Real-rootedness results for triangulation operations inspired by the Tchebyshev polynomials

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June 27, 2014, Stanley@70.

The recent part of the research presented here is joint work with Eran Nevo.

Stanley combination plane

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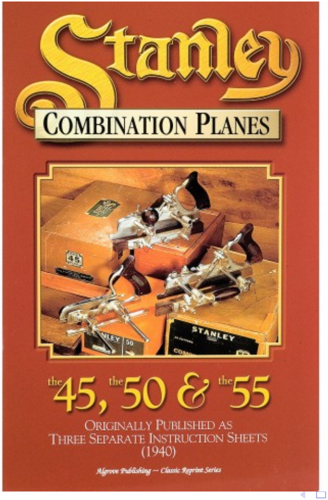

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The resulting poset is the *Tchebyshev transform* of the original.

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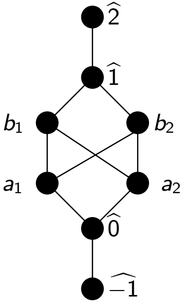

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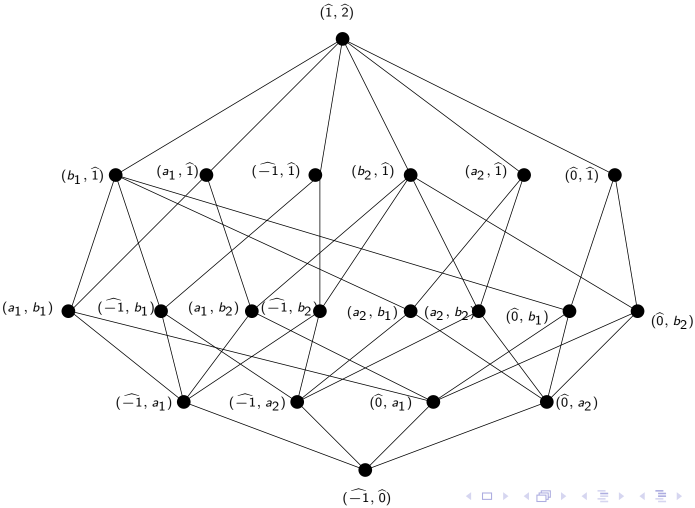

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3. Takes the Cartesian product of posets into the diamond product of their Tchebyshev transforms (Ehrenborg-Readdy)
4. Induces a Hopf algebra endomorphism on the ring of quasisymmetric functions (Ehrenborg-Readdy)

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Note to Richard: For an Eulerian poset P, substituting $c = x$ and $e = 1$ yields into the ce-index $F(\triangle(P \setminus \{\widehat{0}), \widehat{1}\}, x)$.

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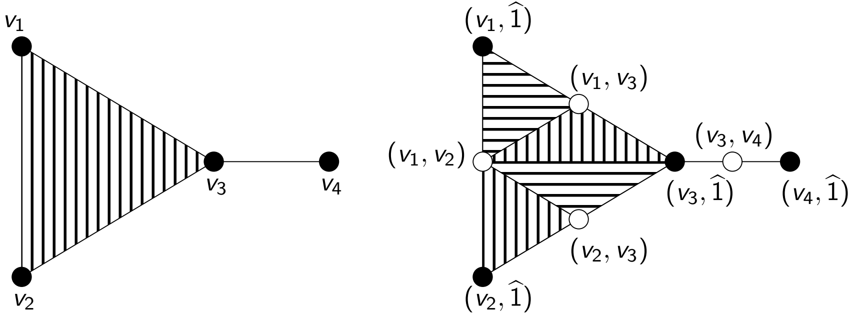

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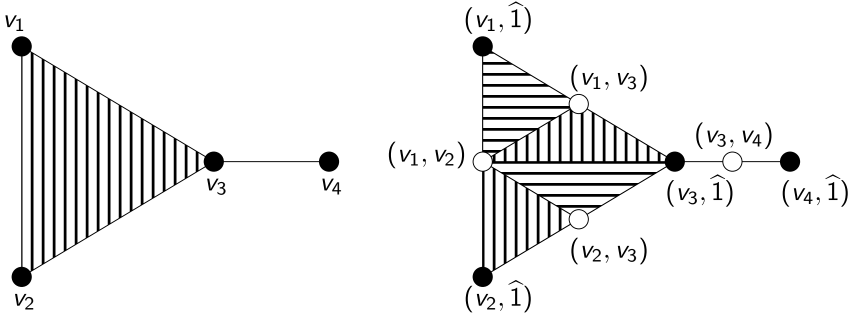

