The topology of the permutation pattern poset

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University of Strathclyde

Work by Jason P. Smith and
joint work with Peter McNamara
and with A. Burstein, V. Jelínek and E. Jelínková

An *occurrence* of a pattern p in a permutation π is a subsequence in π whose letters appear in the same order of size as those in p.

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(No decreasing subsequence of length 4)

The set of all permutations forms a poset $\mathcal{P}$
with respect to pattern containment

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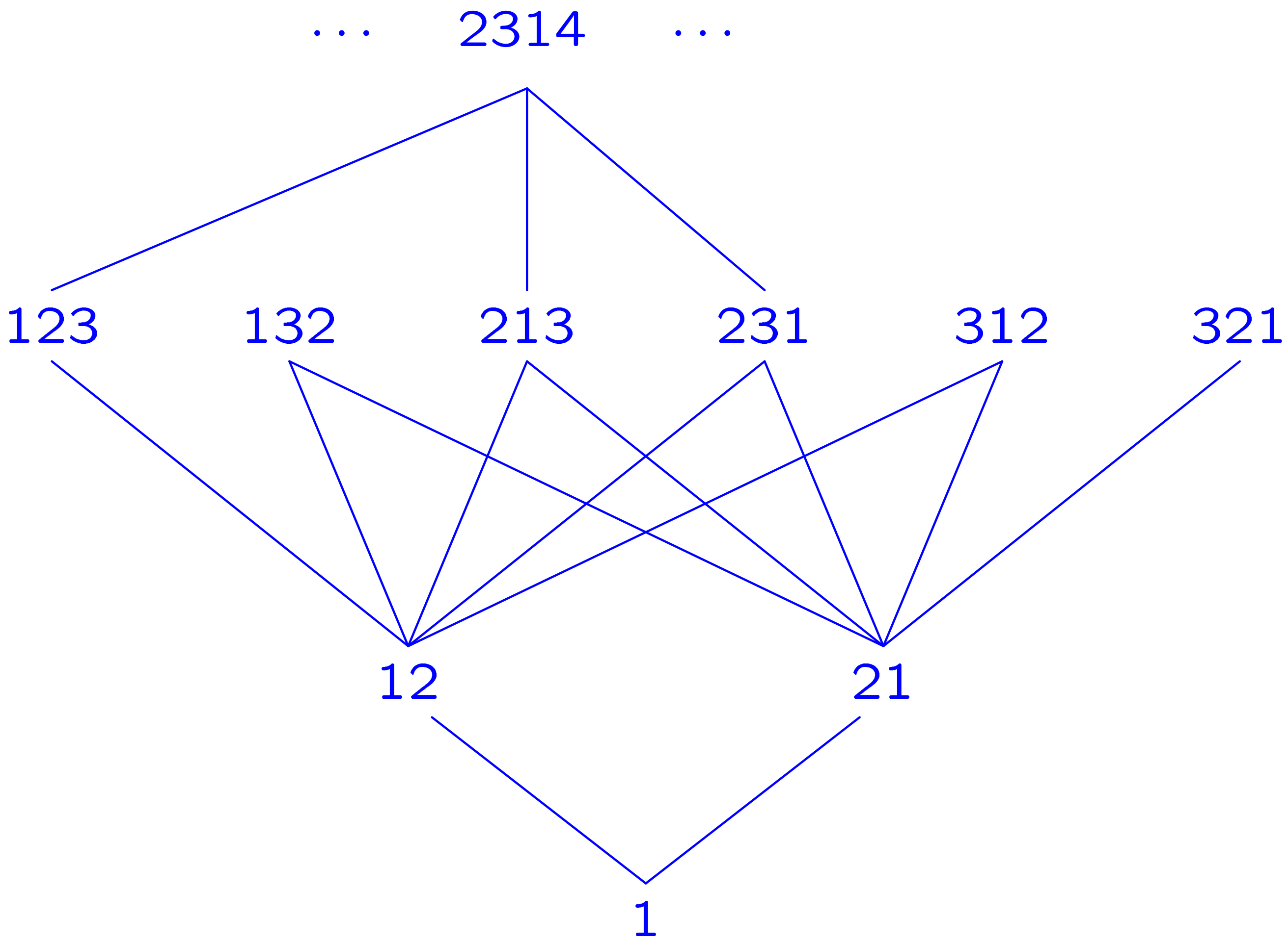
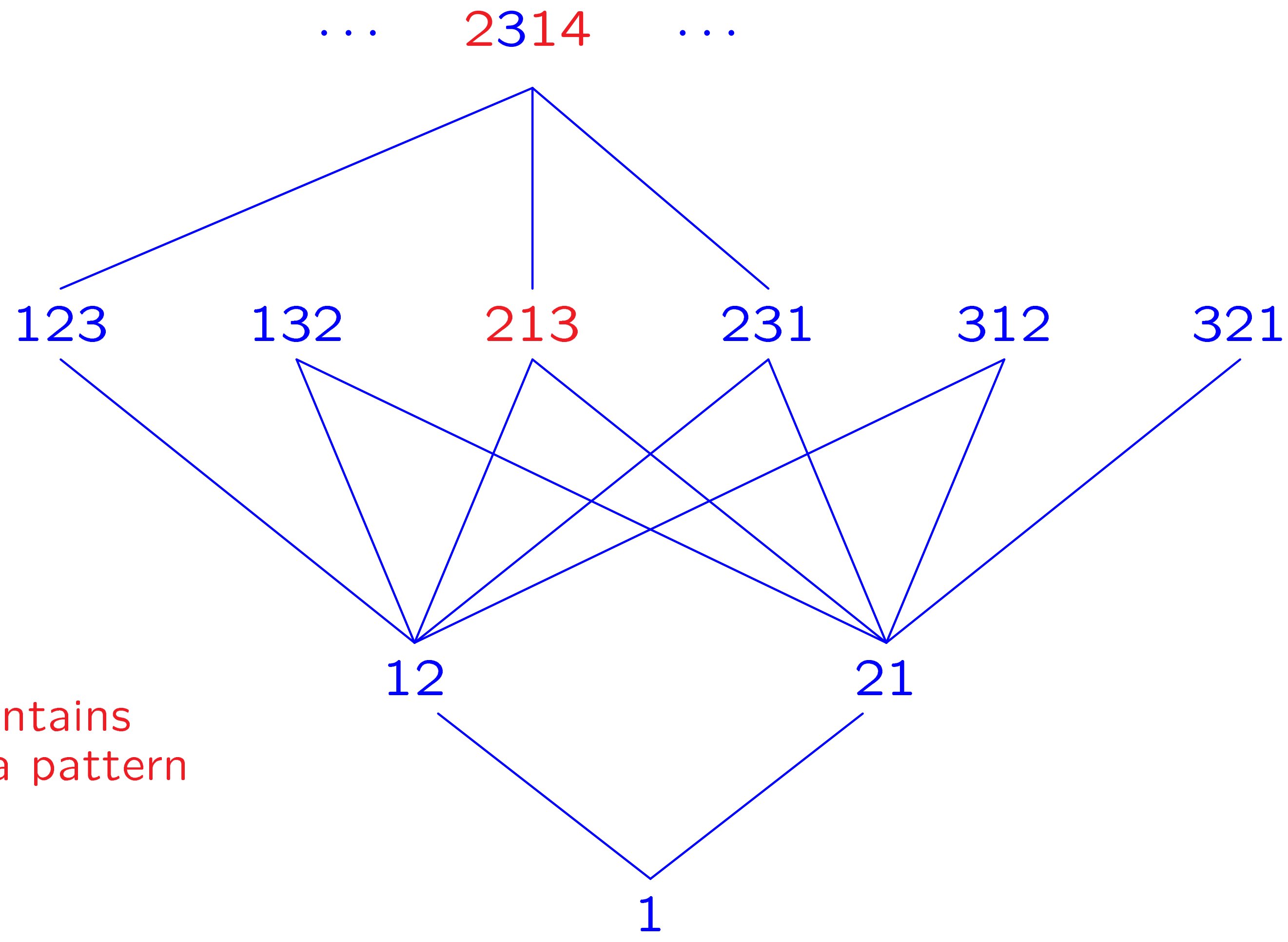
$\sigma \leqslant \tau$ if σ is a pattern in τ

