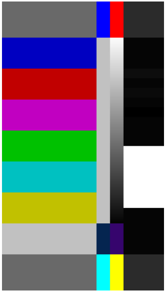
# Rowmotion: Classical & Birational

Tom Roby (University of Connecticut)

*Describing joint research with Darij Grinberg*

Stanley@70
MIT
Cambridge, MA USA



26 June 2014

Slides for this talk are available online (or will be soon) at

http://www.math.uconn.edu/~troby/research.html

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# Abstract

If $P$ is a finite poset, (classical) rowmotion (aka the Fon-der-Flaass map aka Panyushev complementation) is a certain permutation of the set of order ideals (or equivariantly the antichains) of $P$. Various surprising properties of rowmotion have been exhibited in work of Brouwer/Schrijver, Cameron/Fon der Flaass, Panyushev, Armstrong/Stump/Thomas, Striker/Williams, and Propp/R. For example, its order is $p + q$ when $P$ is the product $[p] \times [q]$ of two chains, and several natural statistics have the same average over every rowmotion orbit (i.e., are "homomesic"). Recent work of Einstein/Propp generalizes rowmotion twice: first to the piecewise-linear setting of a poset's "order polytope", defined by Stanley in 1986, and then via detropicalization to the birational setting.

In these latter settings, generalized rowmotion no longer has finite order in the general case. Results of Grinberg and the speaker, however, show that it still has order $p + q$ on the product $[p] \times [q]$ of two chains, and still has finite order for a wide class of forest-like ("skeletal") graded posets and for some triangle-shaped posets. Our methods of proof are partly based on those used by Volkov to resolve the type $AA$ (rectangular) Zamolodchikov Periodicity Conjecture.

## Acknowledgments

This seminar talk discusses recent work with Darij Grinberg, including ideas and results from Arkady Berenstein, David Einstein, Jim Propp, Jessica Striker, and Nathan Williams.

Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains. Darij Grinberg & Jim Propp created many of the other pictures and slides that are used here.

Thanks also to Omer Angel, Drew Armstrong, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, Svante Linusson, Vic Reiner, Richard Stanley, Ralf Schiffler, Hugh Thomas, Pete Winkler, and Ben Young.

## Overview: What to expect in this talk

- Way cool map on $J(P)$ called "rowmotion", and some of unexpected properties of its *order* and *orbits*;
- Great animations by Mike LaCroix to illustrate the above;
- Generalizations of the above to (1) the order polytope of $P$ and (2) arbitrary $\mathbb{K}$-labeling of the nodes of $P$.
- Theorems about the order of these maps for certain classes of posets;
- Allusions to other work that there won't be time to discuss;
- Several jokes; and
- Appearances of the name "Stanley" in certain key places.

Please interrupt with questions!

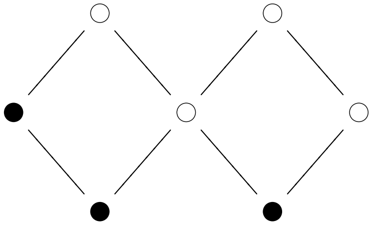
## Classical rowmotion

**Classical rowmotion** is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).

- Let $P$ be a finite poset.
  **Classical rowmotion** is the map $\mathbf{r} : J(P) \to J(P)$ which sends every order ideal $S$ to the order ideal obtained as follows:
  Let $M$ be the set of minimal elements of the complement $P \setminus S$.
  Then, $\mathbf{r}(S)$ shall be the order ideal generated by these elements (i.e., the set of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

**Example:**
Let $S$ be the following order ideal ($\bullet$ = inside order ideal):



# Classical rowmotion: the standard definition

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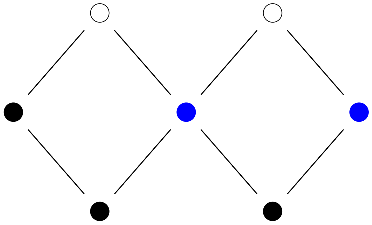
**Example:**
Mark $M$ ($=$ minimal elements of complement) blue.



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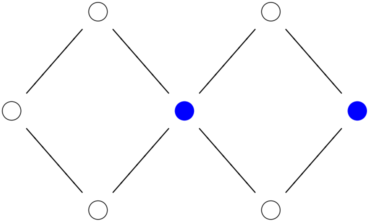
**Example:**

Forget about the old order ideal:



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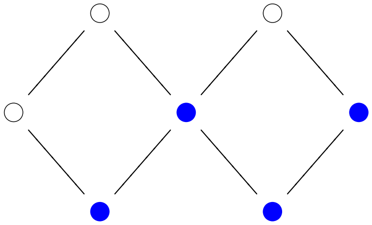
**Example:**
$\mathbf{r}(S)$ is the order ideal generated by $M$ ("everything below $M$"):



## Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

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However, **for some types of** $P$, the order can be explicitly computed or bounded from above.
See Striker-Williams for an exposition of known results.

