Open problem: volumes of flow polytopes

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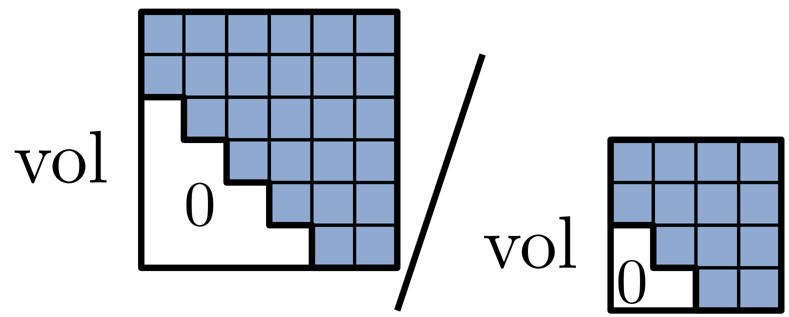


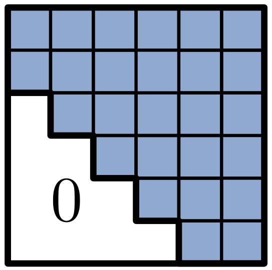
June 23, 2014

joint with: Karola Mészáros, Jessica Striker;
Drew Armstrong, Karola Mészáros, and Brendon Rhoades;
Karola Mészáros

The **Chan-Robbins-Yuen polytope**:

$$\mathcal{CRY}_n := \left\{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{ doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \right\}$$

$$= \text{convex hull } n \times n \text{ permutation matrices}$$



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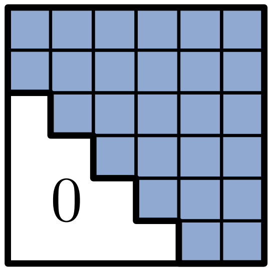

- 2^{n-1} vertices, dimension $\binom{n}{2}$

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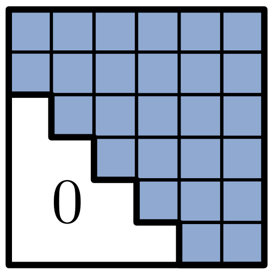

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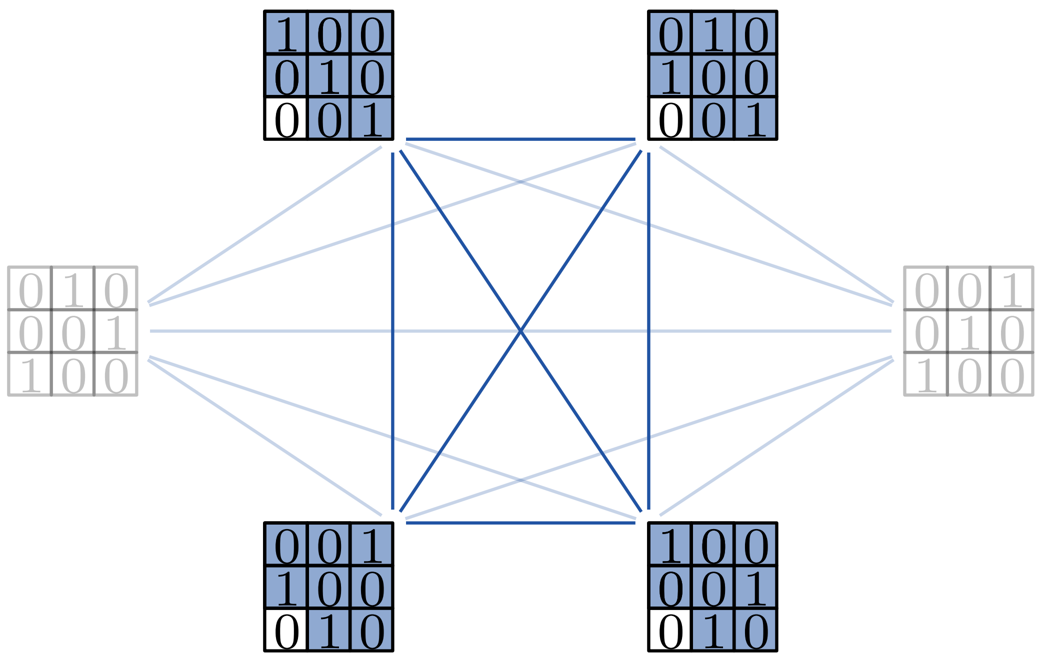
$\mathcal{CRY}_3$