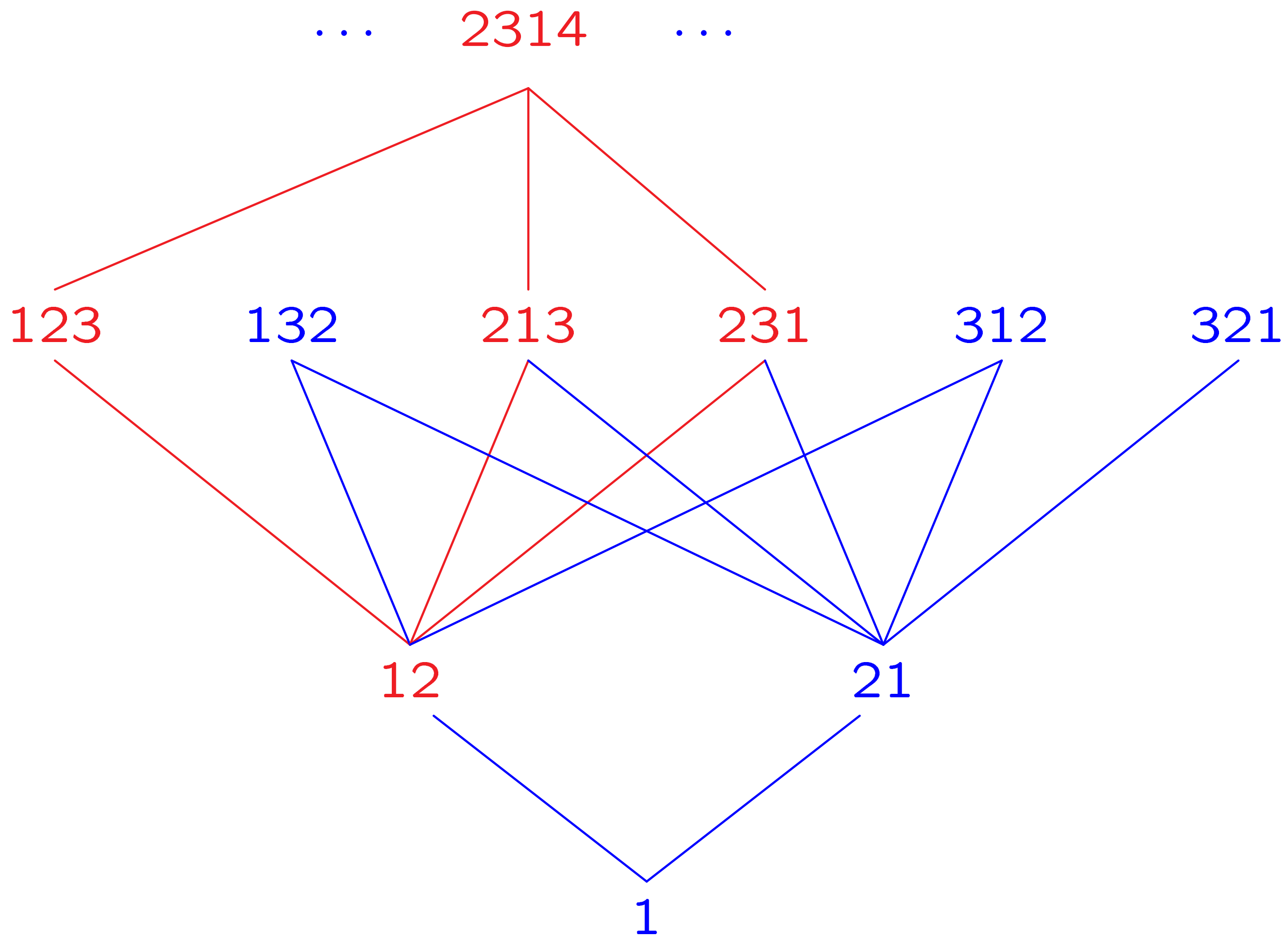


The bottom of the poset $\mathcal{P}$

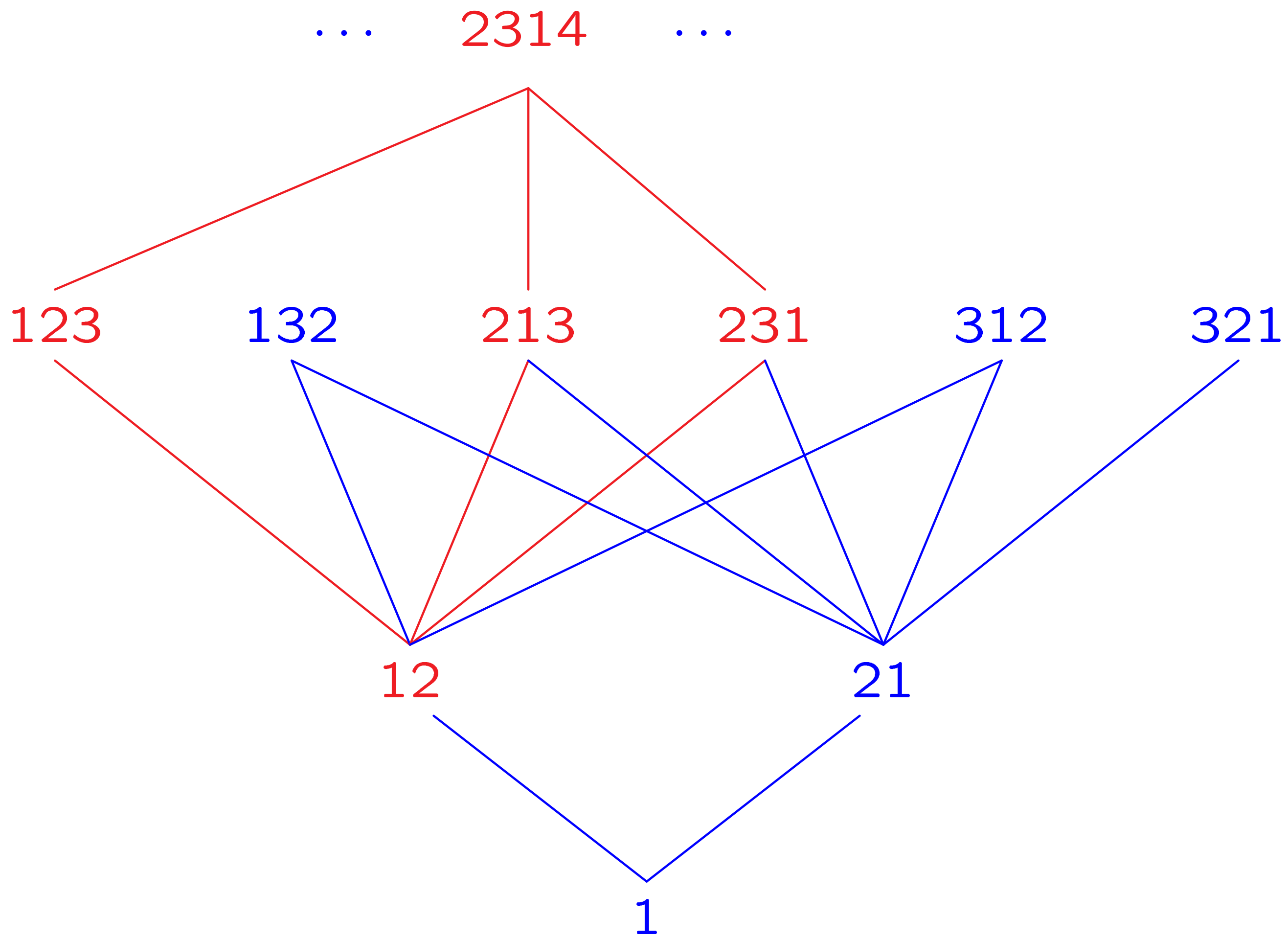


2314 contains
213 as a pattern

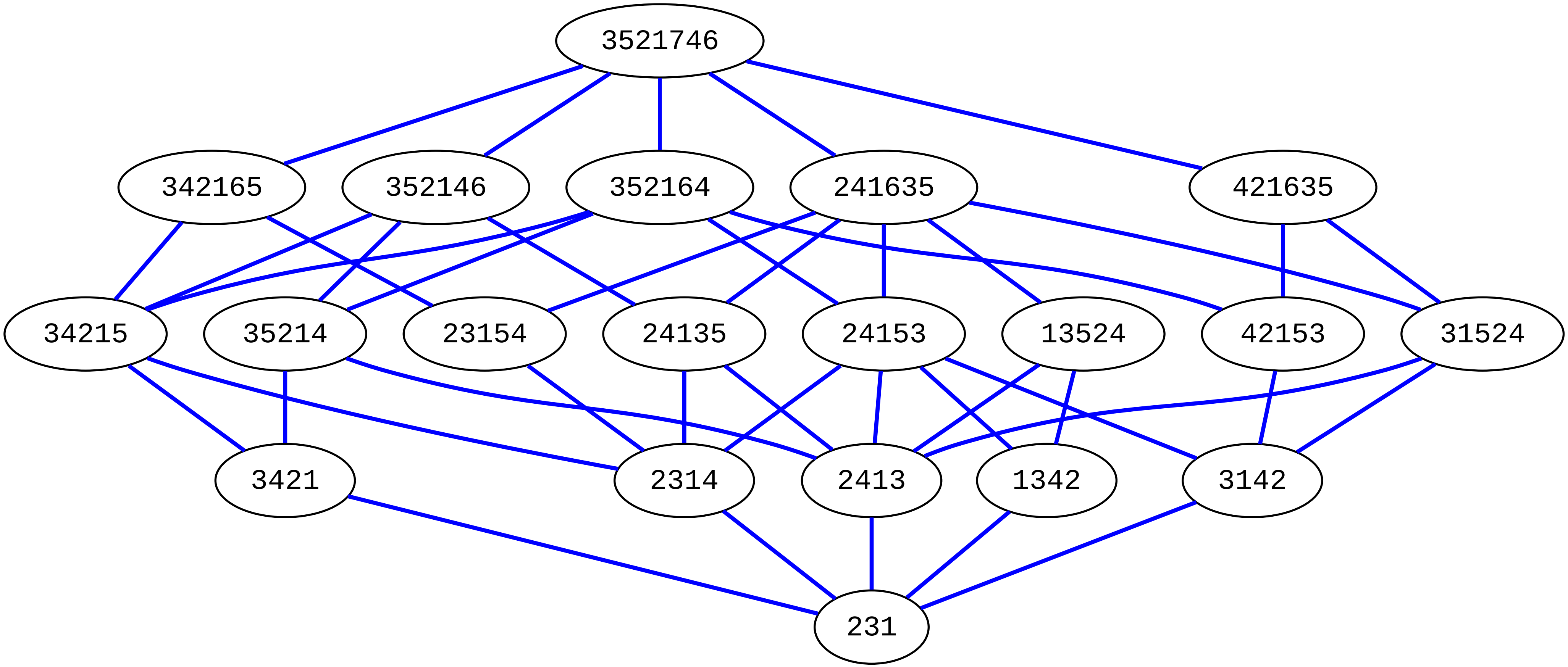
The bottom of the poset $\mathcal{P}$



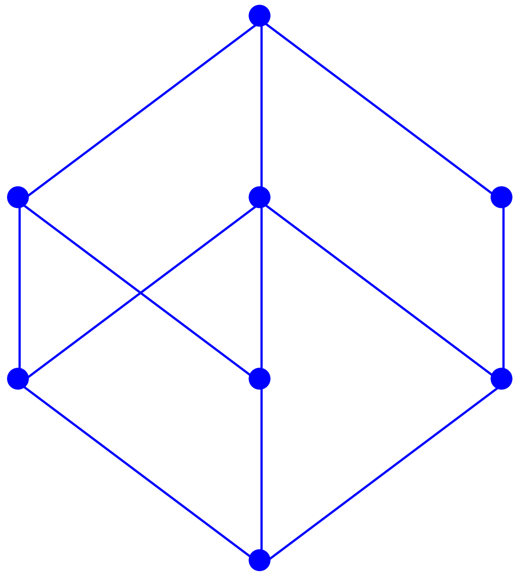
The interval $[12, 2314]$



The interval $[12, 2314] = \{\pi \mid 12 \leqslant \pi \leqslant 2314\}$


3521746
342165
352146
352164
241635
421635
34215
35214
23154
24135
24153
13524
42153
31524
3421
2314
2413
1342
3142
231

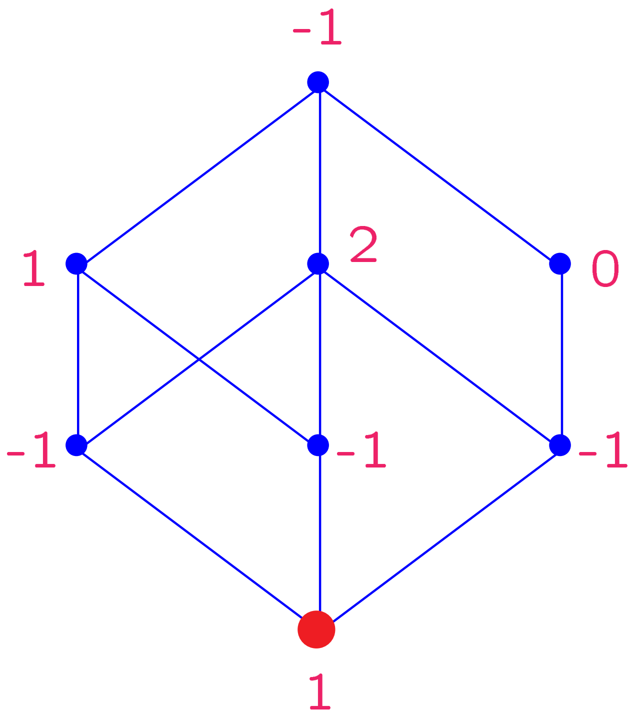
The Möbius function of an interval $\mathcal{I}$



The Möbius function on $\mathcal{I}$ is defined by $\mu(x,x) = 1$ and

$$\sum_{x \leqslant t \leqslant y} \mu(x,t) = 0 \quad \text{if } x < y$$

Computing $\mu(\bullet, y)$ on an interval $\mathcal{I}$



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Computing the Möbius function for the pattern poset

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Wilf (2002): Should be done

Computing the Möbius function for the pattern poset

A very short prehistory

Wilf (2002): Should be done

Wilf (2003): A mess. Don't touch it.

Jason Smith (2014)

A *descent* in a permutation is a letter followed by a smaller one

516792348

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An *adjacency* in a permutation is a maximal increasing sequence of letters adjacent in position and value:

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An occurrence of a pattern in a permutation π is *normal* if the occurrence contains all the tails of π.

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516792348
5   7   34

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The normal occurrences of 3412 in 516792348:

5734

Jason Smith (2014)

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516792348
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5734 and 6734

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Theorem: If σ and τ have the same number of descents, then

$$\mu(\sigma,\tau) = (-1)^{|\tau|-|\sigma|} N(\sigma,\tau),$$

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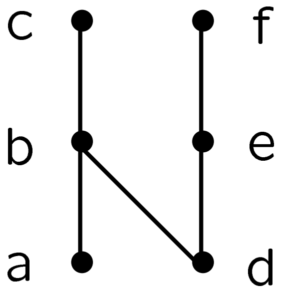
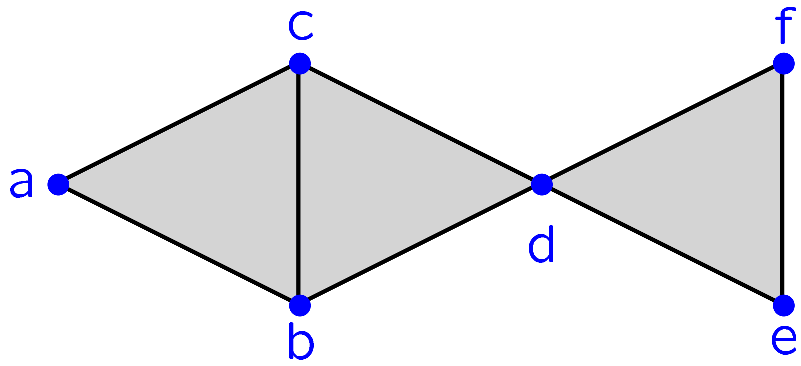
That is, the *order complex* of $[\sigma, \tau]$ is shellable.

$\mathcal{P}$



$\longrightarrow$

Order complex of $\mathcal{P}$



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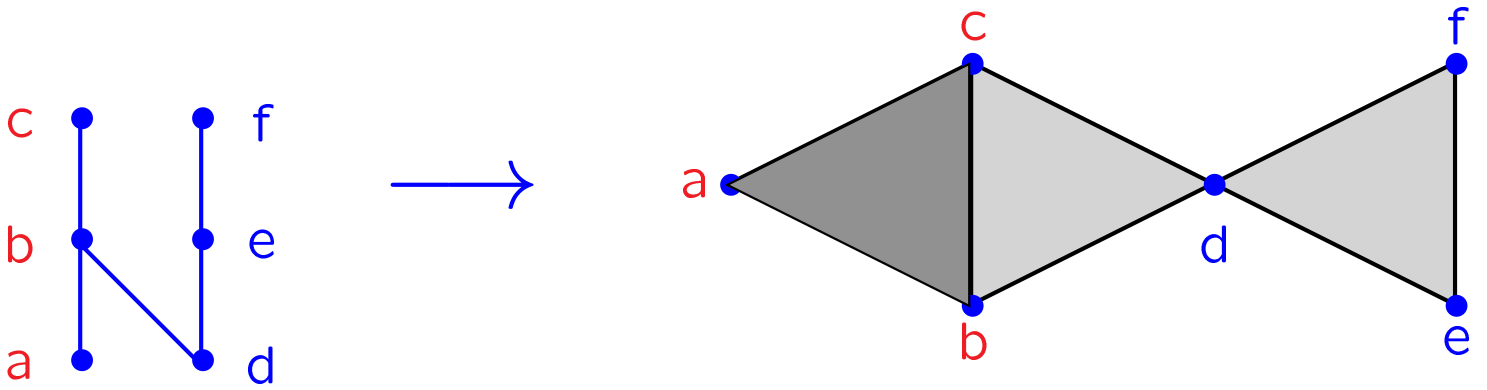