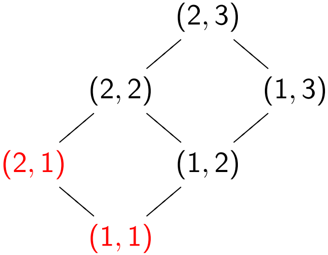
- If $P$ is a $p \times q$-rectangle:

$$\begin{tikzcd}[column sep=small]
& & (2,3) \arrow[dl, dash] \arrow[dr, dash] & \\
& (2,2) \arrow[dl, dash] \arrow[dr, dash] & & (1,3) \arrow[dl, dash] \\
(2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] & \\
& (1,1) & &
\end{tikzcd}$$

(shown here for $p = 2$ and $q = 3$), then $\operatorname{ord}(\mathbf{r}) = p + q$.

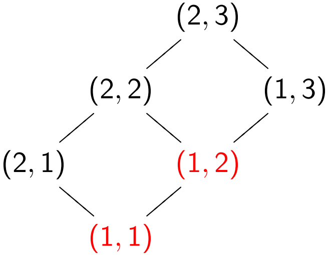
# Classical rowmotion: properties

**Example:**
Let $S$ be the order ideal of the $2 \times 3$-rectangle given by:



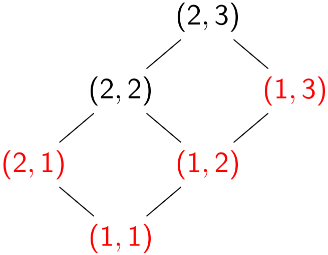
## Classical rowmotion: properties

**Example:**
$\mathbf{r}(S)$ is



# Classical rowmotion: properties

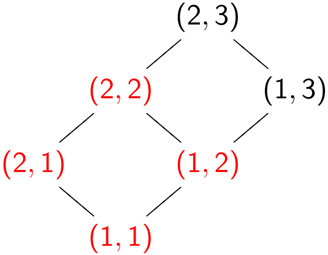
**Example:**
$\mathbf{r}^2(S)$ is



# Classical rowmotion: properties

**Example:**

$\mathbf{r}^3(S)$ is



**Example:**
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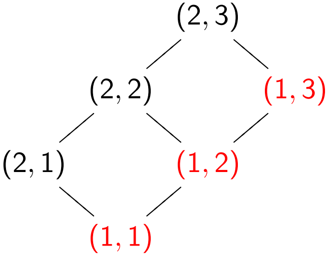
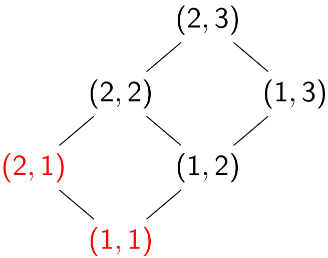


**Example:**

$\mathbf{r}^5(S)$ is

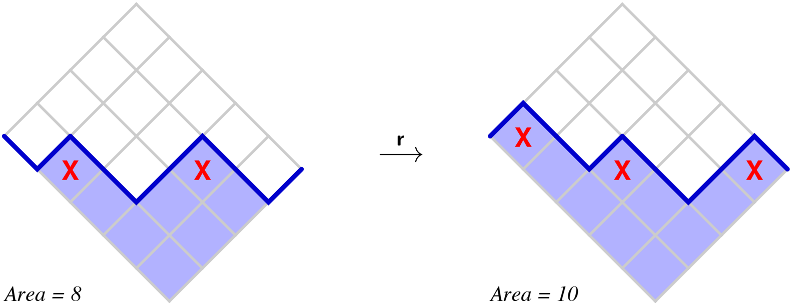


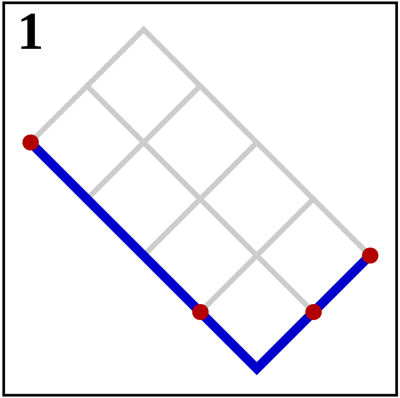

which is precisely the $S$ we started with.