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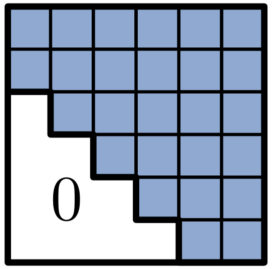

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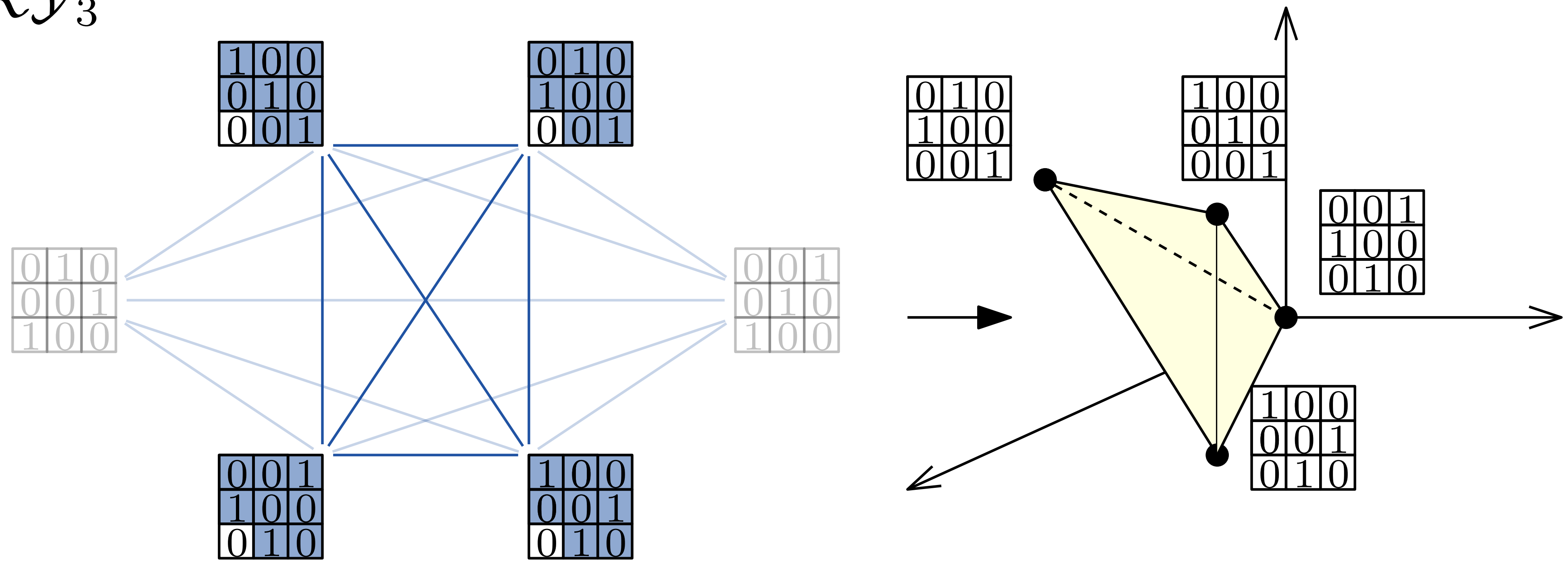
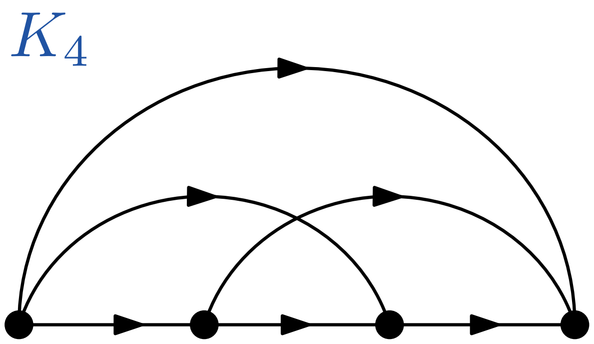


From $\mathcal{CRY}_n$ to a flow polytope

$$\mathcal{CRY}_n := \left\{ (b_{ij}) \in \mathbb{R}^{n^2} \mid \text{ doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \right\}$$

a	b	c
■	d	e
	■	f



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$$a + b + c = 1$$

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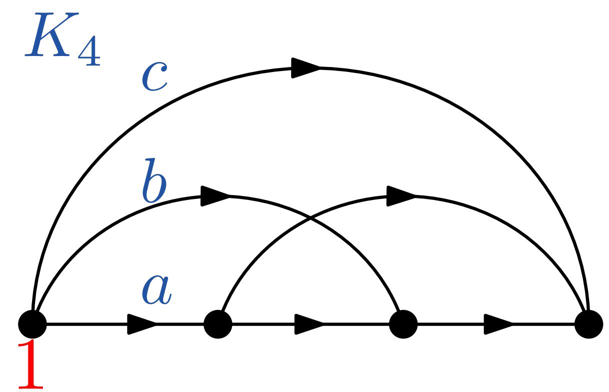

