A conference in honor of Richard P. Stanley's 70th birthday

Massachusetts Institute of Technology

Stanley's Influence on Monomial Ideals

Takayuki Hibi

Osaka University

25 June 2014

Abstract

Following the pioneering work [Sta75] of Stanley, in the late 1970s a new and exciting trend of commutative algebra, the combinatorial study of **squarefree monomial ideals**, broke out. Since then, it has been one of the most active areas of commutative algebra. In my talk a quick survey of monomial ideal theory developed for the last few decades will be supplied.

[Sta75] R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.* **54** (1975), 135–142.

The legend of Richard Stanley, 1975–1985, again

1972 M. Hochster, Rings of invariants of tori, . . .
1975 R. P. Stanley, The upper bound conjecture . . .
1976 G. A. Reisner, Cohen–Macaulay quotients . . .
1977 M. Hochster, Cohen–Macaulay rings, . . .
1978 R. P. Stanley, Hilbert functions of . . .
1980 R. P. Stanley, The number of faces of . . .
1980 D. Eisenbud, Introduction to algebras with . . .
1980 A. Björner, Shellable and Cohen–Macaulay . . .
1983 R. P. Stanley, "Combin. and Commut. Alg."

Commutative Algebra and Combinatorics
US–Japan Joint Seminar, Kyoto, August, **1985**
(Stanley, Björner, Eisenbud, Buchsbaum, . . .)

Commutative Algebra and Combinatorics
ICM 90 Satellite, Nagoya, August, **1990**
(Stanley, Björner, Billera, Greene, ...)

Computational Commutative Algebra and Combinatorics, Osaka, July, **1999**
(Stanley, Kalai, Herzog, Bruns, Procesi, Novik, Babson, Wagner, Hetyei, Duval, ...)

T. Hibi, Ed., "Computational Commutative Algebra and Combinatorics," Adv. Studies in Pure Math., Vol. 33, Math. Soc. Japan, Tokyo, 2002.

- G. Kalai, Algebraic shifting, pp. 121–163.
- J. Herzog, Generic initial ideals . . . , pp. 75–120.

Workshop on Convex Polytopes
RIMS, Kyoto University, July, **2012**
(Stanley, Kalai, Lee, Bayer, Santos, Ziegler, Liu, Panova, Li, Athanasiadis, ...)

The monograph [HH11] invites the reader to become acquainted with current trends on monomial ideals in computational commutative algebra and combinatorics.

[HH11] J. Herzog and T. Hibi, "Monomial Ideals," GTM 260, Springer, 2011.

This book demonstrates current trends in research on combinatorial and computational commutative algebra with a primary emphasis on topics related to monomial ideals.

Providing a useful and quick introduction to areas of research spanning these fields, *Monomial Ideals* is split into three parts. Part I offers a quick introduction to the modern theory of Gröbner bases as well as the detailed study of generic initial ideals. Part II supplies Hilbert functions and resolutions and some of the combinatorics related to monomial ideals including the Kruskal-Katona theorem and algebraic aspects of Alexander duality. Part III discusses combinatorial applications of monomial ideals, providing a valuable overview of some of the central trends in algebraic combinatorics.

Main subjects include edge ideals of finite graphs, powers of ideals, algebraic shifting theory and an introduction to discrete polymatroids. Theory is complemented by a number of examples and exercises throughout, bringing the reader to a deeper understanding of concepts explored within the text.

Self-contained and concise, this book will appeal to a wide range of readers, including PhD students on advanced courses, experienced researchers, and combinatorialists and non-specialists with a basic knowledge of commutative algebra.

Since their first meeting in 1985, Jürgen Herzog (Universität Duisburg-Essen, Germany) and Takayuki Hibi (Osaka University, Japan), have worked together on a number of research projects, of which recent results are presented in this monograph.