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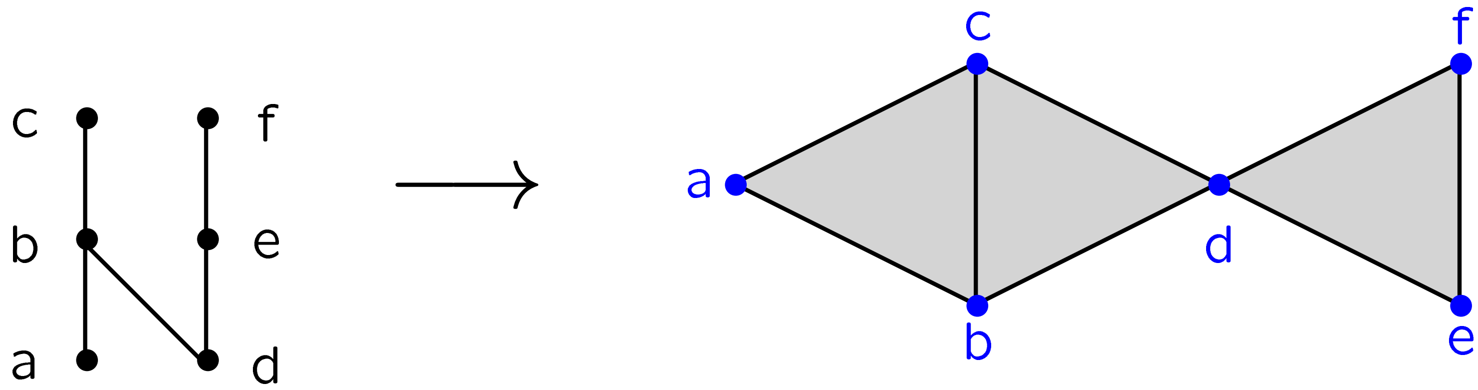

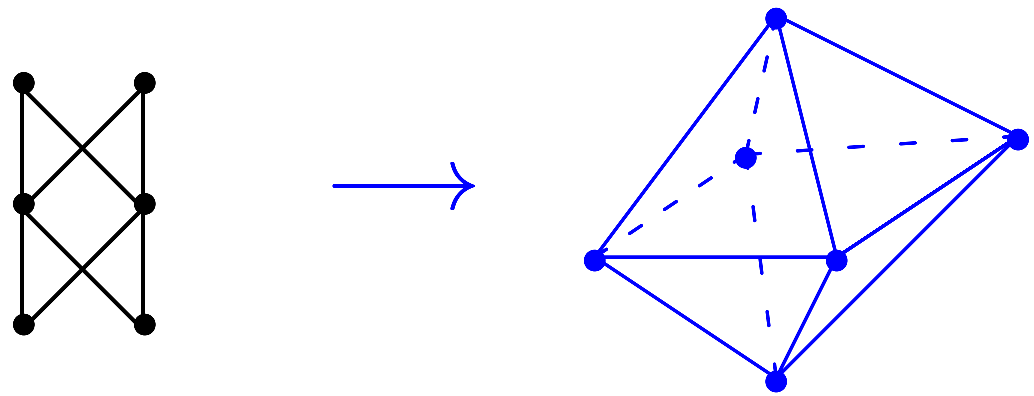

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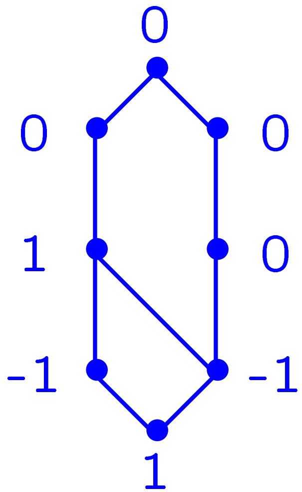

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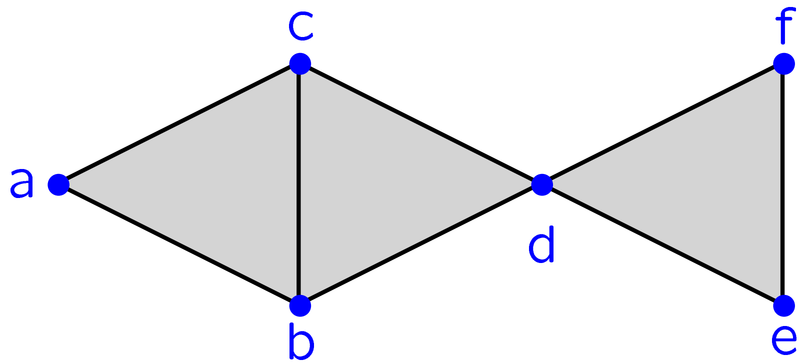
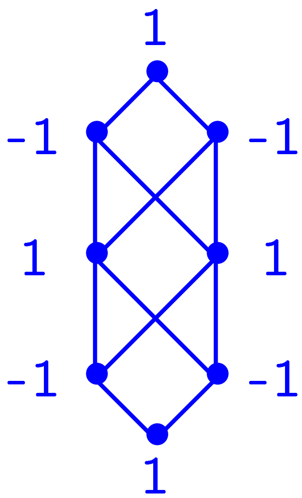
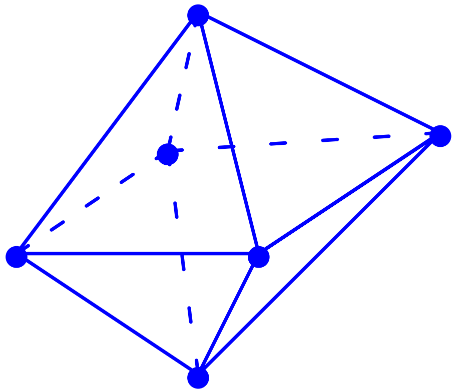


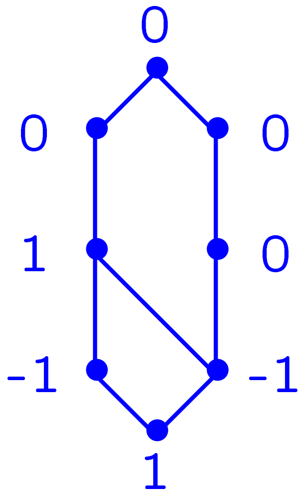


$\longrightarrow$

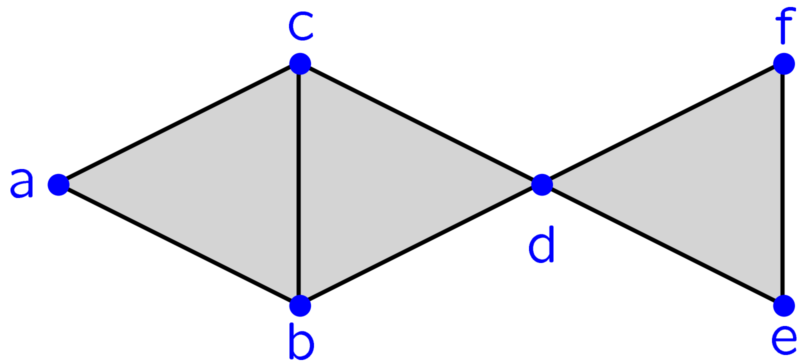


$\widehat{\mathcal{P}}$

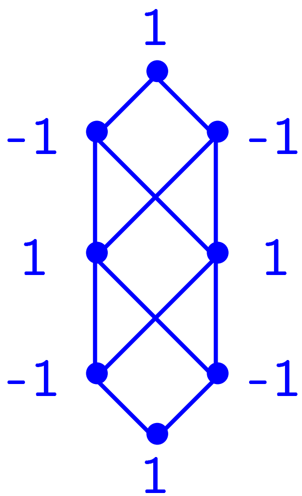
Order complex of $\mathcal{P}$



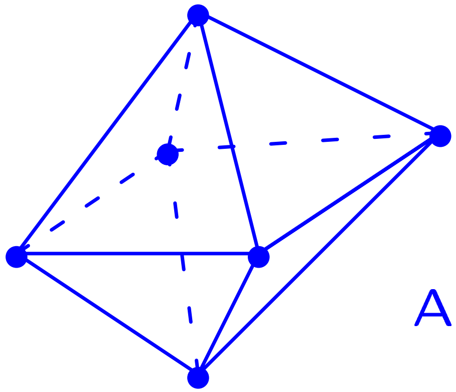
$\longrightarrow$



Contractible

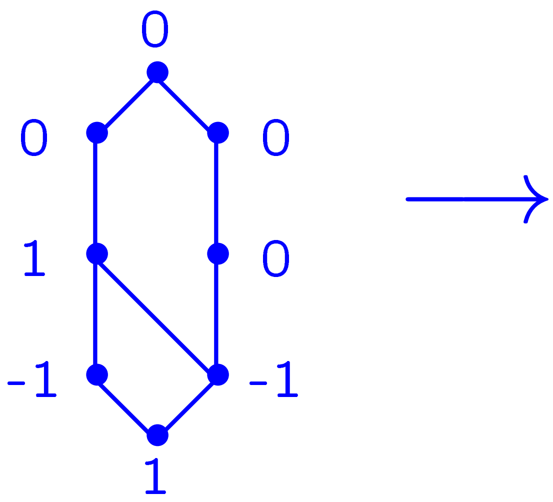


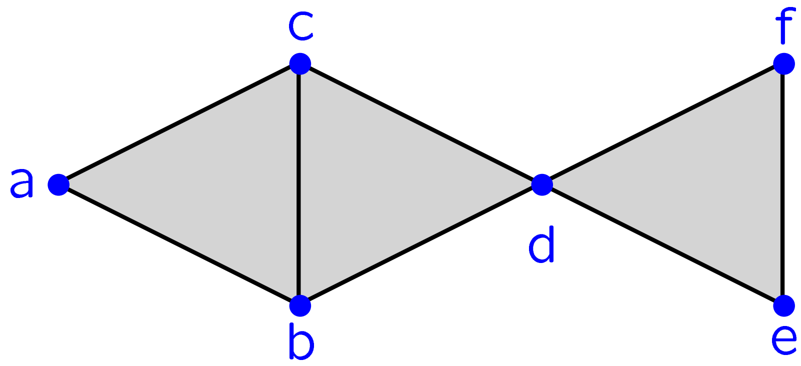
$\longrightarrow$



A sphere

$\widehat{\mathcal{P}}$ Order complex of $\mathcal{P}$





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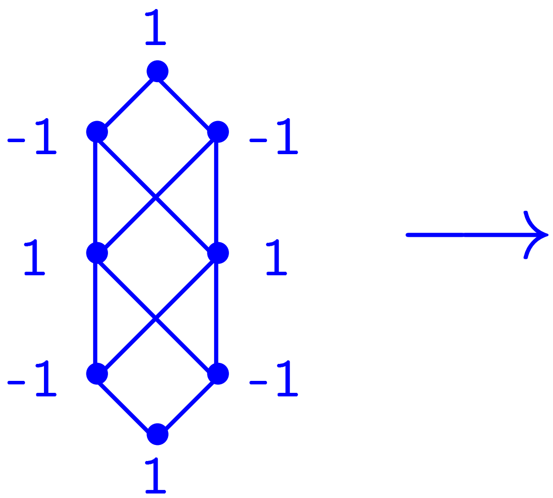
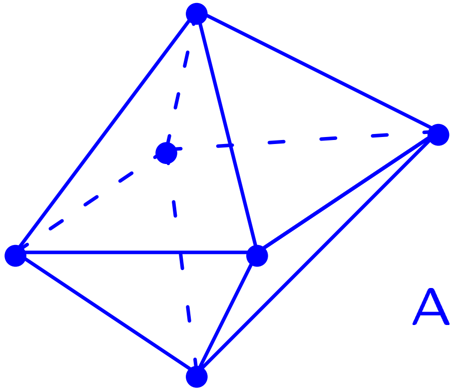
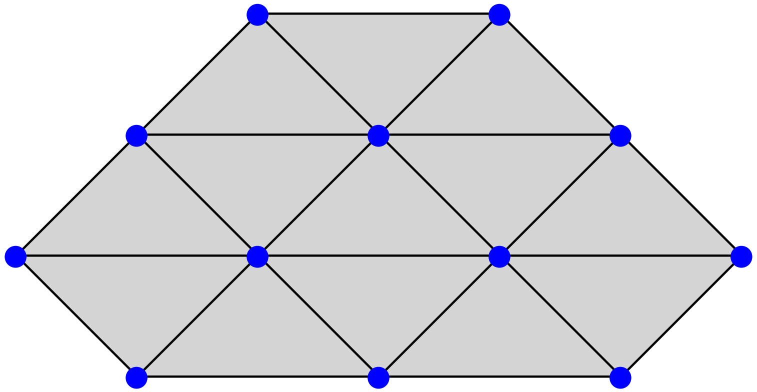


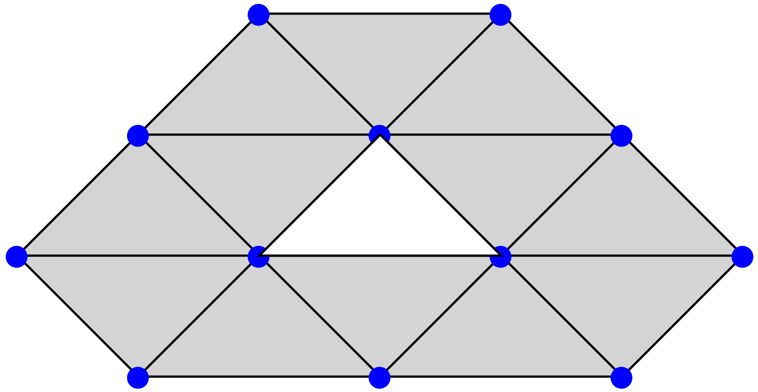


A sphere

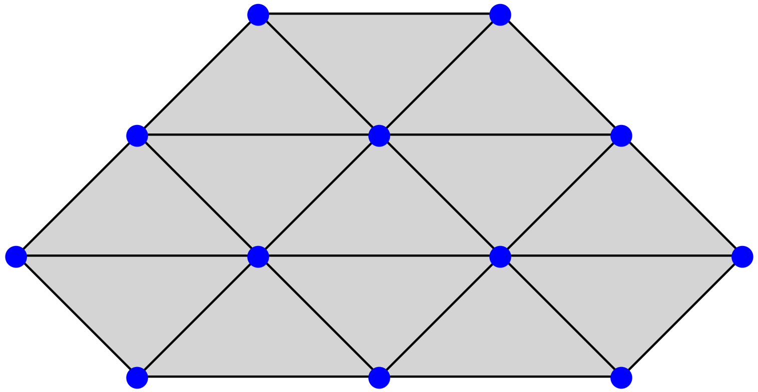
The Möbius function equals the reduced Euler characteristic