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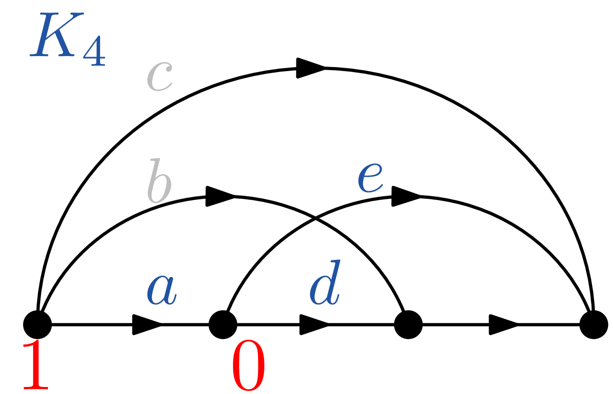
$$d + e - a = 0$$



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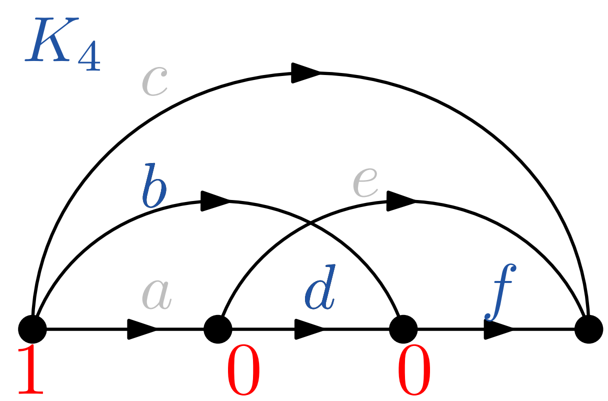

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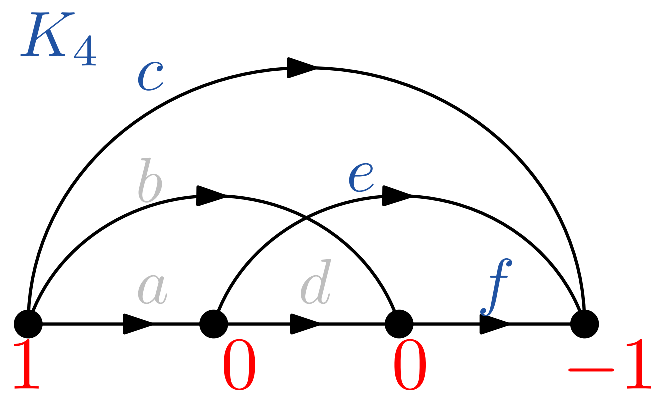

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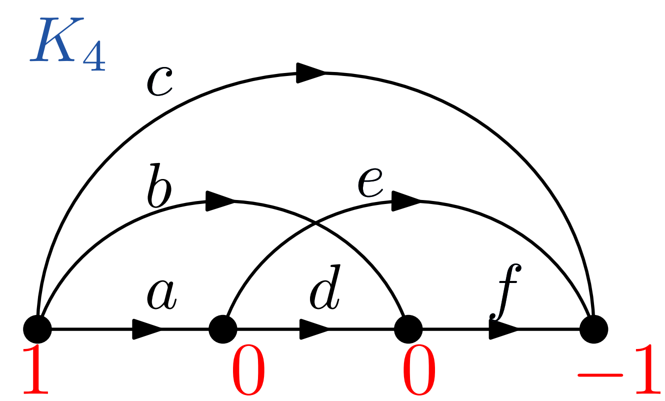

- Correspondence $\mathcal{CRY}_n$ and **flows** in complete graph K_{n+1} with **netflow:** 1 first vertex, -1 last vertex, 0 other vertices.

