcal{R}_{u,v}(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points in the Richardson variety.

They showed that R-polynomials can be defined by the conditions:

1. $R_{u,v}(q) = 0$, if $u \not\leq v$.
2. $R_{u,v}(q) = 1$, if $u = v$.
3. If $vs \lessdot v$ (s a simple reflection) then

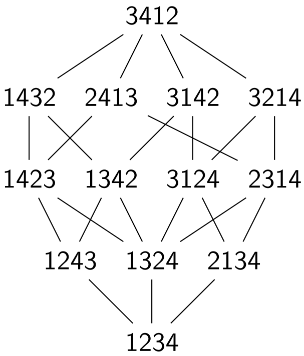
$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } us \lessdot u, \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } us \gtrdot u. \end{cases}$$

Theorem (T.W.)

Let $u, v \in S_n$ with $u \leq v$. Let $t = (ij)$ be inversion-minimal on (u, v). Then $R_{u,v}(q) = qR_{ut,vt}(q) + (q-1)R_{u,vt}(q)$.

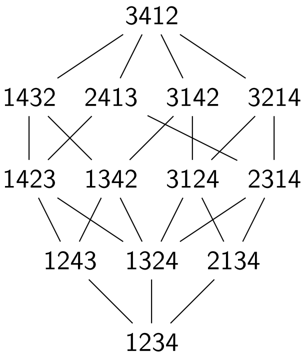
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



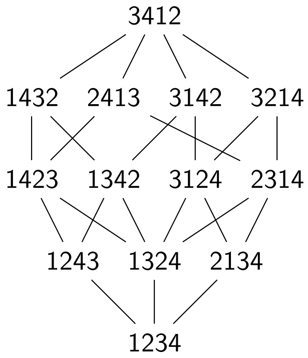
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



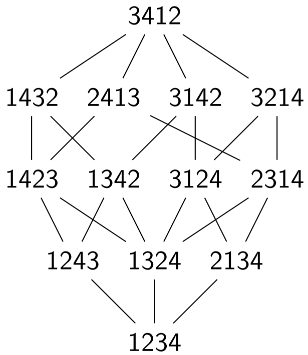
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



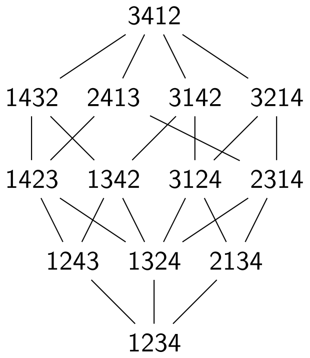
The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \ldots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \ldots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \ldots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



The dimension of Bruhat interval polytopes

Def/Lemma: Let $u \leq v$ in S_n, and let $\mathcal{C} : u = x_{(0)} \lessdot x_{(1)} \dots \lessdot x_{(\ell)} = v$ be any maximal chain in $[u, v]$. Label each edge of $\mathcal{C}$ by the transposition (ab) indicating the positions which are swapped. Then say $a \sim b$ for each edge label on $\mathcal{C}$. Let $B_{u,v} = \{B^1, \dots, B^r\}$ be the blocks of the equivalence relation on $\{1, 2, \dots, n\}$ that $\sim$ generates. Then $B_{u,v}$ is independent of $\mathcal{C}$.



The dimension of Bruhat interval polytopes

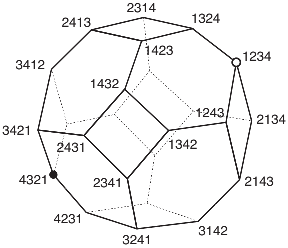
Theorem (T.W.)

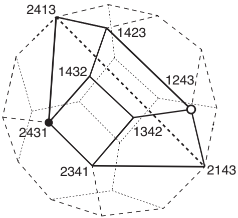
The dimension $\dim \mathrm{Q}_{u,v}$ of the Bruhat interval polytope $\mathrm{Q}_{u,v}$ is

$$\dim \mathrm{Q}_{u,v} = n - \#B_{u,v}.$$

The equations defining the affine span of $\mathrm{Q}_{u,v}$ are

$$\sum_{i \in B^j} x_i = \sum_{i \in B^j} u_i \left(= \sum_{i \in B^j} v_i\right), \quad j = 1, 2, \ldots, \#B_{u,v}.$$





The dimension of Bruhat interval polytopes

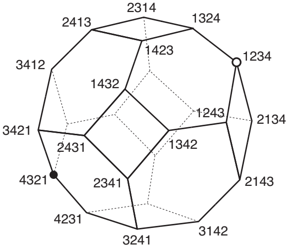
Theorem (T.W.)

