Generalized Stability of Kronecker Coefficients

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1. Introduction

Let I_α be the irrep of S_m indexed by $\alpha \vdash m$.

$$g(\alpha\beta\gamma) := \text{mult. of } I_\gamma \text{ in } I_\alpha \otimes I_\beta = \dim(I_\alpha \otimes I_\beta \otimes I_\gamma)^{S_m}.$$

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Theorem (Murnaghan)

The sequence $g(\alpha + n, \beta + n, \gamma + n)$ converges as $n \to \infty$.

One can also show that the convergence is monotone.
Murnaghan's result is part of a much larger pattern of stability....

Motivation

Why should we care about stability?

- C. Bowman, M. De Visscher and R. Orellana:
  Murnaghan's stable coefficients are related to tensor product multiplicities in the partition algebra.

- T. Church, J. Ellenberg and B. Farb,
  "*FI*-modules: a new approach to stability for S_n-reps."
  A category whose objects are sequences of S_n-modules for $n \geqslant 1$. Finite generation $\Rightarrow$ multiplicities stabilize.

- S. Sam and A. Snowden,
  "Stability patterns in representation theory."
  Many classical groups have representation theories with stable limits.

We will be considering limits that don't necessarily fit into these frameworks...

2. A First Glimpse

Why restrict ourselves to adding columns of length 1?
E.g., why not investigate

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Bad news at $k = 2$: no convergence, no monotonicity.

$$g(nn, nn, nn) = \begin{cases} 1 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

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Fact

The sequence $g(\alpha + n^2, \beta + n^2, \gamma + n^2)$ breaks into monotone convergent subsequences, one for even n, and one for odd n.

Convergence is subtle, but can be reduced to the 2-row case.
For 2-row cases, there are known (messy, ad-hoc) formulas.

And what about $k = 3, 4, 5, \ldots$?
In general these sequences grow without bound.

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Problem (first draft)

Characterize all triples $\alpha\beta\gamma$ such that

$$\lim_{n\to\infty} g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$$

converges for all $\lambda\mu\nu$.

Examples include $\alpha\beta\gamma = (1, 1, 1)$ (Murnaghan) and $(22, 22, 22)$.

3. Monotonicity

Kronecker coefficients also live in the GL-world.

Let $V(\alpha) =$ irrep of $\mathrm{gl}(V)$ with highest weight α.

- Makes sense if $\ell(\alpha) \leqslant \dim V$; 0 otherwise.
- $V(m) = S^m(V)$ (homog. polys of degree m over V).

Fact

Provided that V_1, V_2, V_3 have sufficiently large dimensions, $g(\alpha\beta\gamma)$ is the multiplicity of $V_1(\alpha) \otimes V_2(\beta) \otimes V_3(\gamma)$ in $S^m(V_1 \otimes V_2 \otimes V_3)$ as a $\mathrm{gl}(V_1) \oplus \mathrm{gl}(V_2) \oplus \mathrm{gl}(V_3)$-module.

Equivalently, $g(\alpha\beta\gamma)$ is the dimension of the space of maximal vectors of weight $\alpha \oplus \beta \oplus \gamma$ in $S^*(V_1 \otimes V_2 \otimes V_3)$.

Maximal means killed by the strictly upper triangular part of $\mathrm{gl}(V_1) \oplus \mathrm{gl}(V_2) \oplus \mathrm{gl}(V_3)$.

Key Point: maximal vectors in $S^*(\cdot)$ form a graded subring R.

So if $f_1 \ldots, f_r \in R$ are linearly independent of h.w. $\lambda \oplus \mu \oplus \nu$, and $g \in R$ has h.w. $\alpha \oplus \beta \oplus \gamma$, then $gf_1, \ldots, gf_r \in R$ are linearly independent of h.w. $(\lambda + \alpha) \oplus (\mu + \beta) \oplus (\nu + \gamma)$. This proves...

Proposition

If $g(\alpha\beta\gamma) > 0$, then $g(\lambda\mu\nu + \alpha\beta\gamma) \geqslant g(\lambda\mu\nu)$.