Volume of the $\mathcal{CRY}_n$ polytope

$v_n := \mathrm{volume}(\mathcal{CRY}_n)$

n	2	3	4	5	6	7
v_n	1	1	2	10	140	5880

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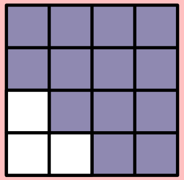
(conjecture Chan-Robbins-Yuen 99)

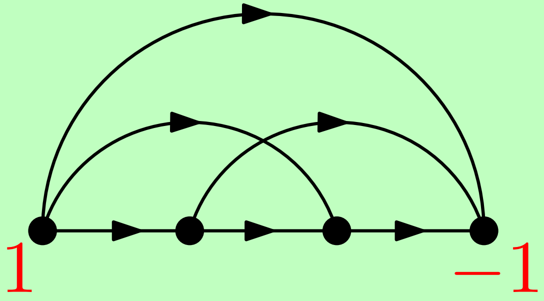
- $v_n = \mathsf{Cat}_0 \mathsf{Cat}_1 \cdots \mathsf{Cat}_{n-2}$ (Zeilberger 99)

$\mathcal{CRY}_n$:

vertices:
permutation matrices



flow polytope complete graph



Variants

$\mathcal{CRY}_n$:

vertices:
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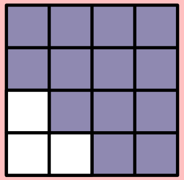
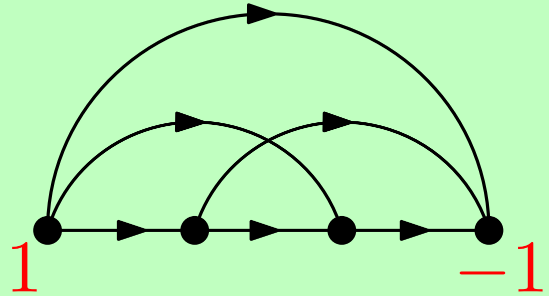


1. vertices: alternating sign matrices

flow polytope complete graph



2. change netflow from $(1, 0, \ldots, 0, -1)$ to $(1, 1, \ldots, 1, -n)$

3. type D analogue of $\mathcal{CRY}_n$

Alternating sign matrices

permutation matrices

- entries are $0, 1$
- rows and columns sum to 1

alternating sign matrices

1	0	0
0	1	0
0	0	1

0	1	0
1	0	0
0	0	1

0	1	0
0	0	1
1	0	0

0	0	1
0	1	0
1	0	0

0	0	1
1	0	0
0	1	0

1	0	0
0	0	1
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Alternating sign matrices

permutation matrices

- entries are $0, 1$
- rows and columns sum to 1

alternating sign matrices

- entries are $0, 1, -1$
- rows and columns sum to 1
- nonzero entries in rows and columns alternate in sign

First enumerated by Zeilberger 92

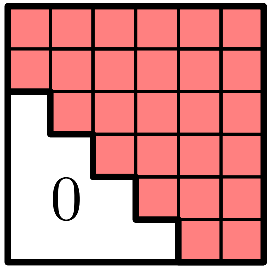
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

1. The $\mathcal{CRY}$ polytope of ASMs

$\mathcal{CRY}_n^{ASM} =$ convex hull $n \times n$ ASMs



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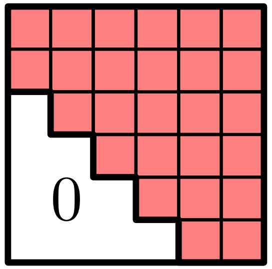

The polytope $\mathcal{CRY}'_n$ of ASMs is an **order polytope** as defined by Stanley 86. **(Mészáros-M-Striker 13+)**

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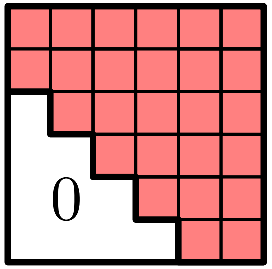

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Example

$.3$	$.4$	$.1$	$.2$
$.7$	$-.2$	$-.1$	$.6$
0	$.8$	$.1$	$.1$
0	0	$.9$	$.1$

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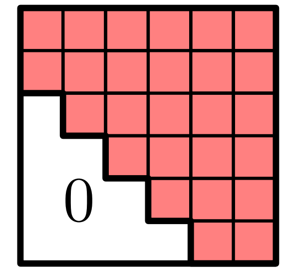