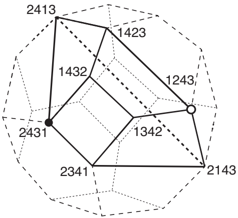
The dimension $\dim \mathrm{Q}_{u,v}$ of the Bruhat interval polytope $\mathrm{Q}_{u,v}$ is

$$\dim \mathrm{Q}_{u,v} = n - \#B_{u,v}.$$

The equations defining the affine span of $\mathrm{Q}_{u,v}$ are

$$\sum_{i \in B^j} x_i = \sum_{i \in B^j} u_i \left(= \sum_{i \in B^j} v_i\right), \quad j = 1, 2, \ldots, \#B_{u,v}.$$





An inequality description for Bruhat interval polytopes

If $\mathcal{M}$ is a matroid on $[n]$ and $A \subset [n]$, let $r_{\mathcal{M}}(A)$ denote the rank of A. Since $\mathrm{Q}_{u,v}$ is a Minkowski sum of matroid polytopes, we get the following.

Theorem (T.W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n-1$, define the matroid $\mathcal{M}_k$ whose bases are

$$\mathcal{B}(\mathcal{M}_k) = \{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ such that } I = \{z(1), \dots, z(k)\}\}.$$

Then

$$\mathrm{Q}_{u,v} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{\mathcal{M}_j}(A) \; \forall \; A \subset [n] \right\}$$

An inequality description for Bruhat interval polytopes

If $\mathcal{M}$ is a matroid on $[n]$ and $A \subset [n]$, let $r_{\mathcal{M}}(A)$ denote the rank of A. Since $\mathrm{Q}_{u,v}$ is a Minkowski sum of matroid polytopes, we get the following.

Theorem (T.W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n-1$, define the matroid $\mathcal{M}_k$ whose bases are

$$\mathcal{B}(\mathcal{M}_k) = \{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ such that } I = \{z(1), \dots, z(k)\}\}.$$

Then

$$\mathrm{Q}_{u,v} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{\mathcal{M}_j}(A) \, \forall \, A \subset [n] \right\}$$

An inequality description for Bruhat interval polytopes

If $\mathcal{M}$ is a matroid on $[n]$ and $A \subset [n]$, let $r_{\mathcal{M}}(A)$ denote the rank of A. Since $\mathrm{Q}_{u,v}$ is a Minkowski sum of matroid polytopes, we get the following.

Theorem (T.W.)

Choose $u \leq v \in S_n$, and for each $1 \leq k \leq n-1$, define the matroid $\mathcal{M}_k$ whose bases are

$$\mathcal{B}(\mathcal{M}_k) = \{I \in \binom{[n]}{k} \mid \exists z \in [u,v] \text{ such that } I = \{z(1), \dots, z(k)\}\}.$$

Then

$$\mathrm{Q}_{u,v} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \binom{n+1}{2}, \sum_{i \in A} x_i \leq \sum_{j=1}^{n-1} r_{\mathcal{M}_j}(A) \,\forall\, A \subset [n] \right\}$$

Bruhat interval polytopes for G/P

When $G = \mathrm{SL}_n(\mathbb{R})$, the Bruhat interval polytope $\mathrm{Q}_{u,v}$ has a natural interpretation in terms of the moment map $\mu : G/B \to \mathbb{R}^n$:

$$\mathrm{Q}_{u,v} = \mu(\mathcal{R}_{u,v}) = \mu(\overline{\mathcal{R}_{u,v}^{>0}}).$$

This leads to the notion of Bruhat interval polytope for G/P.

- G – a semisimple simply connected algebraic group with torus T and Weyl group W
- $P = P_J$ – a parabolic subgroup of G
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.
- Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J.

We define the *Bruhat interval polytope for G/P* to be

$$\mathrm{Q}^J_{u,v} := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}^*_{\mathbb{R}}.$$

Bruhat interval polytopes for G/P

When $G = \mathrm{SL}_n(\mathbb{R})$, the Bruhat interval polytope $\mathsf{Q}_{u,v}$ has a natural interpretation in terms of the moment map $\mu : G/B \to \mathbb{R}^n$:

$$\mathsf{Q}_{u,v} = \mu(\mathcal{R}_{u,v}) = \mu(\overline{\mathcal{R}_{u,v}^{>0}}).$$

This leads to the notion of Bruhat interval polytope for G/P.

- G – a semisimple simply connected algebraic group with torus T and Weyl group W
- $P = P_J$ – a parabolic subgroup of G
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.
- Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J.

We define the *Bruhat interval polytope for G/P* to be

$$\mathsf{Q}_{u,v}^J := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}_{\mathbb{R}}^*.$$

Bruhat interval polytopes for G/P

When $G = \mathrm{SL}_n(\mathbb{R})$, the Bruhat interval polytope $\mathsf{Q}_{u,v}$ has a natural interpretation in terms of the moment map $\mu : G/B \to \mathbb{R}^n$:

$$\mathsf{Q}_{u,v} = \mu(\mathcal{R}_{u,v}) = \mu(\overline{\mathcal{R}_{u,v}^{>0}}).$$

This leads to the notion of Bruhat interval polytope for G/P.