Shellable complex



Nonshellable complex



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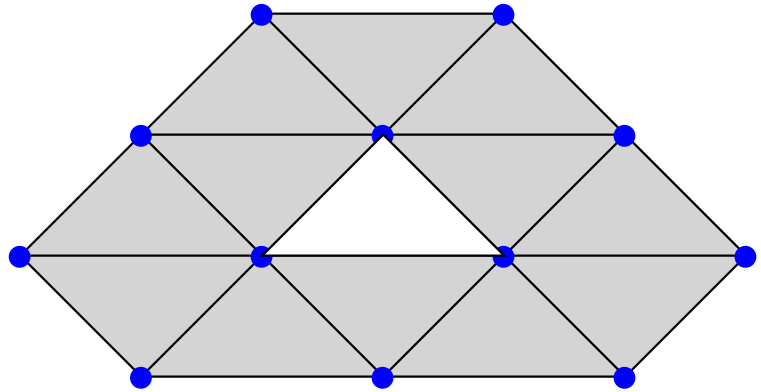

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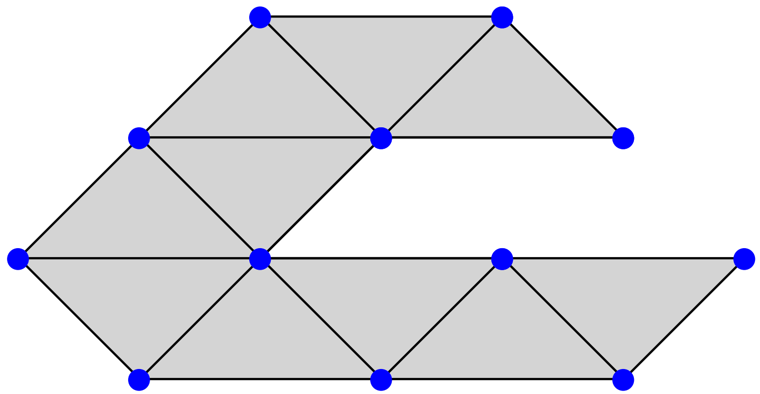

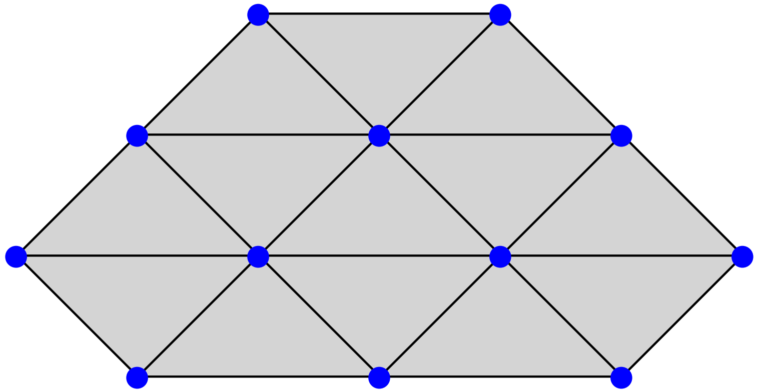

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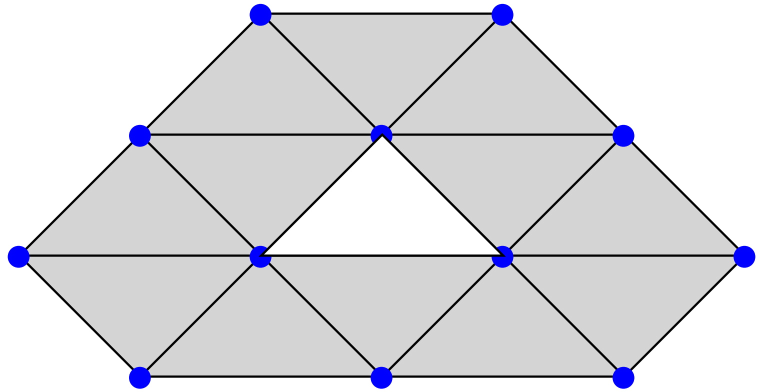

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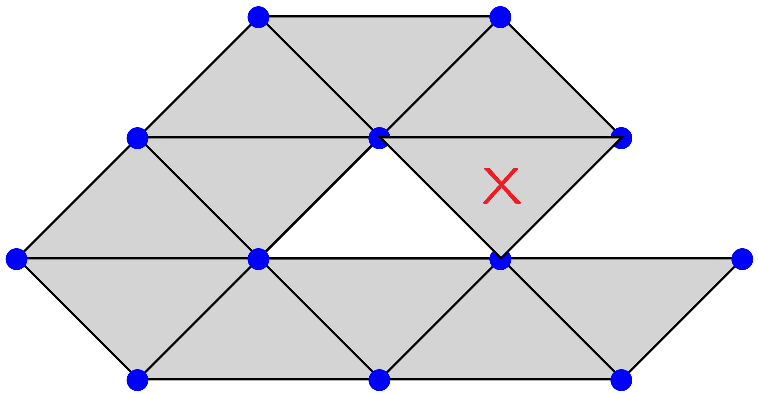

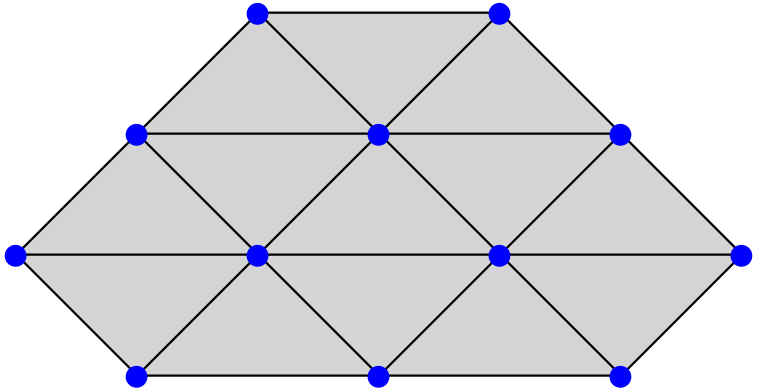

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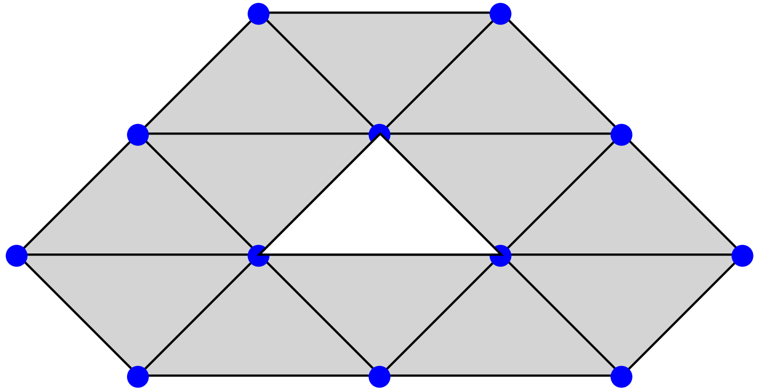

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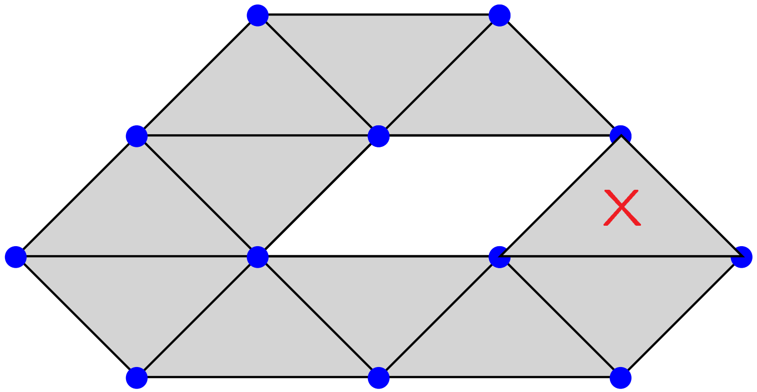

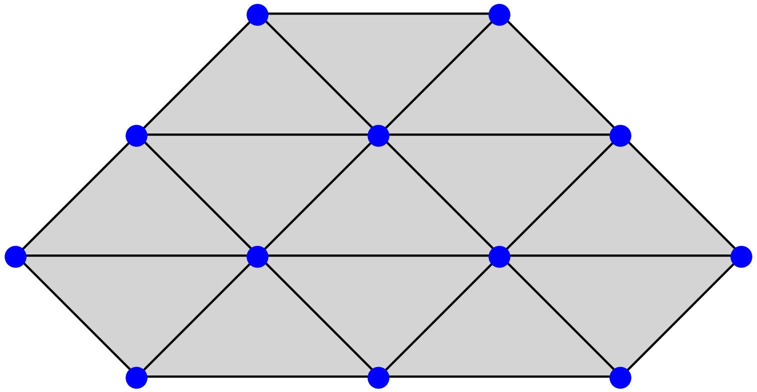

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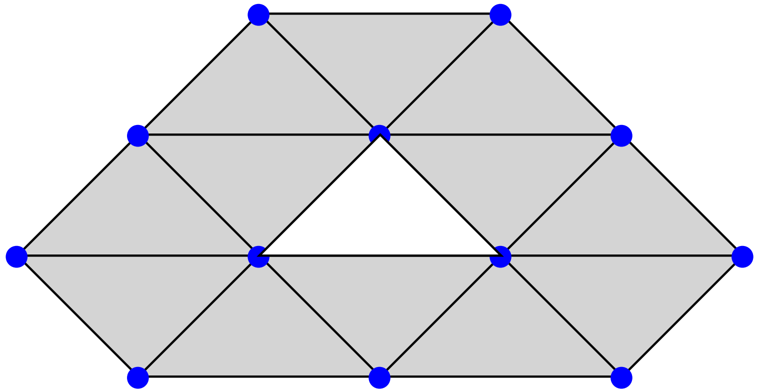

Nonshellable complex

- $\mu(\sigma, \tau)$ equals reduced Euler characteristic of $\Delta((\sigma, \tau))$
- A shellable complex is homotopically a *wedge of spheres*.
- Its reduced Euler characteristic is the number of spheres.
- It has nontrivial homology at most in the top dimension.

Theorem: If σ and τ have the same number of descents, then

$$\mu(\sigma,\tau) = (-1)^{|\tau|-|\sigma|} N(\sigma,\tau),$$

where $N(\sigma,\tau)$ is the number of normal occurrences of σ in τ.

Therefore,

$$|\mu(\sigma,\tau)| \leqslant \sigma(\tau),$$

where $\sigma(\tau)$ is the number of occurrences of σ in τ.

And, the interval $[\sigma,\tau]$ is shellable.

Proof: Biject to *subword order* and use Björner's results (1988).

Jason Smith (2014)

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$$\mu(1,\ 71654823) = 0$$

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In fact, the interval $[1, \pi]$ is contractible.

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There are results/conjectures analogous to the above for the *layered* and *separable* permutations.

Sagan-Vatter (2005): Möbius function for *layered* permutations

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3 2 1 5 4 6 8 7

Sagan-Vatter (2005): Möbius function for *layered* permutations

$$\boxed{3\ \ 2\ \ 1}\ \boxed{5\ \ 4}\ \boxed{6}\ \boxed{8\ \ 7}$$

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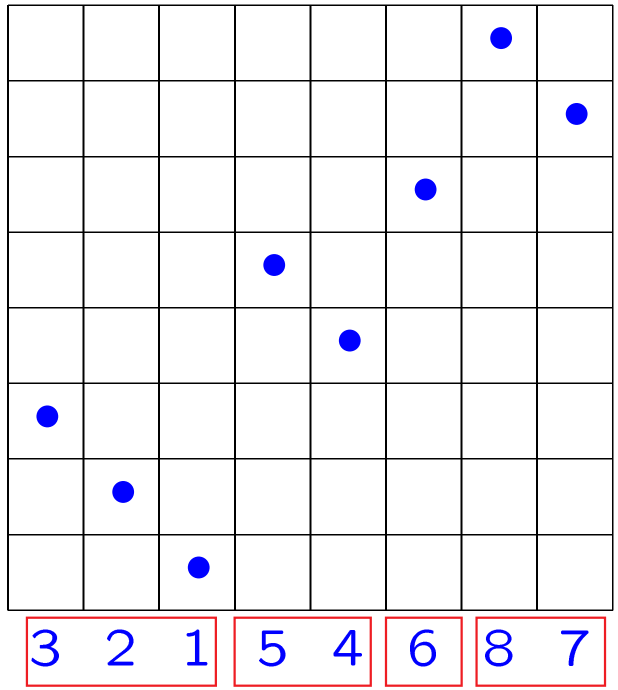
A layered permutation is a concatenation of decreasing sequences, each smaller than the next.