$\mathrm{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$

# Example of rowmotion in lattice cell form

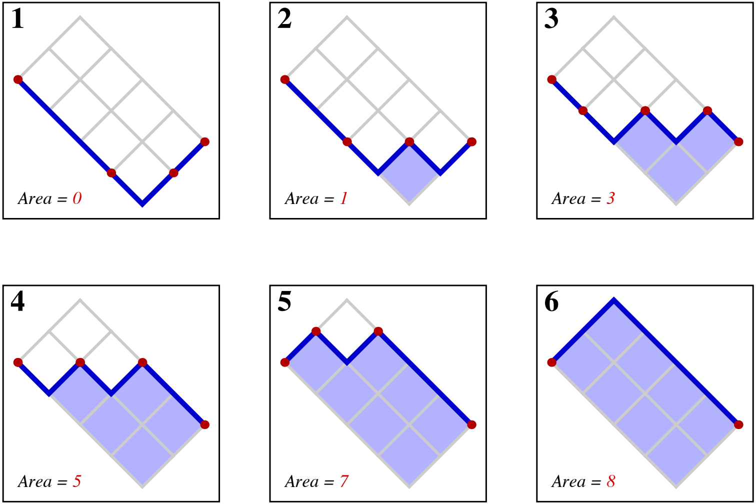
Next we'll take a look at an interesting property of the *orbits* of rowmotion acting on a product of two chains. For the animations which follow, please temporarily take the point of view that: *the elements of the poset are the* **squares** *below* So we would map:



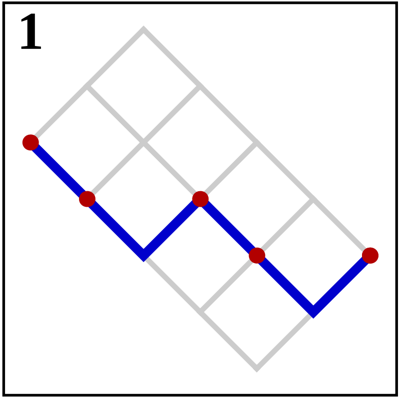



# Rowmotion

# Rowmotion on $[4] \times [2]$ A




$$(0+1+3+5+7+8)\ /\ 6 = 4$$




# Rowmotion

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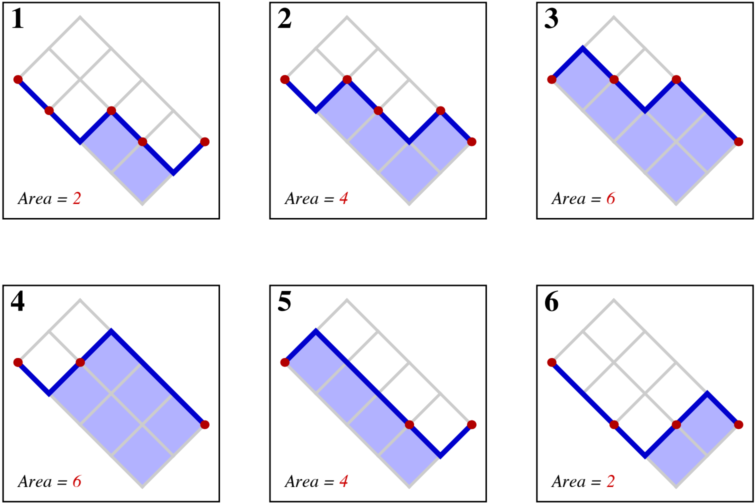
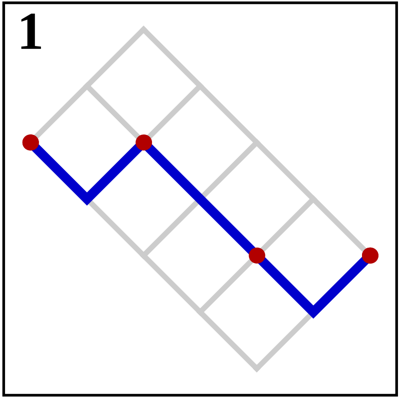



$$(2+4+6+6+4+2)\ /\ 6 = 4$$




# Rowmotion

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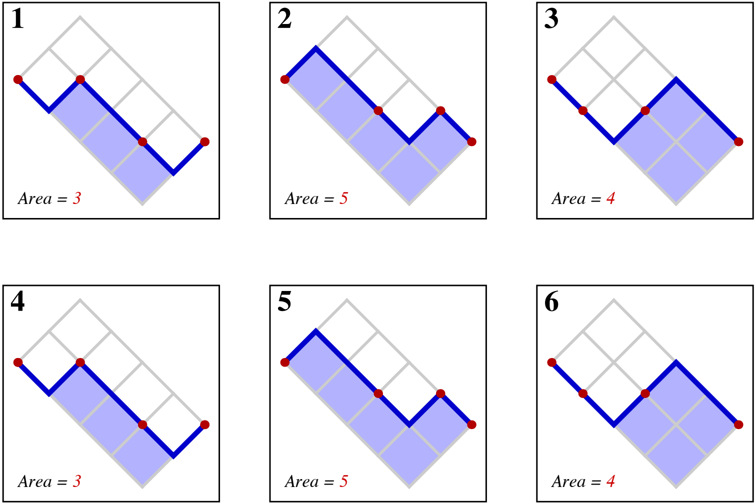


$$(3+5+4+3+5+4)\,/\,6 = 4$$

# What is . . . a Homomesy?

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**DEF:** Given an (invertible) action $\tau$ on a finite set of objects $S$, call a statistic $\varphi : S \to \mathbb{C}$ **homomesic** [Gk., "same middle"] with respect to $(S, \tau)$ iff the average of $\varphi$ over each $\tau$-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} \varphi(s)$ does not depend on the choice of $\mathcal{O}$.