- G – a semisimple simply connected algebraic group with torus T and Weyl group W
- $P = P_J$ – a parabolic subgroup of G
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.
- Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J.

We define the *Bruhat interval polytope for G/P* to be

$$\mathsf{Q}_{u,v}^J := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}_{\mathbb{R}}^*.$$

Bruhat interval polytopes for G/P

When $G = \mathrm{SL}_n(\mathbb{R})$, the Bruhat interval polytope $\mathrm{Q}_{u,v}$ has a natural interpretation in terms of the moment map $\mu : G/B \to \mathbb{R}^n$:

$$\mathrm{Q}_{u,v} = \mu(\mathcal{R}_{u,v}) = \mu(\overline{\mathcal{R}_{u,v}^{>0}}).$$

This leads to the notion of Bruhat interval polytope for G/P.

- G – a semisimple simply connected algebraic group with torus T and Weyl group W
- $P = P_J$ – a parabolic subgroup of G
- ρ_J – sum of fund. weights corresp. to J, so that $G/P \hookrightarrow \mathbb{P}(V_{\rho_J})$.
- Choose $u \leq v$ in W, where v is a min-length coset rep in W/W_J.

We define the *Bruhat interval polytope for G/P* to be

$$\mathrm{Q}_{u,v}^J := \mathrm{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}_{\mathbb{R}}^*.$$

Bruhat interval polytopes for G/P

Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- positroid polytopes

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

Tools in proof:

- work of Rietsh and Marsh-Rietsch,
- the moment map, and the *Gelfand-Serganova* stratification of G/P (which generalizes the matroid stratification of the Grassmannian).

Bruhat interval polytopes for G/P

Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- positroid polytopes

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

Tools in proof:

- work of Rietsh and Marsh-Rietsch,
- the moment map, and the *Gelfand-Serganova* stratification of G/P (which generalizes the matroid stratification of the Grassmannian).

Bruhat interval polytopes for G/P

Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- positroid polytopes

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

Tools in proof:

- work of Rietsh and Marsh-Rietsch,
- the moment map, and the *Gelfand-Serganova* stratification of G/P (which generalizes the matroid stratification of the Grassmannian).

Bruhat interval polytopes for G/P

Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- positroid polytopes

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

Tools in proof:

- work of Rietsh and Marsh-Rietsch,
- the moment map, and the *Gelfand-Serganova* stratification of G/P (which generalizes the matroid stratification of the Grassmannian).

Bruhat interval polytopes for G/P

Bruhat interval polytopes for G/P include:

- Bruhat interval polytopes
- positroid polytopes

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

Tools in proof:

- work of Rietsh and Marsh-Rietsch,
- the moment map, and the *Gelfand-Serganova* stratification of G/P (which generalizes the matroid stratification of the Grassmannian).

Bruhat interval polytopes for G/P

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

As part of the proof, we also show:

Theorem (T.W.)

Each cell of $(G/P)_{\geq 0}$ is contained in one Gelfand-Serganova stratum.[a]

[a] This was conjectured by Rietsch and partially proved in an unpublished manuscript of Marsh-Rietsch.

Moreover, Rietsch's cell decomposition of $(G/B)_{\geq 0}$ is the restriction of the Gelfand-Serganova stratification to G/B. This is analogous to the fact that Postnikov's cell decomposition of $Gr_{k,n}^{+}$ is the restriction of the matroid stratification to $Gr_{k,n}$.

Bruhat interval polytopes for G/P

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

As part of the proof, we also show:

Theorem (T.W.)

Each cell of $(G/P)_{\geq 0}$ is contained in one Gelfand-Serganova stratum.[a]

[a] This was conjectured by Rietsch and partially proved in an unpublished manuscript of Marsh-Rietsch.

Moreover, Rietsch's cell decomposition of $(G/B)_{\geq 0}$ is the restriction of the Gelfand-Serganova stratification to G/B. This is analogous to the fact that Postnikov's cell decomposition of $Gr_{k,n}^{+}$ is the restriction of the matroid stratification to $Gr_{k,n}$.

Bruhat interval polytopes for G/P

Theorem (T.W.)

The face of a Bruhat interval polytopes for G/P is again a Bruhat interval polytope for G/P.

As part of the proof, we also show:

Theorem (T.W.)

Each cell of $(G/P)_{\geq 0}$ is contained in one Gelfand-Serganova stratum.[a]

[a] This was conjectured by Rietsch and partially proved in an unpublished manuscript of Marsh-Rietsch.

Moreover, Rietsch's cell decomposition of $(G/B)_{\geq 0}$ is the restriction of the Gelfand-Serganova stratification to G/B. This is analogous to the fact that Postnikov's cell decomposition of $Gr^+_{k,n}$ is the restriction of the matroid stratification to $Gr_{k,n}$.

Happy birthday Richard!