Sagan-Vatter (2005): Möbius function for *layered* permutations



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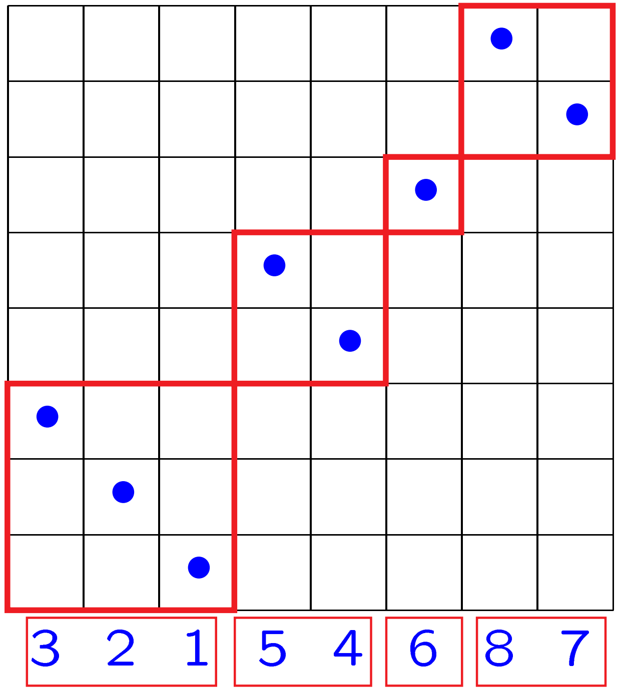

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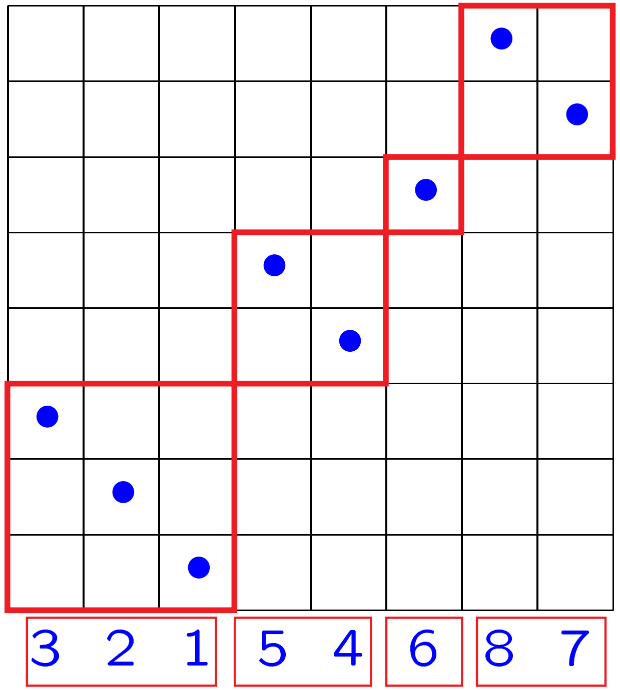
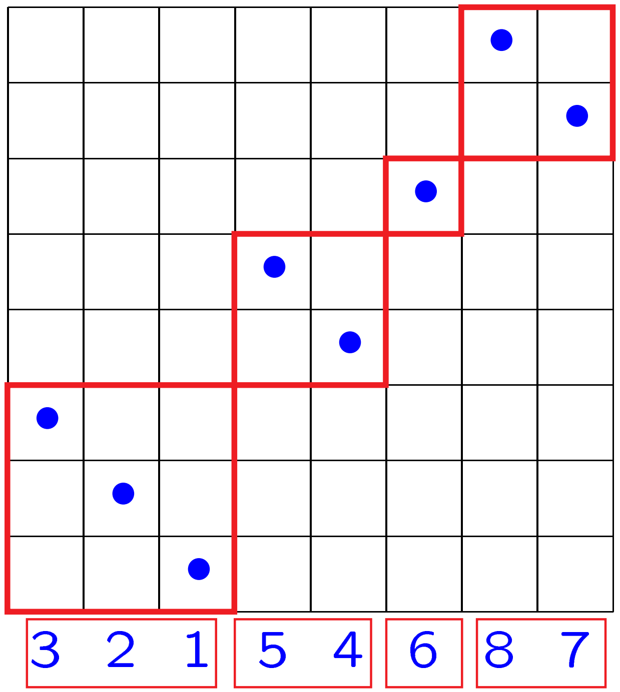


(Any subsequence of a layered permutation is layered)

Sagan-Vatter (2005): Möbius function for *layered* permutations



An effective formula, but too long to fit inside these margins . . .

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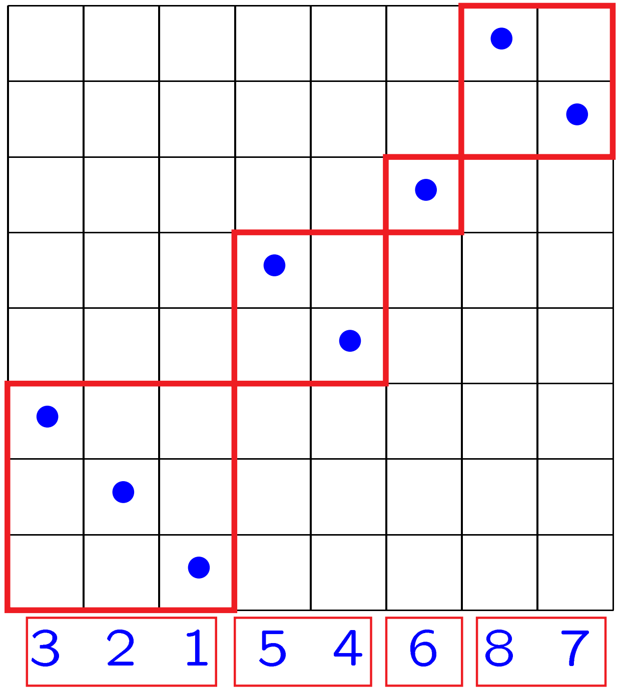
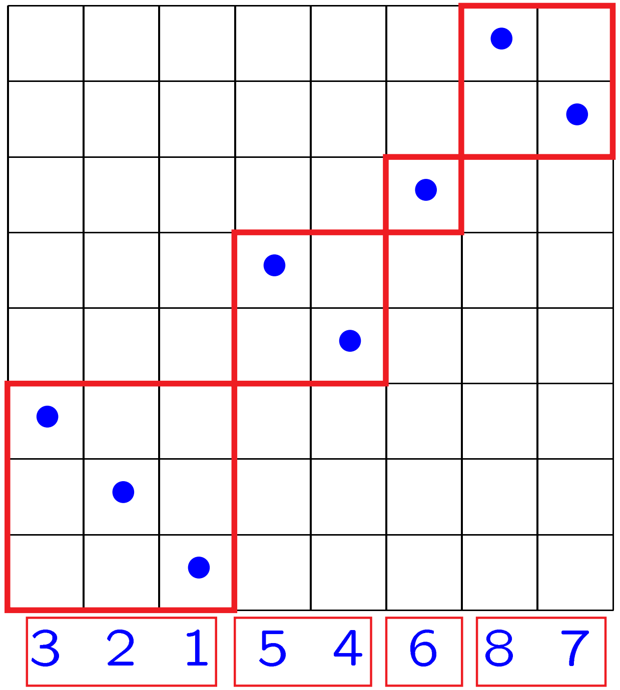


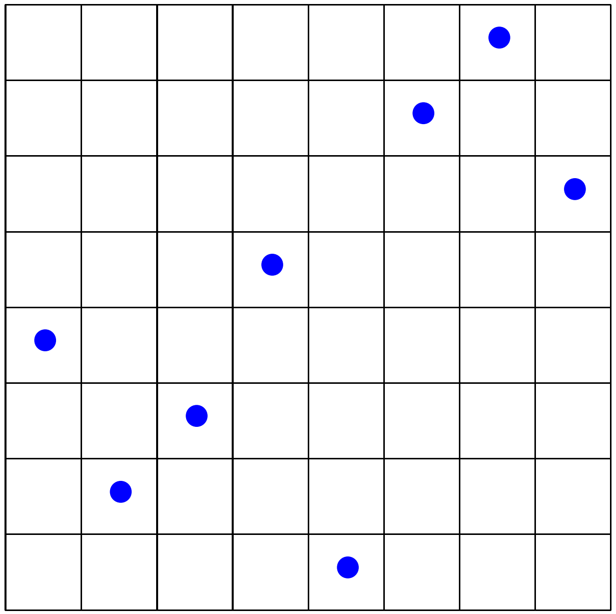
An effective formula, but too long to fit inside these margins . . .

(Similar to permutations with fixed number of descents)

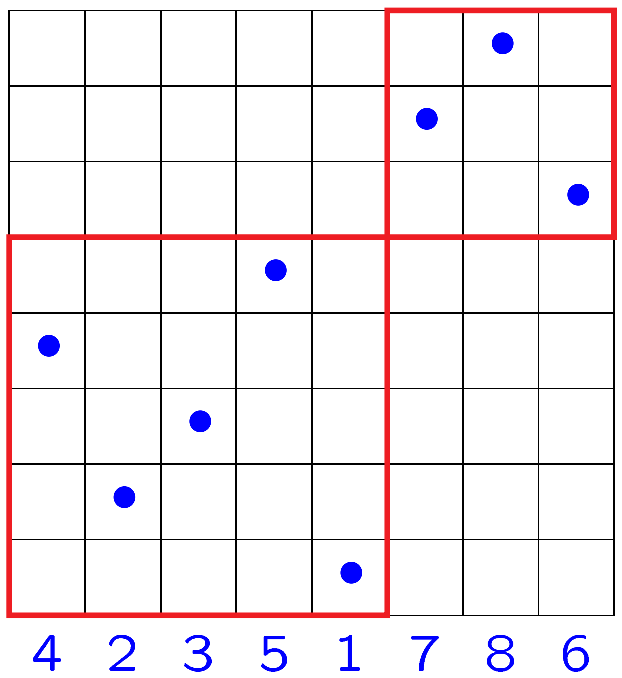
Sagan-Vatter (2005): Möbius function for *layered* permutations

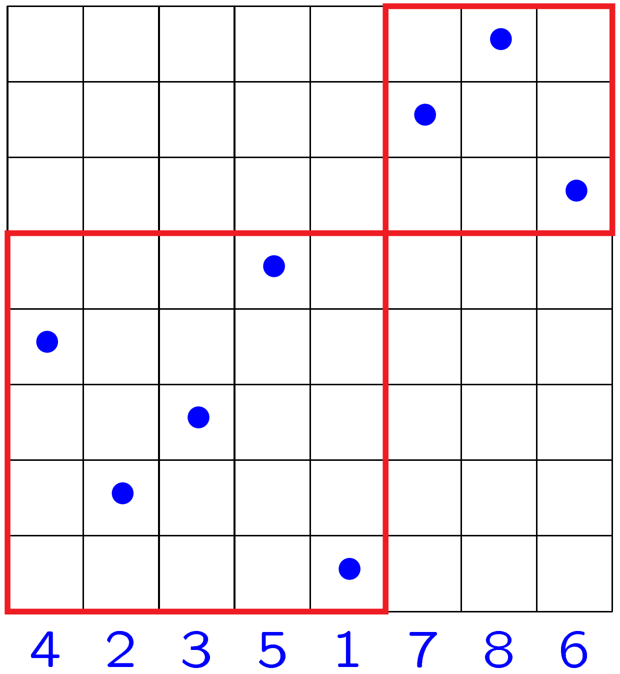


A special case of the *separable* permutations.