We call the triple $(S, \tau, \varphi)$ a **homomesy**.

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We call the triple $(S, \tau, \varphi)$ a **homomesy**.

For example, the statistic $\#I$ (cardinality of the ideal) is *homomesic* with respect to rowmotion, $\mathbf{r}$, acting on $J([4] \times [2])$.

**Theorem (Propp, R.)**

*Let $\mathcal{O}$ be an arbitrary* $\mathbf{r}$*-orbit in* $J([p] \times [q])$*. Then*

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{pq}{2},$$

*i.e., the cardinality statistic is homomesic with respect to the action of rowmotion on order ideals.*

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It turns out that to show a similar statement for rowmotion acting on the *antichains* of $P$, the right tool is an equivariant bijection from Stanley's "Promotion and Evacuation" paper, as rephrased by Hugh Thomas.

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See Jim Propp's talk next week at FPSAC'14 for more information about homomesies in various settings.

# Classical rowmotion: the toggling definition

There is an alternative definition of classical rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
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  ("Try to add or remove $v$ from $S$, as long as the result remains within $J(P)$; otherwise, leave $S$ fixed.")
- More formally, if $P$ is a poset and $v \in P$, then the $v$-**toggle** is the map $\mathbf{t}_v : J(P) \to J(P)$ which takes every order ideal $S$ to:
  - $S \cup \{v\}$, if $v$ is not in $S$ but all elements of $P$ covered by $v$ are in $S$ already;
  - $S \setminus \{v\}$, if $v$ is in $S$ but none of the elements of $P$ covering $v$ is in $S$;
  - $S$ otherwise.

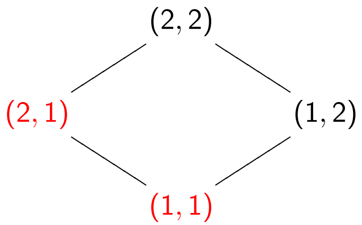
# Classical rowmotion: the toggling definition

- Let $(v_1, v_2, \ldots, v_n)$ be a **linear extension** of $P$; this means a list of all elements of $P$ (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \ldots \circ \mathbf{t}_{v_n}.$$

**Example:**

Start with this order ideal $S$:



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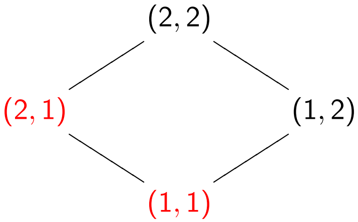
**Example:**
First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



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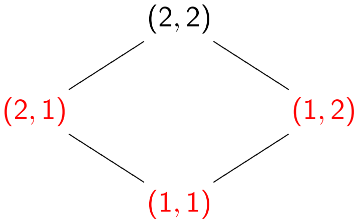
**Example:**

Then apply $\mathbf{t}_{(1,2)}$, which adds $(1, 2)$ to the order ideal:



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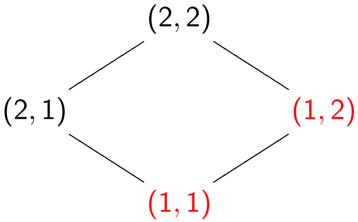
**Example:**
Then apply $\mathbf{t}_{(2,1)}$, which removes $(2,1)$ from the order ideal:



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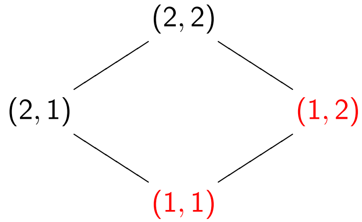
**Example:**
Finally apply $\mathbf{t}_{(1,1)}$, which changes nothing:



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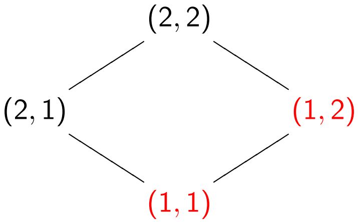
**Example:**
So this is $\mathbf{r}(S)$:



We can generalize this idea of composition of toggles to define a **continuous piecewise-linear (CPL)** version of rowmotion on an infinite set of functions on a poset.