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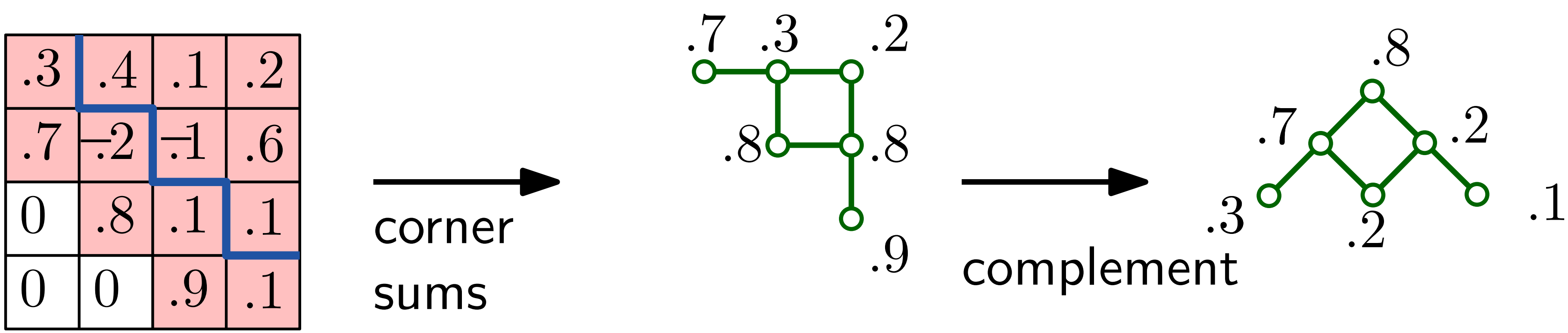

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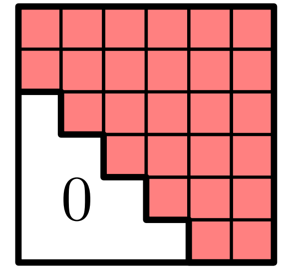

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In EC1

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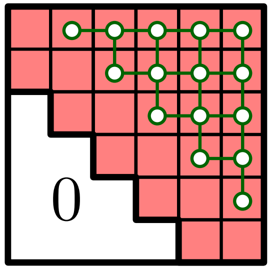
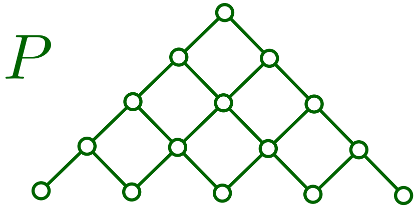


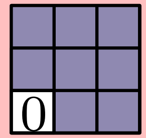


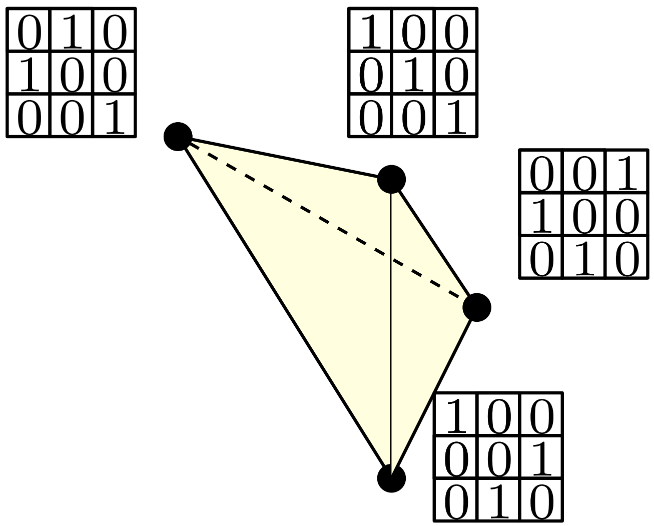
Cat_n vertices

$$\text{volume} = f_{(n-1,n-2,\ldots,1)} = \#SYT(\delta_{n-1})$$

$\mathcal{CRY}_n$:

vertices:
permutation matrices





$\mathcal{CRY}_n^{ASM}$:

vertices:
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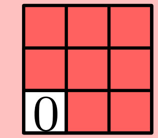
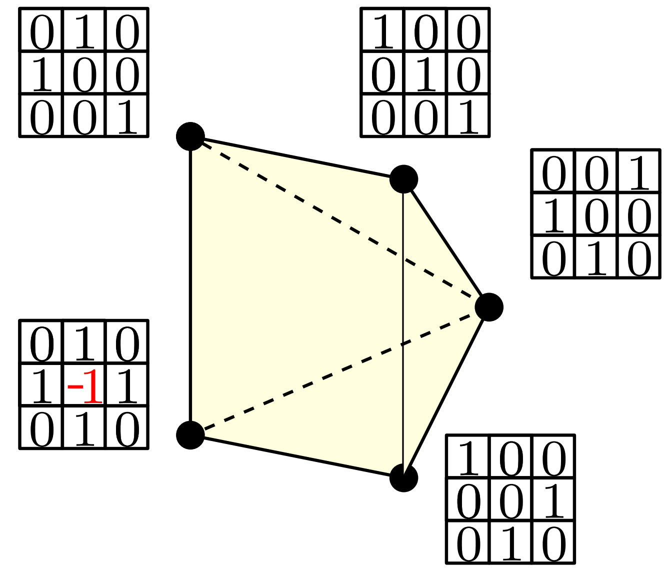