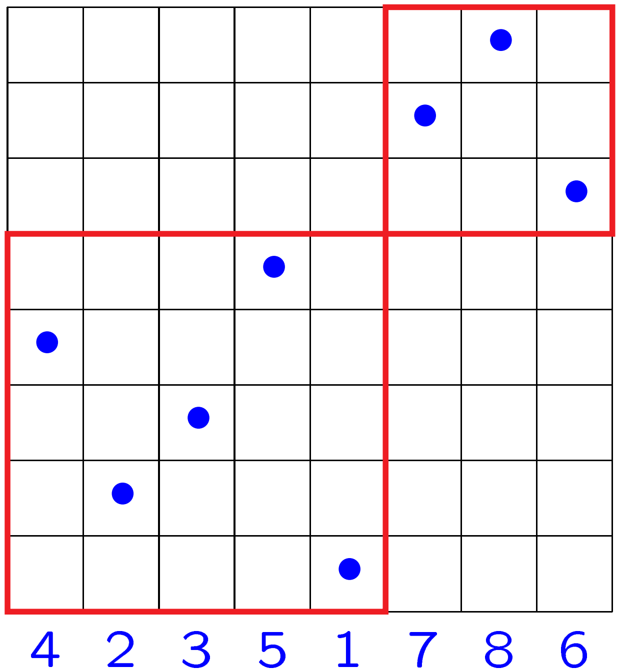
4 2 3 5 1 7 8 6

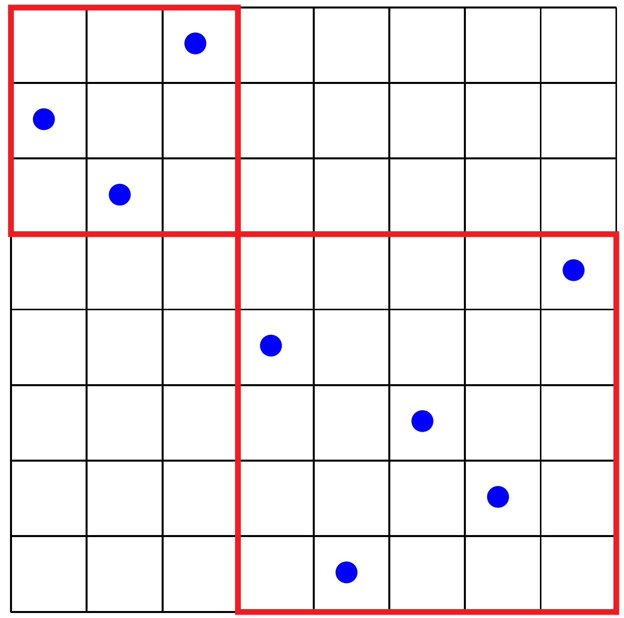

4 2 3 5 1 7 8 6



A *decomposable* permutation
is a *direct sum*

$$42351786 \;=\; 42351 \oplus 231$$



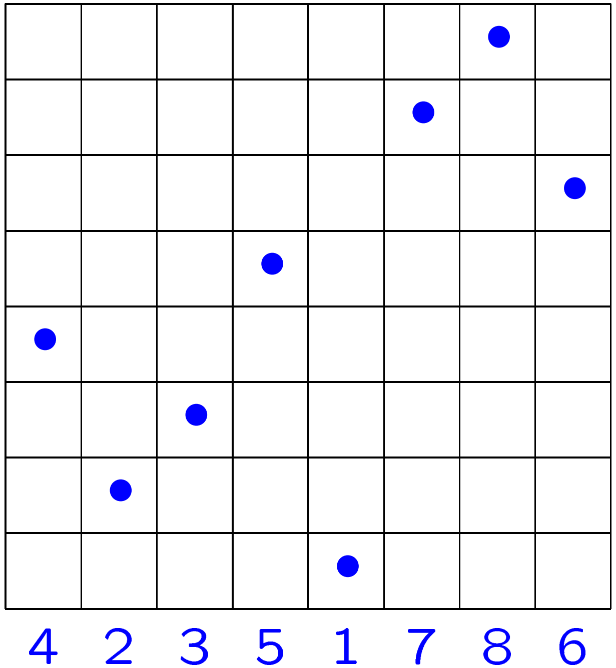


A *decomposable* permutation is a *direct sum*

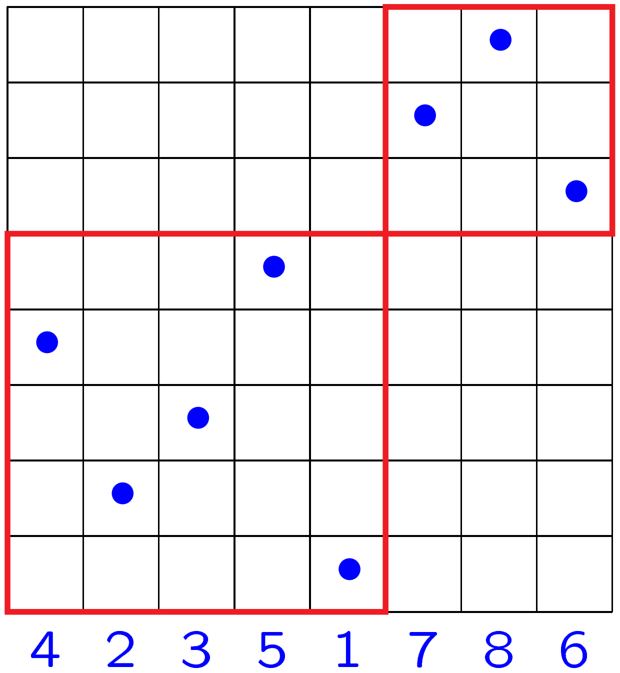
$$42351786 \ = \ 42351 \oplus 231$$

A *skew-decomposable* permutation is a *skew sum*

$$76841325 \ = \ 213 \ominus 41325$$



A permutation is *separable* if it can be generated from 1 by direct sums and skew sums.



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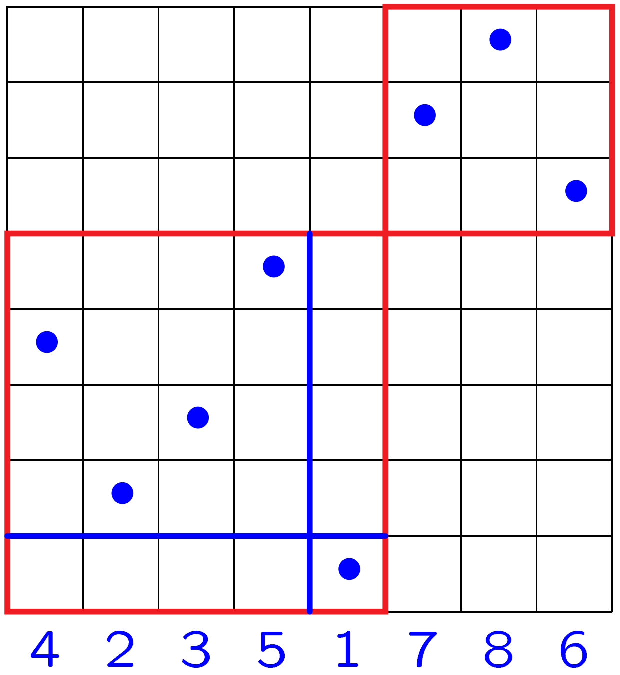

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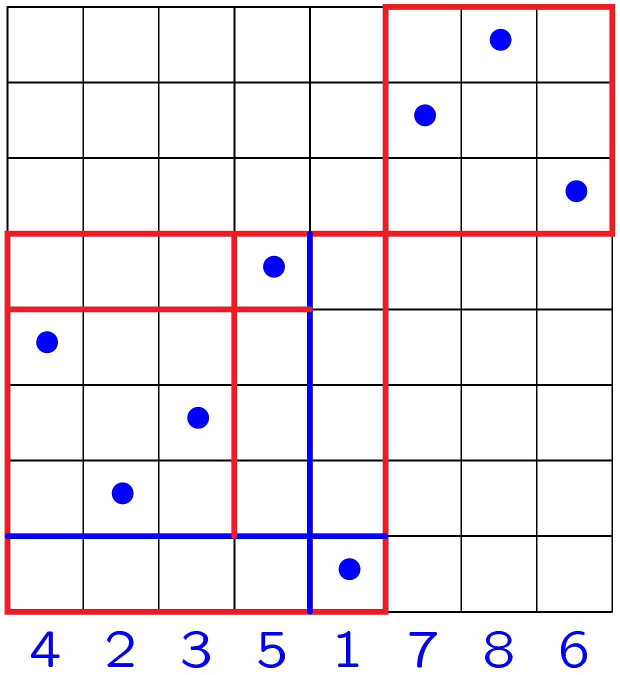

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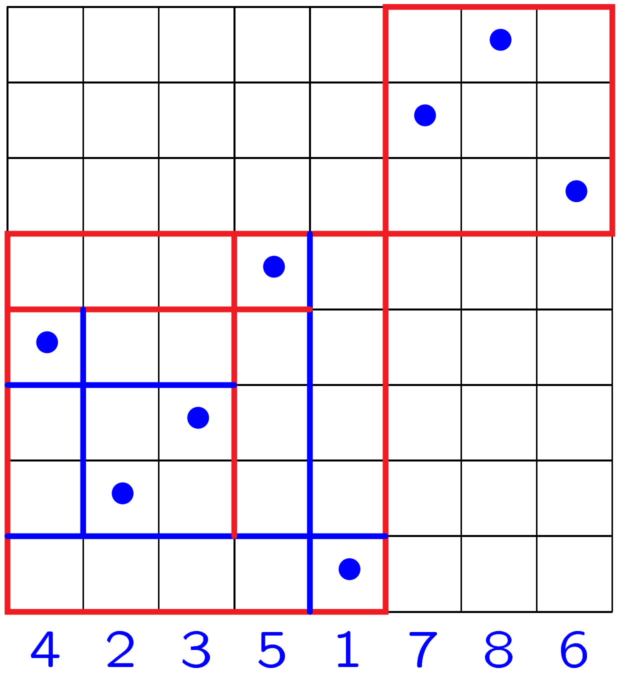

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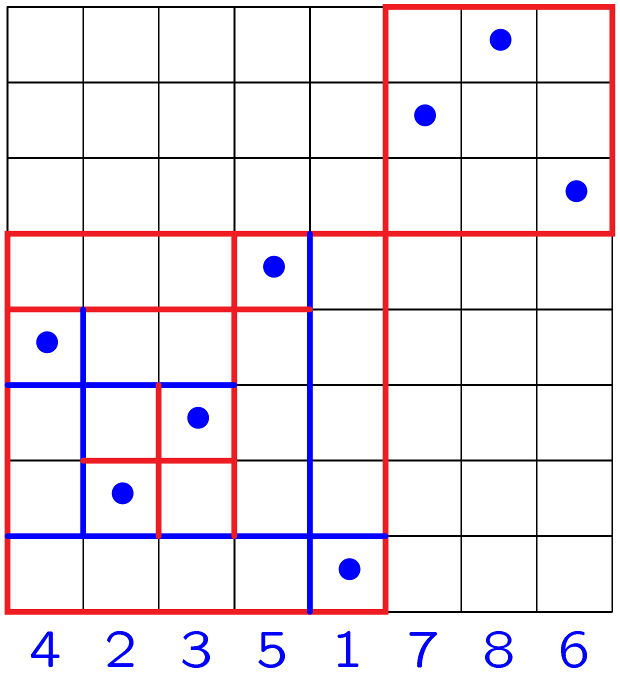

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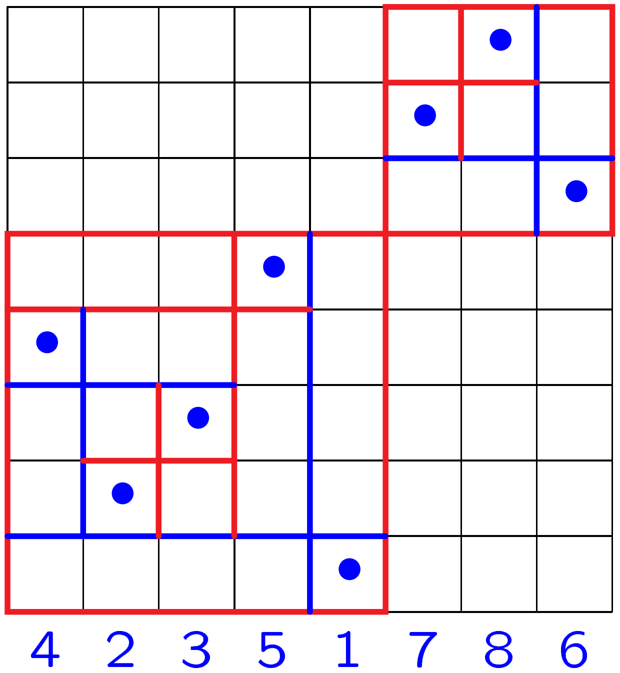

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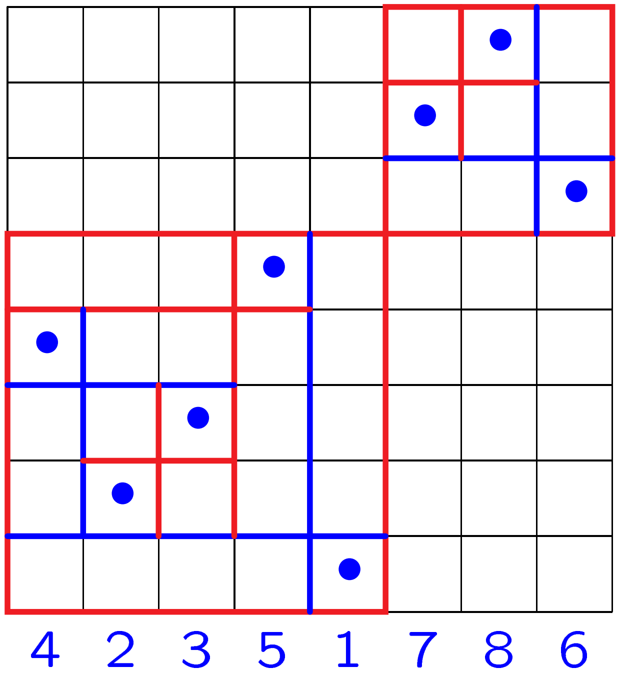

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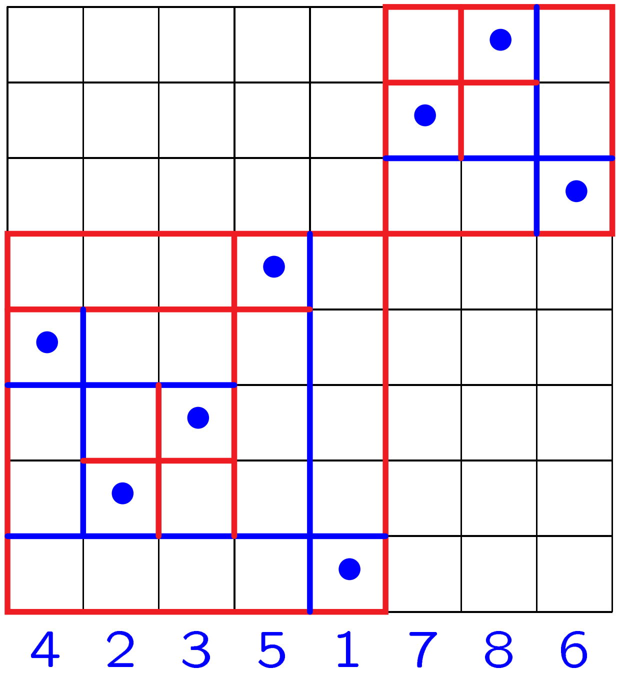