Corollary (probably well-known)

$\mathcal{G} := \{\alpha\beta\gamma : g(\alpha\beta\gamma) > 0\}$ is a semigroup.

Corollary

If $g(\alpha\beta\gamma) > 0$, then $g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$ is weakly increasing. In particular, it converges iff it is bounded.

Example: $g(11, 11, 11) = 0$, $g(22, 22, 22) = 1$ explains the previously observed instance of "alternating" monotonicity.

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Problem (improved)

Characterize in some practical way all stable triples; i.e., all $\alpha\beta\gamma \in \mathcal{G}$ such that $g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)_{n\geqslant 1}$ is bounded (equivalently, convergent) for all $\lambda\mu\nu \in \mathcal{G}$.

Claim

- (α, α, m) is stable for all $\alpha \vdash m$.
- $(\alpha, \alpha', 1^m)$ is stable for all $\alpha \vdash m$.

More examples will be forthcoming...

Claim

If $g(\alpha\beta\gamma) \geqslant 2$, then $g(n \cdot \alpha\beta\gamma) \geqslant n+1$.

This bound can be sharp; e.g., $g(n \cdot (42,42,42)) = n+1$.

Corollary

If $\alpha\beta\gamma$ is stable, then $g(n \cdot \alpha\beta\gamma) = 1$ for all $n \geqslant 1$.

Example: $g(2^3, 2^3, 2^3) = 1$, but $g(4^3, 4^3, 4^3) = 2$,
so $(2^3, 2^3, 2^3)$ is not stable.

Proof of Claim:

Let $f_1, f_2 \in R$ be linearly independent, h.w. $\alpha \oplus \beta \oplus \gamma$.
Then f_1, f_2 are algebraically independent(!).
So $f_1^n, f_1^{n-1} f_2, \ldots, f_2^n \in R$ are linearly independent.
Each has h.w. $n\alpha \oplus n\beta \oplus n\gamma$. $\square$

4. Some Non-Convergence

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Intuition: Suppose we had a positive formula

$$g(\alpha\beta\gamma) = \#(\mathbb{Z}^N \cap P_{\alpha\beta\gamma}),$$

where $P_{\alpha\beta\gamma}$ is a $\mathbb{Q}$-polytope with walls varying linearly with $\alpha\beta\gamma$.

If so, then $g(n \cdot \alpha\beta\gamma)$ would be an Ehrhart quasi-polynomial.
(We do know that $g(n \cdot \alpha\beta\gamma)$ is a quasi-polynomial for large n.)

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Having $g(n \cdot \alpha\beta\gamma) = 1$ for all $n \geqslant 1$ implies that $\dim P_{\alpha\beta\gamma} = 0$ and that the unique point $\mathbf{p} \in P_{\alpha\beta\gamma}$ is a lattice point.

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$\Rightarrow P_{\lambda\mu\nu + n\cdot\alpha\beta\gamma} \subset$ a ball of fixed radius centered at $n\mathbf{p}$.
$\Rightarrow g(\lambda\mu\nu + n \cdot \alpha\beta\gamma)$ is bounded. $\square$

5. Some Convergence/Stability

How to prove that $\alpha\beta\gamma$ is stable?

Idea: Pass to reducible S_m-reps, and represent their tensor product multiplicities using integer points in polytopes.

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Let $M_\alpha := S_m$-action on $S_m/S_{\alpha_1} \times S_{\alpha_2} \times \cdots$ (a perm. rep).

Easy: $M_\alpha \otimes M_\beta \cong \bigoplus_{T \in C(\alpha,\beta)} M_{\mathrm{co}(T)}$, where $C(\alpha, \beta)$ is the set of integer points in the transportation polytope

$$Q(\alpha, \beta) = \left\{ [x_{ij}] : x_{ij} \geqslant 0,\ \textstyle\sum_j x_{ij} = \alpha_i,\ \sum_i x_{ij} = \beta_j \right\},$$

and $\mathrm{co}(T)$ denotes the content of T (a partition).