ISBN 978-0-85729-105-9

9 780857 291059

› springer.com

Herzog · Hibi

GTM

260

Monomial Ideals

Graduate Texts in Mathematics

Jürgen Herzog · Takayuki Hibi

Monomial Ideals

Springer

Keywords

Combinatorics

$[n] = \{1, \ldots, n\}$ vertex set

Δ simplicial complex on $[n]$

$\Delta^\vee = \{[n] \setminus F \,;\, F \notin \Delta\}$ Alexander dual

K field

$S = K[x_1, \ldots, x_n]$ polynomial ring over K

$\deg(x_1) = \cdots = \deg(x_n) = 1$

If $F = \{i_1, i_2, \ldots, i_r\} \subset [n]$, then $u_F = x_{i_1} x_{i_2} \cdots x_{i_r}$

$I_\Delta = (u_F \,;\, F \notin \Delta)$ Stanley–Reisner ideal

$K[\Delta] = S/I_\Delta$ Stanley–Reisner ring

Homological algebra

Roughly speaking, a minimal graded free resolution of a monomial ideal $I \subset S$ contains all information of I.

$\boxed{\text{Example}}$ $S = K[x, y, z]$ $I = (x^2, y^3)$

$$0 \longrightarrow S(-5) \xrightarrow[1 \mapsto (-y^3, x^2)]{\begin{bmatrix} -y^3 & x^2 \end{bmatrix}} S(-2) \oplus S(-3) \xrightarrow{\begin{bmatrix} x^2 \\ y^3 \end{bmatrix}} I \longrightarrow 0$$

$$(1, 0) \mapsto x^2$$
$$(0, 1) \mapsto y^3$$

$$S = S_0 \oplus S_1 \oplus S_2 \oplus S_3 \oplus \cdots \quad \text{where } \deg(1) = 0$$
$$S(-2) = (0) \oplus (0) \oplus S_0 \oplus S_1 \oplus \cdots \quad \text{where } \deg(1) = 2$$

Homological algebra

A minimal graded free resolution of a monomial ideal $I \subset S$ is an exact sequence of graded S-modules

$$0 \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

where $F_i = \oplus_j S(-j)^{\beta_{ij}}$ is nonzero and where $\mathrm{Im}(F_i \longrightarrow F_{i-1}) \subset (x_1, \ldots, x_n)F_{i-1}$

$\boxed{\text{Example}}$ $I = (x_4x_5x_6, x_1x_5x_6, x_1x_2x_6, x_1x_2x_5)$

$$0 \to S(-4)^3 \xrightarrow{\begin{bmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & -x_5 & 0 \\ 0 & 0 & x_5 & -x_6 \end{bmatrix}} S(-3)^4 \longrightarrow I \to 0$$

Homological algebra

A minimal graded free resolution of a monomial ideal $I \subset S$ is an exact sequence of graded S-modules

$$0 \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

where $F_i = \oplus_j S(-j)^{\beta_{ij}}$ is nonzero and where $\mathrm{Im}(F_i \longrightarrow F_{i-1}) \subset (x_1, \ldots, x_n)F_{i-1}$

- $\beta_i = \sum_j \beta_{ij}$ $(= \mathrm{rank}(F_i))$ $\quad i$ th Betti number
- $\mathrm{reg}(I) = \max\{\, j\,;\, \beta_{i,i+j} \neq 0,\ \exists i \,\}$ $\quad$ regularity
- $h = \mathrm{proj\,dim}(I)$ $\quad$ projective dimension
- $\mathrm{depth}(S/I) = n - h - 1$ $\quad$ depth of S/I if $I \neq 0$

Homological algebra

A minimal graded free resolution of a monomial ideal $I \subset S$ is an exact sequence of graded S-modules

$$0 \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

where $F_i = \oplus_j S(-j)^{\beta_{ij}}$ is nonzero and where $\mathrm{Im}(F_i \longrightarrow F_{i-1}) \subset (x_1, \ldots, x_n)F_{i-1}$

$\boxed{\text{DEF}}$ We say that I has a linear resolution if its minimal graded free resolution is of the form

$$0 \longrightarrow S(-d-h)^{\beta_h} \longrightarrow \cdots$$
$$\longrightarrow S(-d-1)^{\beta_1} \longrightarrow S(-d)^{\beta_0} \longrightarrow I \rightarrow 0$$

Topics on Monomial Ideals

(a) Alexander duality

1996 **N. Terai and T. Hibi**

By virtue of Alexander duality theorem it is shown that the first Betti number of a Stanley–Reisner ideal is independent of the characteristic of the base field. (Adv. Math. 124, 332–333)