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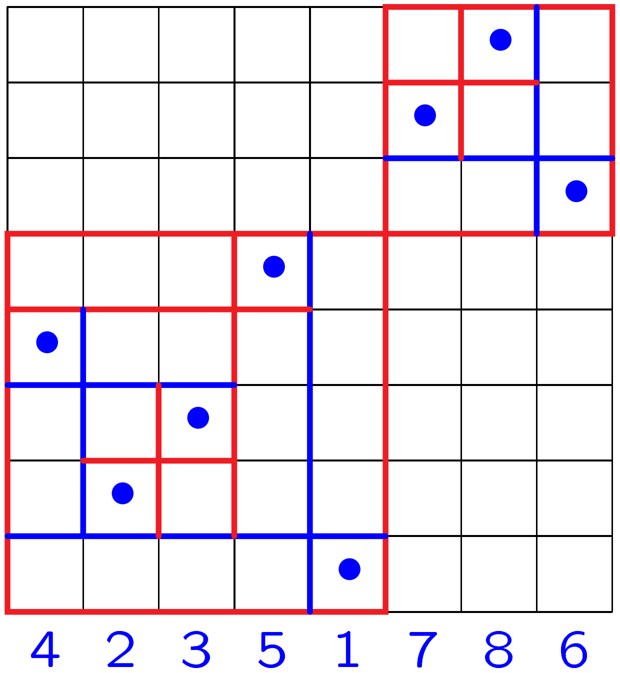
Decomposes by skew/direct sums into singletons



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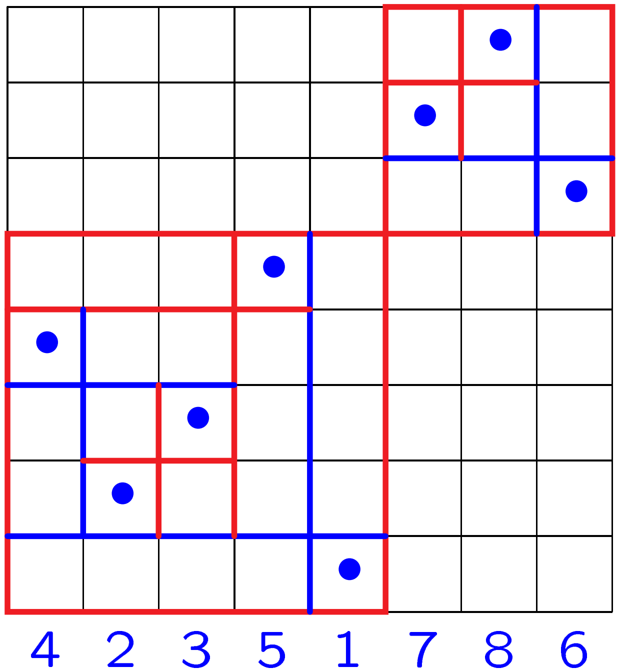

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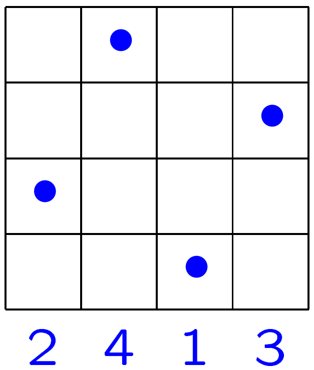

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Theorem: If σ and τ are separable permutations, then

$$\mu(\sigma,\tau) = \sum_{\mathsf{X}\in\mathcal{OP}} (-1)^{\mathrm{parity}(\mathsf{X})}$$

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Bruce Sagan $\longrightarrow$ S

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E. Babson, A. Björner, L, V. Welker, J. Shareshian

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Abbreviating your last name to a single letter implies everybody should remember your name.

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Let's put an end to this immodesty!

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(Unless your name is Central Shipyard, in which case you may be forgiven)

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(This computes $\mu(\sigma,\tau)$ in polynomial time)

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(A generalization of a conjecture of Tenner and Steingrímsson)

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$$\mu(135\ldots(2k\text{-}1)\,(2k)\ldots42,\, 135\ldots(2n\text{-}1)\,(2n)\ldots42) = \binom{n+k-1}{n-k}$$

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$$\mu(1342, 13578642) = \binom{8/2+4/2-1}{8/2-4/2} = \binom{5}{2}$$

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Neither corollary true in general

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$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \bullet & \bullet & \bullet & \bullet & \cdots \end{array}$$

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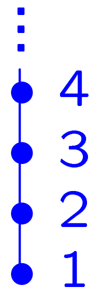
P chain:



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P chain:

$\vdots$

4

3

2

1

$1343 \leqslant_P 113414$ $(3 <_P 4)$

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• • • • $\cdots$

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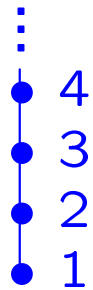

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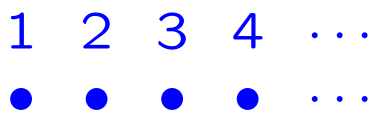

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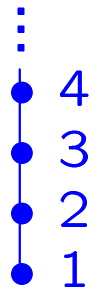

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Layered

Is there a family of intervals of permutations interpolating between these two extremes (that are shellable, or at least with a tractable Möbius function)?

McNamara-Steingrímsson (reformulation of BJJS):

Theorem: Let $\tau = \tau_1 \oplus \cdots \oplus \tau_k$ be finest decomposition. Then

$$\mu(\sigma, \tau) = \sum_{\sigma = \sigma_1 \oplus \ldots \oplus \sigma_k} \prod_m \mu(\sigma_m, \tau_m) + \epsilon_m$$

where $\quad \epsilon_m = \begin{cases} 1, & \text{if } \sigma_m = \emptyset \text{ and } \tau_{m-1} = \tau_m \\ 0, & \text{else} \end{cases}$

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(only sometimes this is because $[\sigma, \tau]$ is a direct product)

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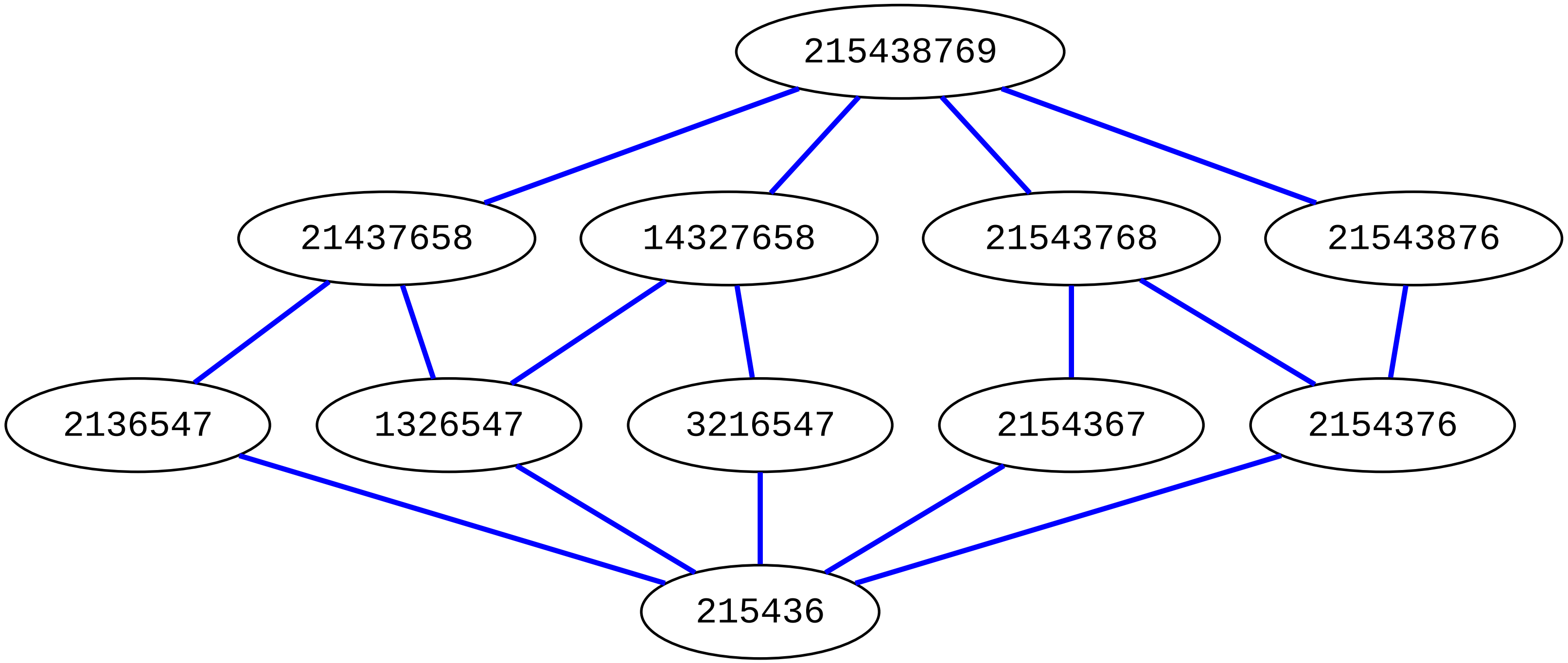

A disconnected interval of rank 3

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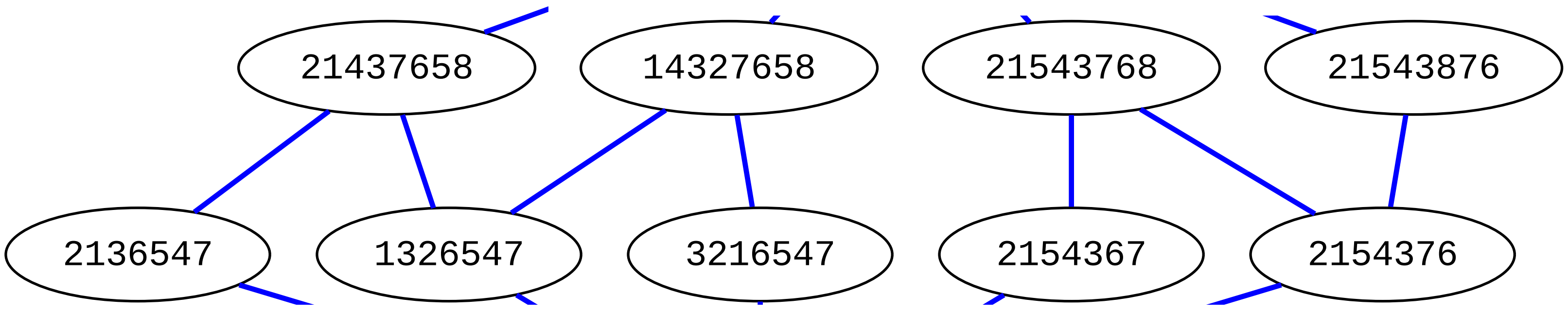

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> The number of permutations avoiding any given pattern p grows only exponentially.

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Follows from the *Marcus-Tardos theorem:*

> The number of permutations avoiding any given pattern p grows only exponentially.

Thus, almost every interval $[\sigma, \tau]$ (for τ large enough) contains the subintervals $[\pi, \pi \oplus \pi]$ and $[\pi, \pi \ominus \pi]$ for some $\pi > 1$, one of which is disconnected.