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Let $P$ be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.
The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

## Flipping-maps in the order polytope

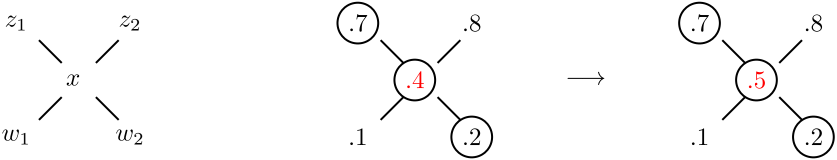
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending $f$ to the unique $f'$ satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w <\cdot\, x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means $z$ covers $x$ and $w <\cdot\, x$ means $x$ covers $w$.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w <\cdot\, x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w <\cdot\, x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

# Example of flipping at a node

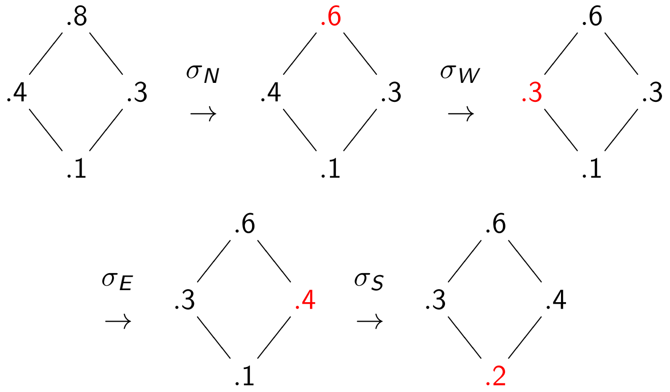



$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

# Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

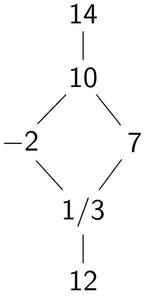



(Here we successively flip values at the North, West, East, and South.)

## Birational rowmotion: definition

- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of $P$** will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

**Example:** This is a $\mathbb{Q}$-labelling of the $2 \times 2$-rectangle:



# Birational rowmotion: definition

- For any $v \in P$, define the **birational $v$-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$\left(T_v f\right)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \cdot \dfrac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \widehat{P}$.

- That is,
  - **invert** the label at $v$,
  - **multiply** by the **sum** of the labels at vertices **covered by** $v$,
  - **multiply** by the **harmonic sum** of the labels at vertices **covering** $v$.

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for all $w \in \widehat{P}$.

- Notice that this is a **local change** to the label at $v$; all other labels stay the same.
- We have $T_v^2 = \mathrm{id}$ (on the range of $T_v$), and $T_v$ is a birational map.

- We define **birational rowmotion** as the rational map

$$R := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of $P$.

- This is indeed independent on the linear extension, because:

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- This is indeed independent on the linear extension, because:
  - $T_v$ and $T_w$ commute whenever $v$ and $w$ are incomparable (even when one doesn't cover another other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

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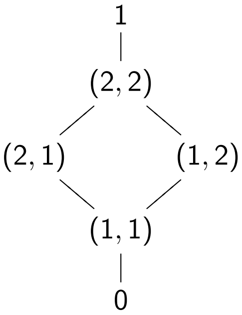
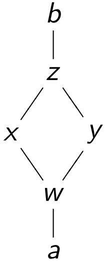
  - $T_v$ and $T_w$ commute whenever $v$ and $w$ are incomparable (even when one doesn't cover another other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- For more information about the lifting of rowmotion from classical to CPL to birational, see, Einstein-Propp [EiPr13], where $R$ is denoted $\rho_B$.

**Example:**

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| poset | labelling |
|---|---|
|   |   |

## Birational rowmotion: example

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Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| poset | labelling |
|---|---|
| $\begin{tikzcd}[column sep=small] & 1 \arrow[d, dash] & \\ & (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\ (2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] \\ & (1,1) \arrow[d, dash] & \\ & 0 & \end{tikzcd}$ | $\begin{tikzcd}[column sep=small] & b \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}$ |

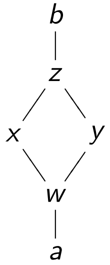
We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).

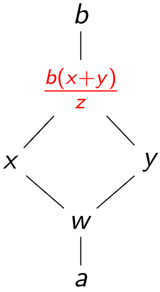
That is, toggle in the order “top, left, right, bottom”.

**Example:**

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| original labelling $f$ | labelling $T_{(2,2)} f$ |
| --- | --- |

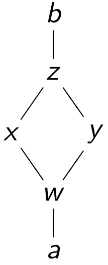


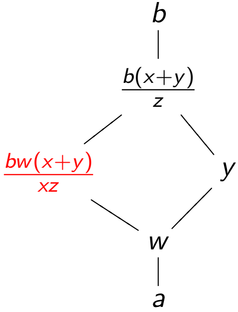




We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ \color{red}{T_{(2,2)}}$.