1998 **J. A. Eagon and V. Reiner**

$\boxed{\textbf{Theorem}}$ The Stanley–Reisner ideal I_Δ of a simplicial complex Δ has a linear resolution if and only if the Alexander dual $\Delta^\vee$ of Δ is Cohen–Macaulay. (J. Pure Appl. Alg. 130)

Related results of Eagon–Reiner theorem

- **N. Terai** gives the formula
  $\operatorname{proj\,dim}(I_\Delta) = \operatorname{reg}(K[\Delta^\vee])\ (= \operatorname{reg}(I_{\Delta^\vee}) - 1)$
  which generalizes Eagon–Reiner theorem.
- **J. Herzog** proves that I_Δ has linear quotients if and only if $\Delta^\vee$ is shellable.
- It is known that I_Δ is componentwise linear if and only if $\Delta^\vee$ is sequentially Cohen–Macaulay.
- **E. Miller** studies Alexander duality for arbitrary monomial ideals.
- **T. Römer** and **K. Yanagawa** independently discuss Alexander duality for squarefree modules.
- By virtue of E–R theorem, the Cohen–Macaulay bipartite graphs can be classified ([HH, JAC 22]).

(b) Powers of monomial ideals

2004 **J. Herzog, T. Hibi and X. Zheng**

$\boxed{\textbf{Theorem}}$ Let $I \subset K[x_1, \ldots, x_n]$ be an ideal generated by quadratic squarefree monomials. Then I^N has a linear resolution for $N = 1, 2, \ldots$ if and only if the finite graph on $\{1, \ldots, n\}$ whose edges are those $\{i, j\}$ with $x_i x_j \notin I$ is a chordal graph. (Math. Scand. 95)

(c) Limit depth

Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal.

$$f(k) = \text{depth}(S/I^k),\ k = 1, 2, \ldots \quad \text{depth function}$$

It is known that $f(k)$ is constant for $k \gg 0$. Thus one has $\lim_{k\to\infty} f(k)$ which is called the limit depth of I.

A depth function is not necessarily monotone.

$\boxed{\textbf{Example}}$ (Bandari–Herzog–Hibi, **2013**)

- $S = K[a, b, c, d, x_1, y_1, \ldots, x_n, y_n]$
- $I \subset S$ the monomial ideal generated by

$$a^6,\ a^5b,\ ab^5,\ b^6,\ a^4b^4c,\ a^4b^4d$$

$$a^4x_1y_1^2,\ b^4x_1^2y_1,\ \ldots,\ a^4x_ny_n^2,\ b^4x_n^2y_n.$$

- Then
  depth$(S/I^k) = 0$ if k is odd with $k \leq 2n+1$;
  depth$(S/I^k) = 1$ if k is even with $k \leq 2n$;
  depth$(S/I^k) = 2$ if $k > 2n+1$.

$\boxed{\textbf{Example}}$ Let $2 \leq d < n$ and $I_{n,d}$ the squarefree Veronese ideal of degree d in $S = K[x_1, \ldots, x_n]$. Thus $I_{n,d}$ is generated by all squarefree monomials of degree d in $x_1, \ldots, x_n$. Then

$$\mathrm{depth}(S/I_{n,d}^k) = \max\{\, 0, n - k(n-d) - 1 \,\}$$

$\boxed{\textbf{Conjecture}}$ (a) The depth function of a squarefree monomial is nonincreasing.

(b) Given a nonincreasing function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ which is eventually constant, there exists a squarefree monomial ideal $I \subset K[x_1, \ldots, x_n]$ for $\exists n$ with $f(k) = \mathrm{depth}(S/I^k)$ for all $k \geq 1$.

2005 **J. Herzog and T. Hibi**

$\boxed{\textbf{Theorem}}$ Given a bounded nondecreasing function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$, there exists a monomial ideal $I \subset K[x_1, \ldots, x_n]$ for $\exists n$ with $f(k) = \mathrm{depth}(S/I^k)$ for all $k \geq 1$. (J. Alg. 291)

$\boxed{\textbf{Conjecture}}$ Given an **arbitrary** function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ which is eventually constant, there exists a monomial ideal $I \subset K[x_1, \ldots, x_n]$ for $\exists n$ with $f(k) = \mathrm{depth}(S/I^k)$ for all $k \geq 1$.

The influence of

R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.* **54** (1975), 135–142.

is really big !