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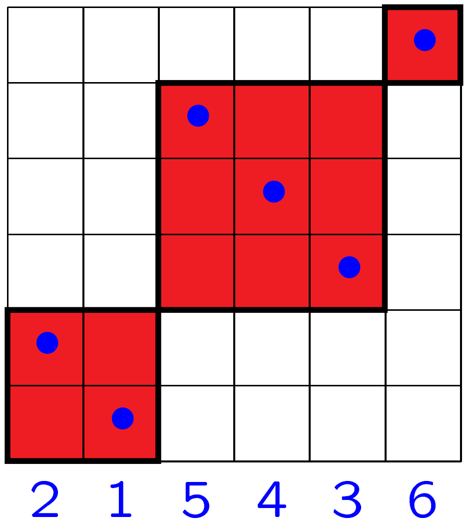

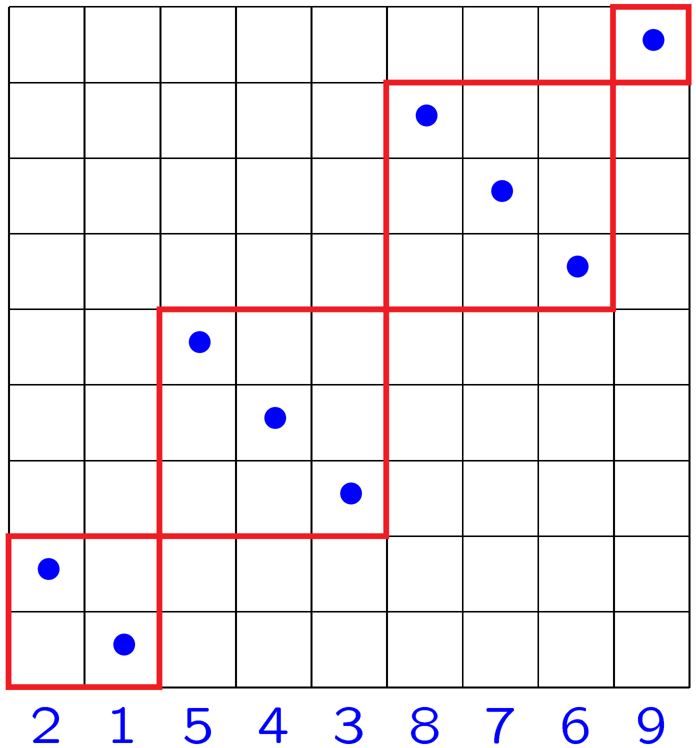

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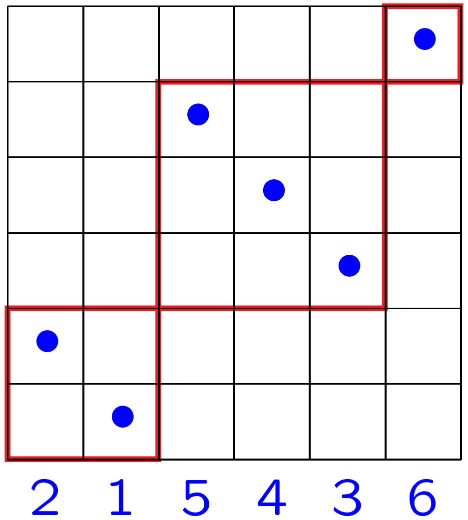
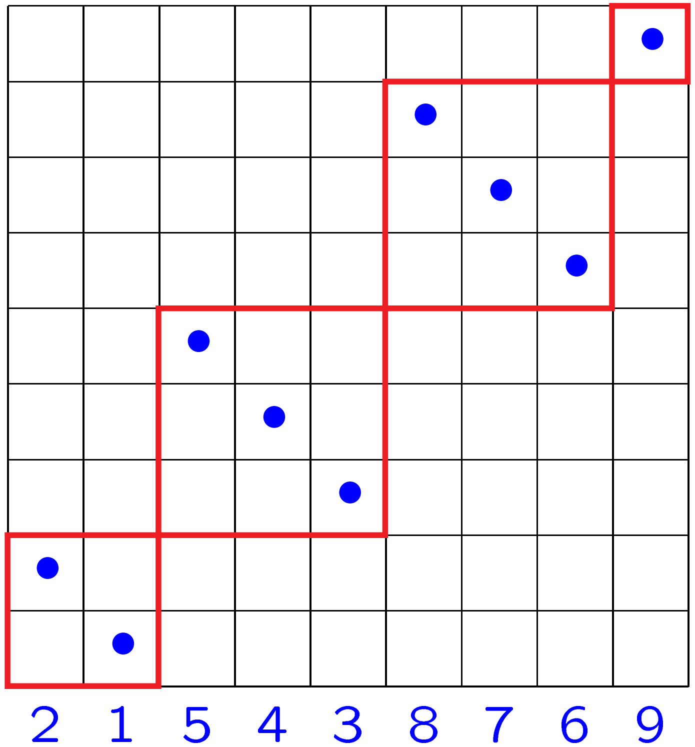
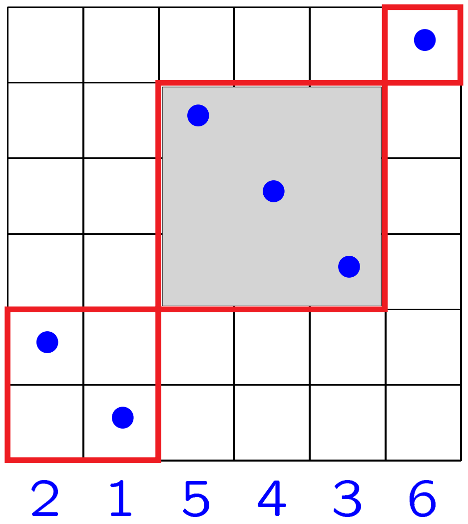
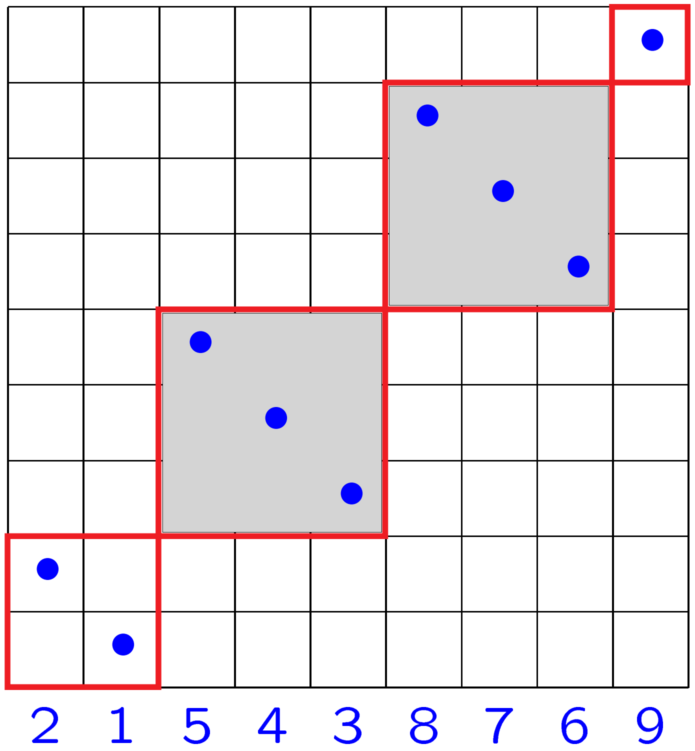




McNamara and Steingrímsson:

Theorem: An interval $[\sigma, \tau]$ of layered permutations is disconnected if and only if σ and τ differ by a repeated layer of size at least 3.





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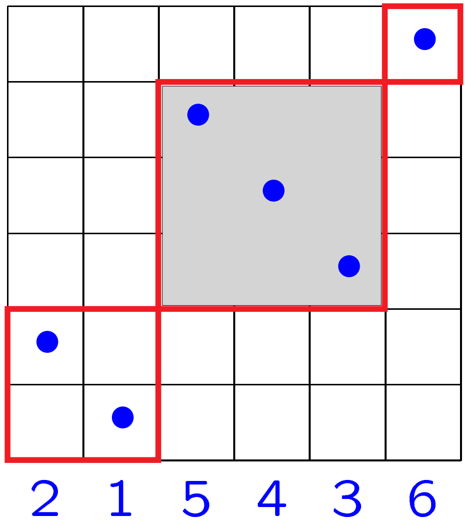

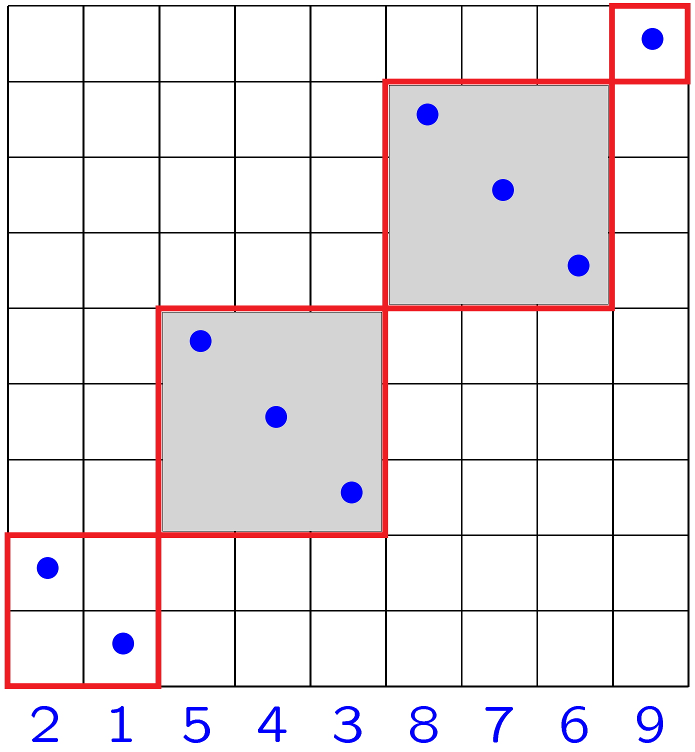

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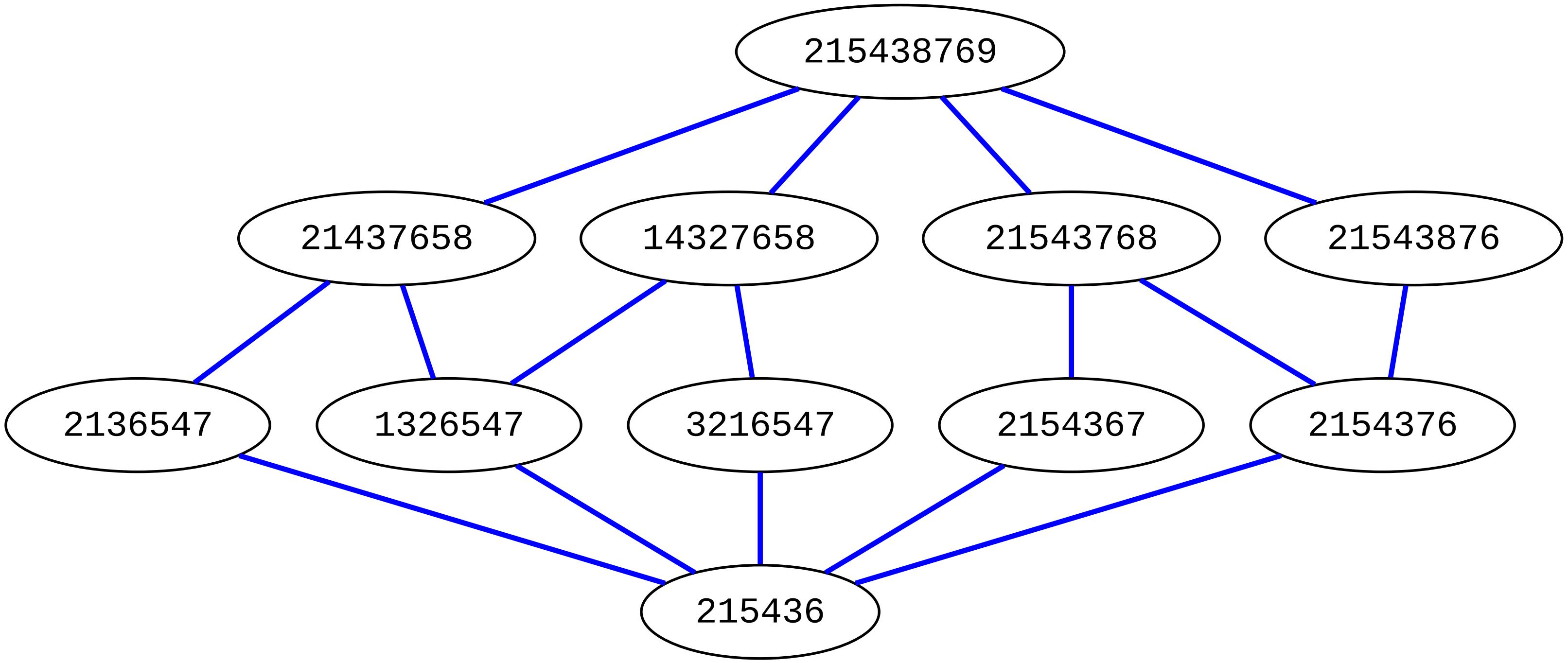

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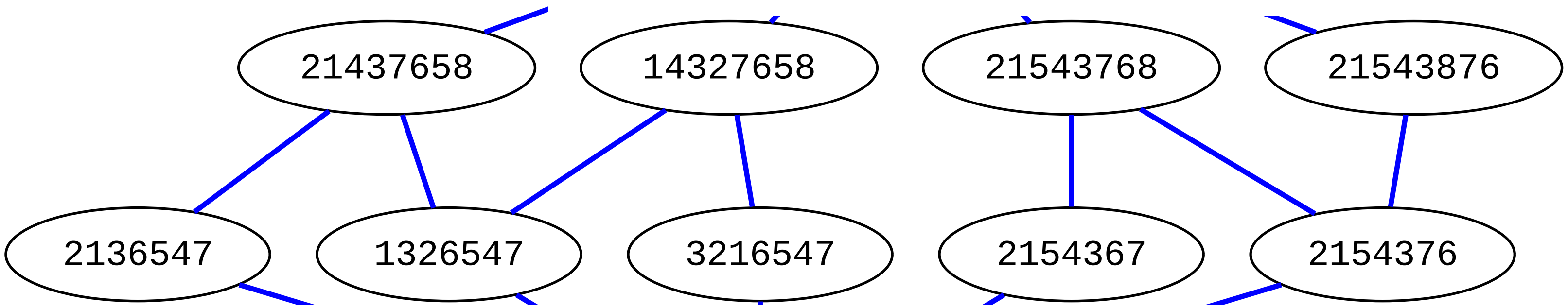

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Conjecture: The same is true of separable permutations.

The interval

$$[123, 3416725]$$

has no non-trivial disconnected subintervals, and alternating Möbius function, but homology in different dimensions.

Betti numbers: 0, 1, 2.

Some questions:

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- Is the rank function of every interval unimodal?
- How does $\max(|\mu(1, \pi)|)$ grow with the length of π?

Thanks, Richard!

(and you all $\ddot\smile$)