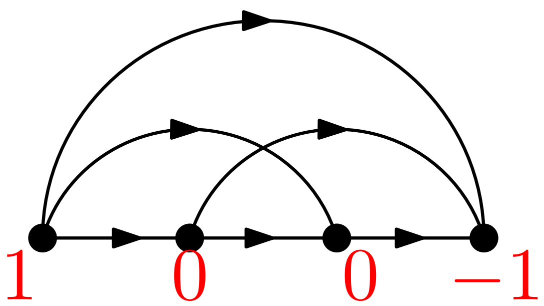
Question

What can we learn about $\mathcal{CRY}_n$ from $\mathcal{CRY}_n^{ASM}$?

2. The Tesler polytope

$\mathcal{CRY}_n$:

flow polytope complete graph

$\mathcal{CRY}_3$: 

- 2^{n-1} vertices
- dimension $\binom{n}{2}$

Theorem (Zeilberger 99):

$$\text{volume} = \prod_{i=0}^{n-2} Cat_i$$

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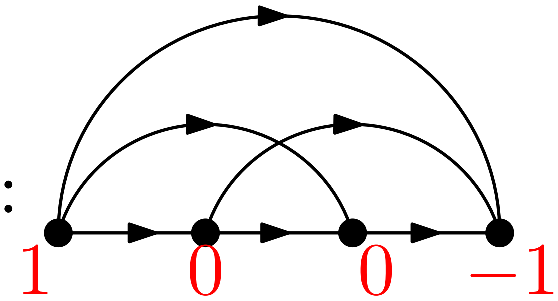
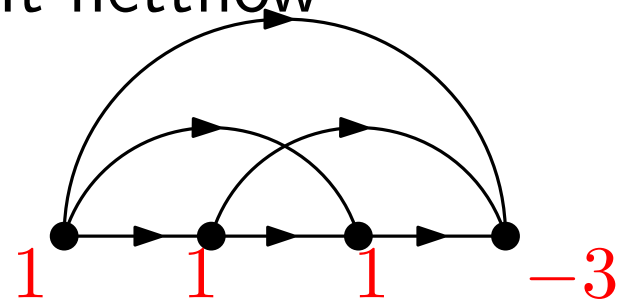


flow polytope complete graph
different nettflow

$\mathcal{T}_3$:



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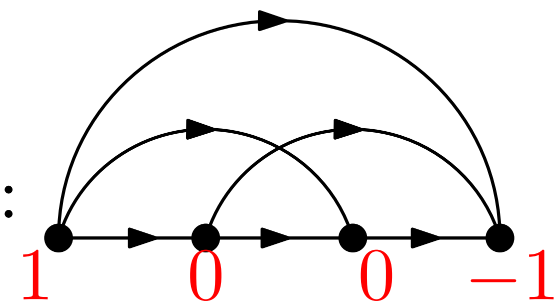

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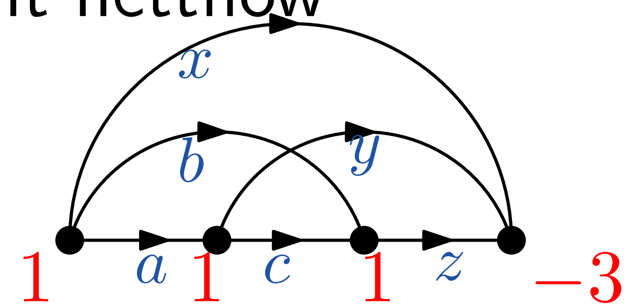

x	a	b
	y	c
		z

- lattice points are **Tesler matrices**

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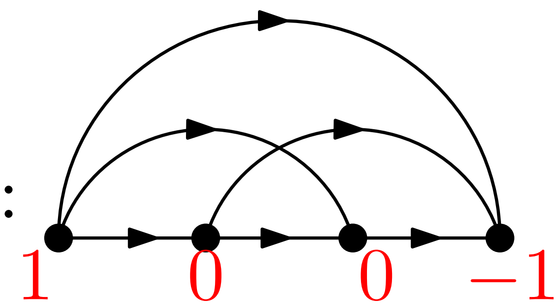