Definition: $h(\alpha\beta\gamma) :=$ multiplicity of I_γ in $M_\alpha \otimes M_\beta$.
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Better: $h(\alpha\beta\gamma) = \sum_T K_{\gamma,\mathrm{co}(T)}$, where
$K_{\gamma,\delta} =$ multiplicity of I_γ in M_δ (Kostka number),
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$$Q(\alpha,\beta;\gamma) := \left\{[x_{ij}] \in Q(\alpha,\beta) : \mathrm{co}(x_{ij}) \leqslant \gamma\right\}.$$

Theorem

If $\alpha\beta\gamma \in \mathcal{G}$ and $h(n\cdot\alpha\beta\gamma) = 1$ for all $n \geqslant 1$, then $\alpha\beta\gamma$ is stable.

Note: $h(n\cdot\alpha\beta\gamma) = 1$ for all $n \geqslant 1$ iff $Q(\alpha,\beta;\gamma) = \{T\}$ for some integer table T such that $K_{\gamma,\mathrm{co}(T)} = 1$.

Proof: Recycle the "proof" that $g(n\cdot\alpha\beta\gamma) = 1 \Rightarrow$ stability. $\square$

Remarks

- How effective is this theorem at finding stable triples?

n	$\#\mathcal{G}^n/\sim$	#stable	
1	1	1	
2	2	2	
3	5	4	$(21, 21, 21)$ not stable
4	15	11	Theorem$\Rightarrow$ all but 2
5	40	18	Theorem$\Rightarrow$ all but 3

- Theorem $\Rightarrow$ (m, α, α) is stable; $Q(m, \alpha)$ is 0-dimensional.
- Theorem $\Rightarrow$ $(\alpha, \alpha', 1^m)$ is stable; here

$$Q(\alpha, \alpha'; 1^m) = \{T\}, \quad T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Conjecture

If $h(\alpha\beta\gamma) = 1$, then $h(n \cdot \alpha\beta\gamma) = 1$ for all $n \geqslant 1$.

- Confirmed for partitions of size $\leqslant 10$.
- Having $h(\alpha\beta\gamma) = 1$ is equivalent to having (1) a unique integer $T \in Q(\alpha, \beta; \gamma)$, and (2) $K_{\gamma,\mathrm{co}(T)} = 1$.
- $Q(\alpha, \beta; \gamma)$ need not be a lattice polytope even if it contains only one lattice point; i.e., (2) is necessary.

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Problem

Identify the isolated integer points $T \in Q(\alpha, \beta)$; i.e., all T such that no other integer $T' \in Q(\alpha, \beta)$ has $\mathrm{co}(T') \leqslant \mathrm{co}(T)$.

- If the conjecture is true and $T \in Q(\alpha, \beta)$ is isolated and has content γ, then $\alpha\beta\gamma$ is stable.
- If T is isolated, then T must be a plane partition. The converse is true for $\ell(\alpha) \leqslant 2$ but not in general.

Stanley 700 battery jump starter

An Algebra Digression

It would be nice to have growth bounds that go beyond linear.

Definition/Problem

Define $\delta(n, d, r)$ to be the minimum, over all sequences $f_1, \ldots, f_r$ of linearly independent homogeneous polynomials of degree d, of

$$\dim \operatorname{Span}\{f_1^{i_1} \cdots f_r^{i_r} : i_1 + \cdots + i_r = n\}.$$

The problem is to determine $\delta(n, d, r)$.

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Consequence:
If $g(\alpha\beta\gamma) = r$ and $\alpha, \beta, \gamma \vdash d$, then $g(n \cdot \alpha\beta\gamma) \geqslant \delta(n, d, r)$.

Intuition:
The optimal case should be to take $f_1, \ldots, f_r$ to be monomials in as few variables as possible.
Example: If $d = 5$, $r = 3$, take $f_1, f_2, f_3 = x^5, x^4 y, x^3 y^2$.