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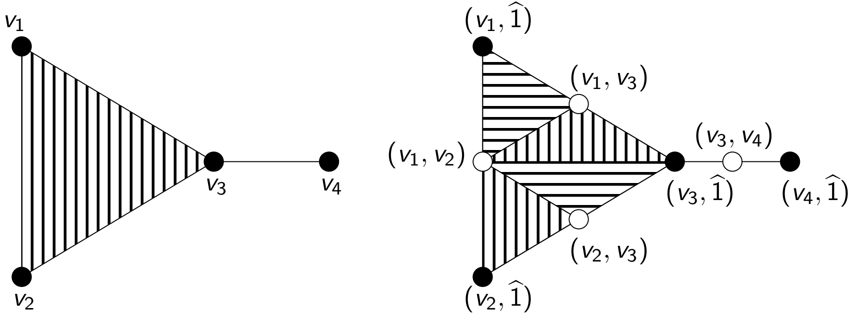

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$$F(T(\triangle), x) = T(F(\triangle, x)),$$

where $T(x^n) = T_n(x)$.

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Combinatorial interpretation?

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(For each root of multiplicity m also use $t^k e^{\lambda t}$ for $k = 0, 1, \ldots, m$.) $\lim_{t\to\infty} t^k e^{\lambda t} = 0$ iff λ has negative real part.

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The *Möbius transformation* $z \mapsto \frac{z+1}{z-1}$ takes the left t-halfplane into the unit disk.

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$$(1-t)^d \cdot F_{\triangle}\left(\frac{1+t}{1-t}\right) = (1-t)^d \sum_{j=0}^{d} f_j \left(\frac{t}{1-t}\right)^j = h_{\triangle}(t).$$

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Corollary

If $F(\triangle, x)$ *is* Schur stable *(=its zeros are inside the disk* $|x| < 1$*) then* $h(\triangle, t)$ *is* Hurwitz stable *(=its zeros are inside the left t-halfplane). The converse also holds for homology spheres (or whenever* $\deg h(\triangle, t) = \deg F(\triangle, x)$*).*

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The join $\triangle_1 * \triangle_2$ *is* S*-stable (*H*-stable) if and only if both* $\triangle_1$ *and* $\triangle_2$ *are* S*-stable (*H*-stable).*

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Conjecture

The direct product of S-stable graded posets is S-stable.

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The proof uses *Lucas' theorem* stating that the roots of the derivative are in the convex hull of the roots of the original polynomial.

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Corollary

All Boolean algebras $B_n = I \times I \times \cdots \times I$ are S-stable.

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The order complex of a Boolean algebra is the barycentric subdivision of a simplex. The h-vector entries being all positives, all roots must be real and negative.

An application to the derivative polynomials for tangent and secant

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They are defined by

$$\frac{d^n}{dx^n}\tan(x) = P_n(\tan x) \quad \text{and} \quad \frac{d^n}{dx^n}\sec(x) = Q_n(\tan x)\cdot\sec(x).$$

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Proposition

The zeros of $P_n(x)$ and $Q_n(x)$ are pure imaginary, have multiplicity 1*, belong to the line segment $[-\mathbf{i},\mathbf{i}]$ and are interlaced with $-\mathbf{i}$ and $\mathbf{i}$ being zeros of $P_n(x)$.*

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Let $T_n^B(x)$ be the F-polynomial of the Tchebyshev transform of the boolean algebra B_n. Then we have

$$T_n^B(x) = (-1)^n \widetilde{Q}_n(x),$$

where $\widetilde{Q}_n(x)$ is the derivative polynomial for hyperbolic secant.

Elements of the proof

Define $U, T : \mathbb{R}[x] \to \mathbb{R}[x]$ as the linear maps, sending x^n into $T_n(x)$ and $U_{n-1}(x)$, respectively. (Tchebyshev polynomials of the first, resp. second kind.)

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The proof uses *Schelin's theorem* "backwards".

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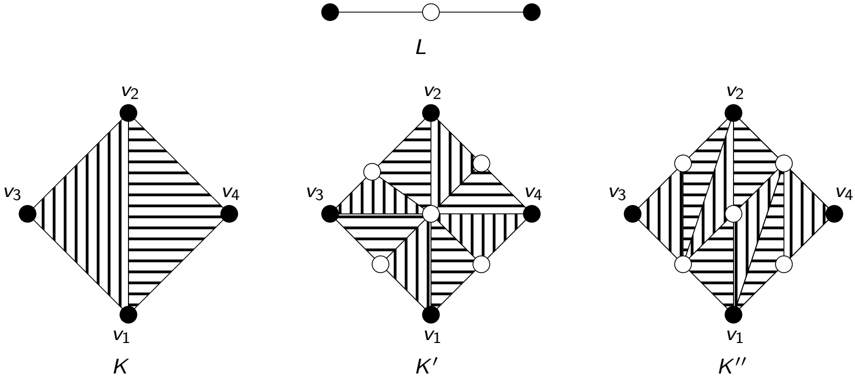