**Example:**
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| original labelling $f$ | labelling $T_{(2,1)} T_{(2,2)} f$ |
|---|---|




We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

# Birational rowmotion: example

**Example:**

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| original labelling $f$ | labelling $T_{(1,2)} T_{(2,1)} T_{(2,2)} f$ |
|---|---|

$$\begin{tikzcd}[column sep=small] & b \arrow[d, no head] & \\ & z \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[dr, no head] & & y \arrow[dl, no head] \\ & w \arrow[d, no head] & \\ & a & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & b \arrow[d, no head] & \\ & \dfrac{b(x+y)}{z} \arrow[dl, no head] \arrow[dr, no head] & \\ \dfrac{bw(x+y)}{xz} \arrow[dr, no head] & & \color{red}{\dfrac{bw(x+y)}{yz}} \arrow[dl, no head] \\ & w \arrow[d, no head] & \\ & a & \end{tikzcd}$$

We are using $R = T_{(1,1)} \circ \color{red}{T_{(1,2)}} \circ T_{(2,1)} \circ T_{(2,2)}$.

# Birational rowmotion: example

**Example:**
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| original labelling $f$ | labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$ |
|---|---|
| $\begin{tikzcd} & b \arrow[d, dash] & \\ & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w \arrow[d, dash] & \\ & a & \end{tikzcd}$ | $\begin{tikzcd} & b \arrow[d, dash] & \\ & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{bw(x+y)}{xz} \arrow[dr, dash] & & \frac{bw(x+y)}{yz} \arrow[dl, dash] \\ & \color{red}{\frac{ab}{z}} \arrow[d, dash] & \\ & a & \end{tikzcd}$ |

We are using $R = \color{red}{T_{(1,1)}} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

# Birational rowmotion: order

- Let $\operatorname{ord} \phi$ denote the order of a map or rational map $\phi$. This is the smallest positive integer $k$ such that $\phi^k = \mathrm{id}$ (on the range of $\phi^k$), or $\infty$ if no such $k$ exists.
- The above shows that $\operatorname{ord}(\mathbf{r}) \mid \operatorname{ord}(R)$ for every finite poset $P$.
- Do we have equality?

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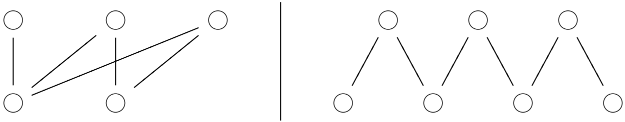
  **No!** Here are two posets with $\operatorname{ord}(R) = \infty$:



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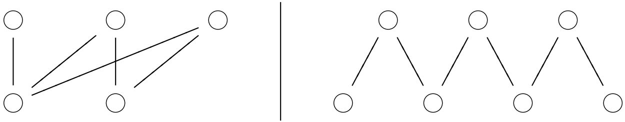

- **Nevertheless**, equality holds for many special types of $P$.

**Example:**

Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^0 f =$

$$\begin{tikzcd}
 & b \arrow[d, dash] & \\
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & w \arrow[d, dash] & \\
 & a &
\end{tikzcd}$$

**Example:**

Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^1 f =$

$$\begin{tikzcd}[column sep=small]
 & b \arrow[d, dash] & \\
 & \frac{b(x+y)}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{bw(x+y)}{xz} \arrow[dr, dash] & & \frac{bw(x+y)}{yz} \arrow[dl, dash] \\
 & \frac{ab}{z} \arrow[d, dash] & \\
 & a &
\end{tikzcd}$$

## Birational rowmotion: example of finite order

**Example:**
Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.
$R^2 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& \frac{bw(x+y)}{xy} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ab}{y} \arrow[dr, dash] & & \frac{ab}{x} \arrow[dl, dash] \\
& \frac{az}{x+y} \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

# Birational rowmotion: example of finite order

**Example:**

Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^3 f =$

$$\begin{tikzcd}
& b \arrow[d, dash] & \\
& \frac{ab}{w} \arrow[dl, dash] \arrow[dr, dash] & \\
\frac{ayz}{w(x+y)} \arrow[dr, dash] & & \frac{axz}{w(x+y)} \arrow[dl, dash] \\
& \frac{xy}{aw(x+y)} \arrow[d, dash] & \\
& a &
\end{tikzcd}$$

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Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

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& b \arrow[d, no head] & \\
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x \arrow[dr, no head] & & y \arrow[dl, no head] \\
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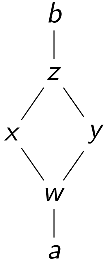