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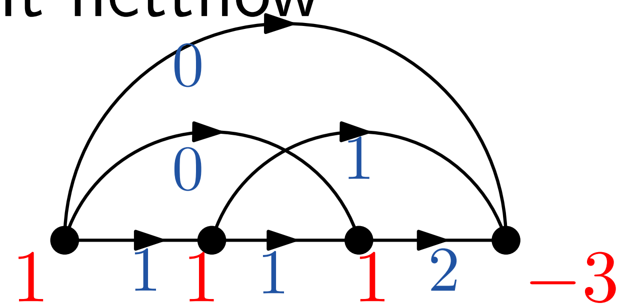

0	1	0
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		2

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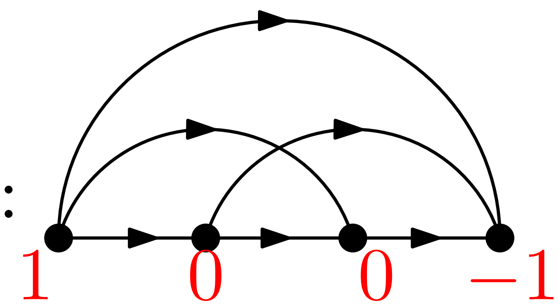

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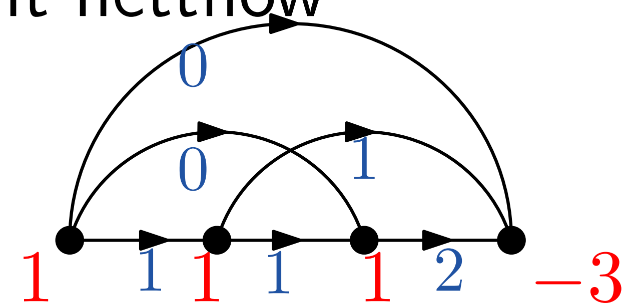

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- $n!$ vertices
- dimension $\binom{n}{2}$

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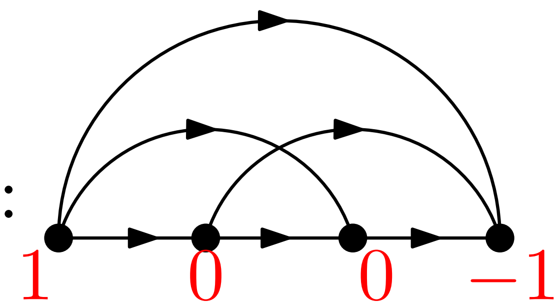

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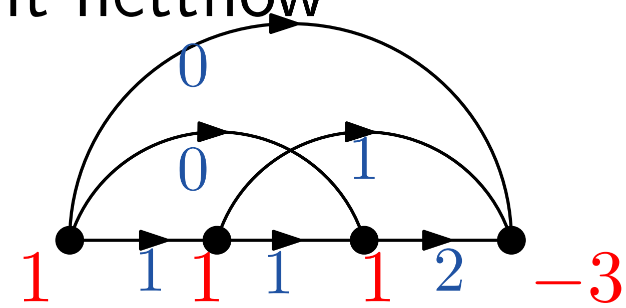

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Theorem

(Armstrong-Mészáros-M-Rhoades 14+)

$$\text{vol} = f_{(n-1,n-2,\dots,1)} \cdot \prod_{i=0}^{n-1} Cat_i$$

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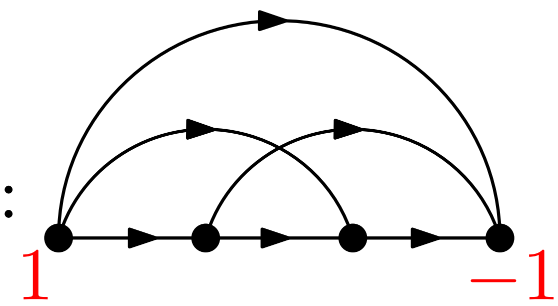
$\mathcal{CRY}_n$:

flow polytope complete graph

$\mathcal{CRY}_3$: 

- 2^{n-1} vertices
- dimension $\binom{n}{2}$

Theorem (Zeilberger 99):

$$\text{volume} = \prod_{i=0}^{n-2} Cat_i$$

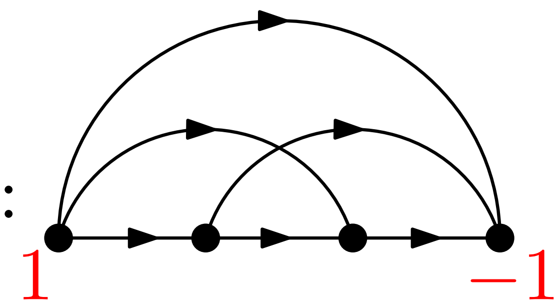
Second generalization $\mathcal{CRY}_n$:

3. The type D $\mathcal{CRY}$ polytope

$\mathcal{CRY}_n$:

flow polytope complete graph

$\mathcal{CRY}_3$:



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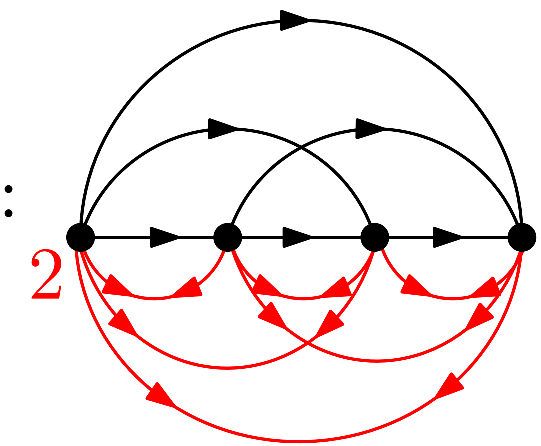
Second generalization $\mathcal{CRY}_n$:

flow polytope complete **signed** graph

$\mathcal{CRY}_3^D$:

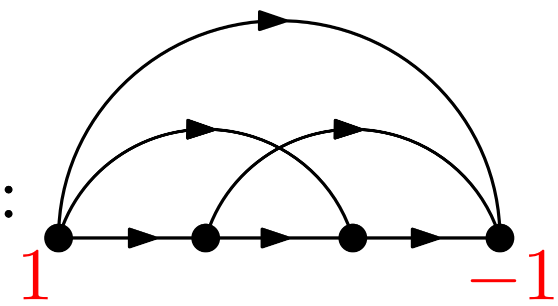


3. The type D $\mathcal{CRY}$ polytope

$\mathcal{CRY}_n$:

flow polytope complete graph

$\mathcal{CRY}_3$:



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flow polytope complete **signed** graph

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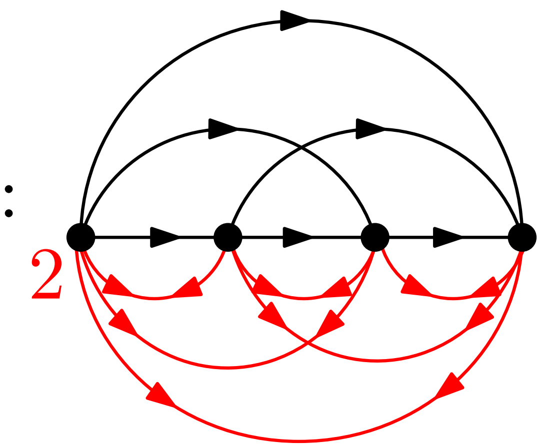


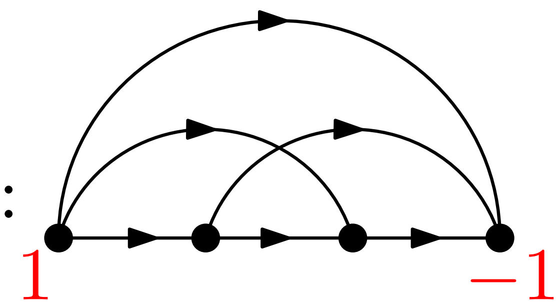
- $3^n - 2^n$ vertices
- dimension $n^2 - 1$

3. The type D $\mathcal{CRY}$ polytope

$\mathcal{CRY}_n$:

flow polytope complete graph

$\mathcal{CRY}_3$:



- 2^{n-1} vertices
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Second generalization $\mathcal{CRY}_n$:

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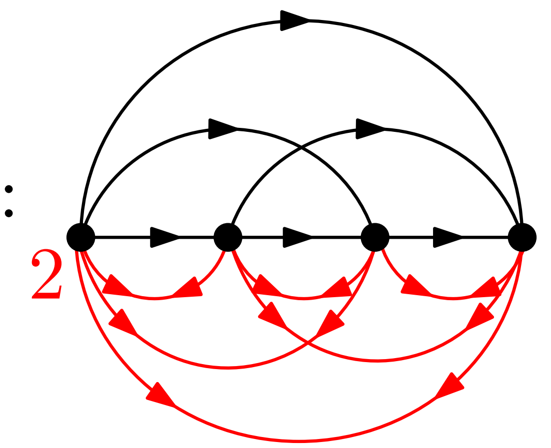

- $3^n - 2^n$ vertices
- dimension $n^2 - 1$

Conjecture: (Mészáros-M 12)

$$\text{volume} = 2^{(n-1)^2} \cdot \prod_{i=0}^{n-1} Cat_i$$

Happy birthday Richard!