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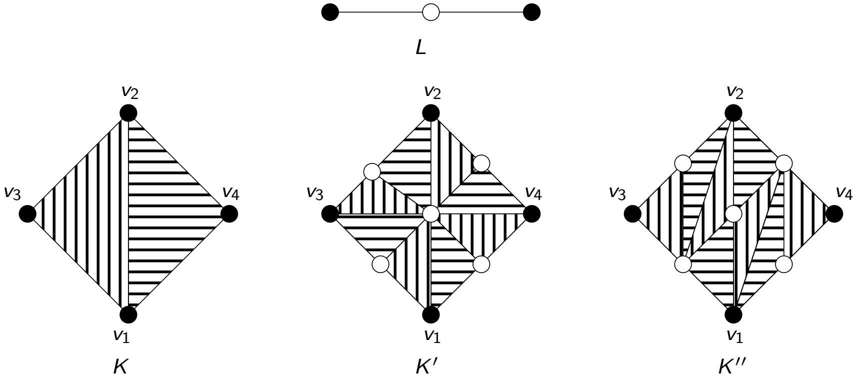

Different triangulations, same face numbers.

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Theorem (H.-Nevo)

The face numbers of K' do not depend on the order of the k-faces and they depend on the face numbers of K in a linear fashion.

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3. $(-1)^n T_n^L(-x) = T_n^L(x)$ (Dehn-Sommerville equations).
4. All real roots of $T_n^L(x)$ belong to the interval $(-1, 1)$. (Nonnegativity of the h-numbers.)

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1. Let L be the subdivision of the 1-simplex by s interior vertices. Then

$$T_n^L(x) = \left(\sqrt{x^2 + s(1-x^2)}\right)^n \cos(n\alpha(x)),$$

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2. Let L be the simplex obtained from a tetrahedron just by adding one new interior vertex and connecting it to all four original vertices. Then $T_6^L(x) = 6 - 9x^2 - 60x^4 + 64x^6$ has only 4 real roots.

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We have $T_0(x) = 1$, $T_1(x) = 1$, $T_2(x) = x^2$ and

$$\begin{aligned} T_n^L(x) &= 3xT_{n-1}^L(x) + ((e-3)x^2 - e)T_{n-2}^L(x) \\ &\quad + ((2m+1-e)\cdot x^3 + (e-2m)x)\cdot T_{n-3}^L(x) \\ &\quad \text{for } n \geq 3. \end{aligned}$$

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(It is true in general that $T_n^L(x)$ satisfies $T_n^L(x) = x^n$ for $n \leq \dim L$, and a "Fibonacci type recurrence".)

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Cardano's formula gives

$$q_j(x) = x + \omega^j \sqrt[3]{(x-1)(x+1)^2} + \omega^{2j} \sqrt[3]{(x-1)^2(x+1)}$$

where $j \in \{0, 1, 2\}$ and $\omega = e^{i2\pi/3}$.

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We rewrite this as

$$T^L_n(x)/x = \frac{||q_1(x)||^{n-1}}{3}\left(\left(\frac{q_0(x)}{||q_1(x)||}\right)^{n-1} + \frac{q_1(x)^{n-1} + \overline{q_1(x)^{n-1}}}{||q_1(x)||^{n-1}}\right).$$

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Equivalently,

$$T_n^L(x)/x = \frac{||q_1(x)||^{n-1}}{3}\left(\left(\frac{q_0(x)}{||q_1(x)||}\right)^{n-1} + 2\cos((n-1)\alpha(x))\right),$$

where $\alpha(x)$ is the argument of $q_1(x)$.

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The function $2\cos((n-1)\alpha(x))$ has at least $(n-1)$ zeros inside the interval $(-1,1)$. Before the least zero, between two consecutive zeros, and after the largest zero this attains 2 or -2, thus leaving (and, with the exception of the segment after the largest zero, reentering) the region between the lines $y = -1$ and $y = 1$.

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The continuous function

$$-\left(\frac{q_0(x)}{||q_1(x)||}\right)^{n-1} : [-1,1] \to [-1,1]$$

never leaves this horizontal region, thus its graph must intersect the graph of $2\cos((n-1)\alpha(x))$ at least $n-1$ times.

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The proof of the general case is similar, but more complicated.

The end (?)

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HAPPY BIRTHDAY, RICHARD!