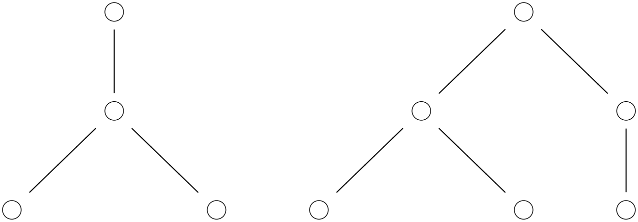

So we are back where we started.

$$\operatorname{ord}(R) = 4.$$

- **Theorem.** Assume that $n \in \mathbb{N}$, and $P$ is a poset which is a forest (made into a poset using the "descendant" relation) having all leaves on the same level $n$ (i.e., each maximal chain of $P$ has $n$ vertices). Then,

$$\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, \ldots, n+1).$$

**Example:**
This poset



has $\operatorname{ord}(R) = \operatorname{ord}(\mathbf{r}) \mid \operatorname{lcm}(1, 2, 3, 4) = 12$.

# Birational rowmotion: the rectangle case

- **Theorem (periodicity):** If $P$ is the $p \times q$-rectangle (i.e., the poset $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$ with coordinatewise order), then
$$\operatorname{ord}(R) = p + q.$$

**Example:** For the $2 \times 2$-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

## Birational rowmotion: the rectangle case

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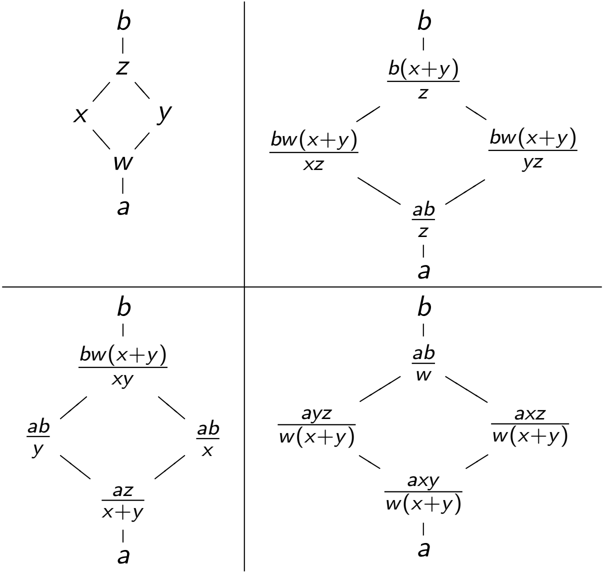
**Example:** For the $2 \times 2$-rectangle, this claims $\operatorname{ord}(R) = 2 + 2 = 4$, which we have already seen.

- **Theorem (reciprocity):** If $P$ is the $p \times q$-rectangle, and $(i, k) \in P$ and $f \in \mathbb{K}^{\widehat{P}}$, then
$$f\left(\underbrace{(p+1-i, q+1-k)}_{\substack{=\text{antipode of } (i,k) \\ \text{in the rectangle}}}\right) = \frac{f(0) f(1)}{\left(R^{i+k-1} f\right)((i, k))}.$$

- These were conjectured (independently) by James Propp and R.

**Example:** Here is the generic $R$-orbit on the $2 \times 2$-rectangle again:



# Birational rowmotion: the rectangle case, example

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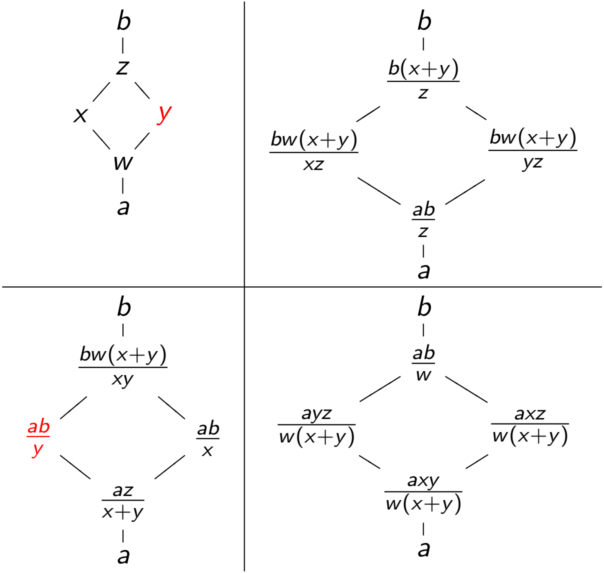

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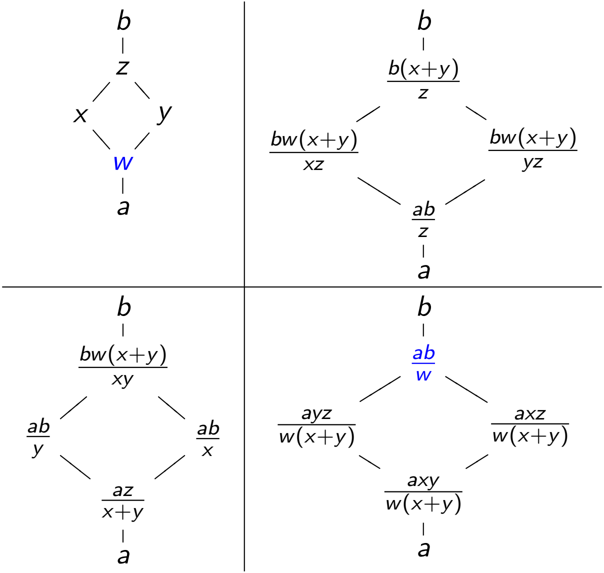


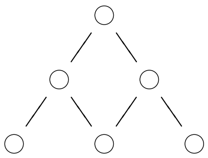
## Birational rowmotion: the rectangle case, proof idea

- Inspiration: Alexandre Yu. Volkov, *On Zamolodchikov's Periodicity Conjecture*, `arXiv:hep-th/0606094`.
- We reparametrize our assignments $f : \widehat{P} \to \mathbb{K}$ through $p \times (p+q)$-matrices in such a way that birational rowmotion corresponds to "cycling" the columns of the matrix.
- This uses a 3-term Plücker relation.
- Lots of technicalities to be managed, particularly around birational maps not necessarily being defined everywhere.

## Birational rowmotion: the $\Delta$-triangle case

- **Theorem (periodicity):** If $P$ is the triangle $\Delta(p) = \{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i+k > p+1\}$ with $p > 2$, then
$$\operatorname{ord}(R) = 2p.$$

**Example:** The triangle $\Delta(4)$:



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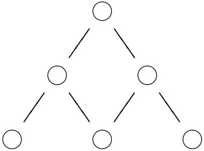

- **Theorem (reciprocity):** $R^p$ reflects any $\mathbb{K}$-labelling across the vertical axis.
- These are precisely the same results as for classical rowmotion.

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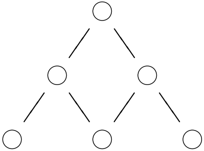
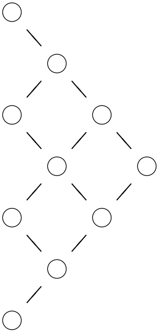


- **Theorem (reciprocity):** $R^p$ reflects any $\mathbb{K}$-labelling across the vertical axis.
- These are precisely the same results as for classical rowmotion.
- The proofs use a "folding"-style argument to reduce this to the rectangle case.

- **Theorem (periodicity):** If $P$ is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k\}$, then

$$\operatorname{ord}(R) = 2p.$$

**Example:** For $p = 4$, this $P$ has the form:



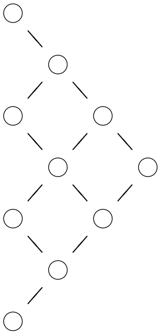
.

## Birational rowmotion: the $\triangleright$-triangle case

- **Theorem (periodicity):** If $P$ is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k\}$, then

$$\operatorname{ord}(R) = 2p.$$

**Example:** For $p = 4$, this $P$ has the form:

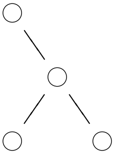


- Again this is reduced to the rectangle case.

- **Conjecture (periodicity):** If $P$ is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ i+k > p+1\}$, then

$$\operatorname{ord}(R) = p.$$

**Example:** For $p = 4$, this $P$ has the form:



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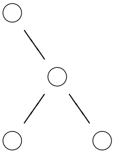
# Birational rowmotion: the right-angled triangle case

- **Conjecture (periodicity):** If $P$ is the triangle $\{(i,k) \in \{1,2,...,p\} \times \{1,2,...,p\} \mid i \leq k;\ \ i+k > p+1\}$, then
$$\operatorname{ord}(R) = p.$$

**Example:** For $p = 4$, this $P$ has the form:



.

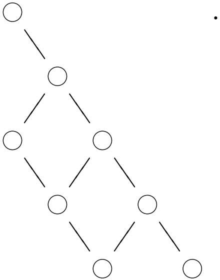
- We proved this for $p$ odd.
- Note that for $p$ even, this is a type-B positive root poset. Armstrong-Stump-Thomas did this for classical rowmotion.

## Birational rowmotion: the trapezoid case (Nathan Williams)

- **Conjecture (periodicity):** If $P$ is the trapezoid $\{(i,k) \in \{1,2,\ldots,p\} \times \{1,2,\ldots,p\} \mid i \leq k;\ i+k > p+1;\ k \geq s\}$ for some $0 \leq s \leq p$, then

$$\operatorname{ord}(R) = p.$$

**Example:** For $p = 6$ and $s = 5$, this $P$ has the form:



- This was observed by Nathan Williams and verified for $p \leq 7$.
- Motivation comes from Williams's "Cataland" philosophy.

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